OPERATION REGIMES OF CERENKOV–SMITH-PURCELL FREE ELECTRON LASERS AND T.W. AMPLIFIERS *

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We present a unified linear analysis which describes free electron lasers based on Cerenkov and Smith-Purcell effects and T.W. amplifiers. In the single electron interaction regime the analysis reproduces the exponential and non exponential (interferential) gain expressions. It is shown that the first expression is dominant and that considerable increase in the gain can be attained by using electron beams with small velocity spread.

There has been recently considerable theoretical and experimental interest in the concept of free electron lasers [1–11]. Laser amplification at 10.6 μm wavelength [2] and laser oscillation at 3.4 μm [3], were demonstrated using "magnetic bremsstrahlung" radiation scheme, proving that coherent stimulated emission from accelerated free electrons is possible at optical frequencies.

Free electron lasers based on the Smith-Purcell effect [12] or Cerenkov radiation have been suggested [4,9]. Such free electron lasers if realized could be made with much more compact device structures compared to the bremsstrahlung laser, and therefore may be interesting for applications.

A gain expression for the Smith-Purcell–Cerenkov free electron laser was derived in ref. [4] where it was shown that at the single electron interaction regime the exponential gain is proportional to the electron density n_0 and to the derivative of the electron beam distribution function at velocity corresponding to the phase velocity (ω/β) of the amplified electromagnetic wave $-f_0'(v)|v=\omega/\beta$, ω and β are the angular frequency and the wavenumber of the electromagnetic wave respectively. It was also shown there that in the collective interaction regime the dispersion equation of the free electron laser reduces to that of the conventional T.W. amplifier [13]. In ref. [14] the Cerenkov amplifier was analyzed in the collective regime and it was shown that in this regime the device operates essentially as a T.W. amplifier and its gain at synchronism is proportional to $n_0^{1/3}$. In another place [9] the analysis was extended to a warm beam where again it was shown that the gain is proportional to n_0 and to the derivative of the distribution function.

Recently Yariv [10] and Wachtell [11] independently developed different analyses of the Smith-Purcell laser, showing that the gain is *not* exponential and is proportional to a function

$$F(\theta) \equiv \frac{d}{d\theta}\left(\frac{\sin(\theta/2)}{\theta/2}\right)^2, \qquad \theta \equiv (\omega/v_0 - \beta)l, \qquad (1,2)$$

where v_0 is the electron beam velocity and l is the interaction length. This function is the same function which appears in the analysis of the bremsstrahlung free electron laser [6]. Similar gain expressions were previously derived by Russian workers for a device called "Orotron" which is closely related to the Smith-Purcell laser [15,16]. Unfortunately, their theoretical work did not appear in periodical English translations and it was hardly familiar to scientists in the west.

In the present letter we show that the different gain expressions obtained by the different workers and the conventional expressions of the T.W.T. gain are all valid in different operational regimes, and can be derived as different limits of one comprehensive theory. Using this comprehensive model we will compare the different operation regimes and find the best condition

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and highest optical gain which can be obtained from the free electron laser.

Assume a Cerenkov or Smith-Purcell laser amplifier which consists of a slow wave structure (waveguide) and an electron beam passing along it. In the first case the slow-waveguide is a dielectric waveguide and the electron beam passes in vacuum close to its boundary or through a bore in its center so that the electrons experience the longitudinal electric field of a TM mode propagating along it. In the latter case the slow wave structure is a waveguide with periodic corrugation of its boundaries and the electron beam interacts with a slow space harmonic of a TM Floquet mode which propagates through the waveguide [5]. The z axis is oriented at the waveguide axis and electron beam direction and its origin is at the waveguide entrance.

The interaction of the electron beam and an electromagnetic wave propagating along the waveguide is governed by Maxwell and Vlasov-equations

$$\nabla \times H = -\mathrm{i}\omega\epsilon E + J, \qquad \nabla \times E = \mathrm{i}\omega\mu H \tag{3,4}$$

$$-\mathrm{i}\omega f_1(z,p) + v\,\partial f_1(z,p)/\partial z = e\,[\mathrm{d}f_0(p)/\mathrm{d}p]\,E_z . \tag{5}$$

where $f_0(p)$ is the unperturbed one dimensional electron beam distribution normalized to the electron density [15]

$$n_0 = \int_{-\infty}^{\infty} f_0(p)\,\mathrm{d}p, \qquad J_z = -\frac{e}{m}\int_{-\infty}^{\infty} f_1(z,p)\,p\,\mathrm{d}p.$$

To simplify the problem it is desirable to reduce the three dimensional Maxwell eqs. (3), (4) into a differential equation in z only like (5). This is done following a technique derived in [17,18]. It will be presented in greater detail in a later publication. Taking first the case of Cerenkov laser (dielectric waveguide), the derivation is based on substituting the expansion of the transverse electric field $E_t(x,y,z)$ in terms of the waveguide eigenmodes $(\mathcal{E}_m(x,y)\,\mathrm{e}^{\mathrm{i}\beta_m z})$

$$E_t = \sum_m a_m(z)\mathcal{E}_m(x,y) \tag{8}$$

into the Maxwell equations (3), (4) and using the ortogonality of the modes. One obtains

$$E_z(x,y,z) = \sum_m a_m(z)\mathcal{E}_{mz}(x,y) + \frac{1}{\mathrm{i}\omega\epsilon}J_z \tag{9}$$

$$\mathrm{d}a_m(z)/\mathrm{d}z - \mathrm{i}\beta_m a_m(z)$$
$$= \frac{1}{\mathcal{N}_m}\iint_{-\infty}^{\infty} J\cdot\mathcal{E}_{-m}(x,y)\exp(-\mathrm{i}\beta_{-m}z)\,\mathrm{d}x\,\mathrm{d}y \tag{10}$$

$$\mathcal{N}_m \equiv \iint_{-\infty}^{\infty}(\mathcal{E}_m\times\mathcal{H}_{-m} - \mathcal{E}_{-m}\times\mathcal{H}_m)\cdot e_z\,\mathrm{d}x\,\mathrm{d}y . \tag{11}$$

We simplify the problem further by assuming a single mode m in the waveguide and that the electron beam cross section area A_e is small so that the field across it can be considered constant $\mathcal{E}_m(x,y)\approx\mathcal{E}_m(x_e,y_e)$. Thus the problem was reduced to the solution of two coupled linear first order differential equations (5), (10). These can be readily solved after these equations and (7), (9) are Laplace transformed $(\bar{g}(s)\equiv\int_0^\infty \mathrm{e}^{-sz}g(z)\mathrm{d}z)$:

$$-\mathrm{i}\omega\bar{f}_1(s,p) + vs\bar{f}_1(s,p) = ef_0'(p)\bar{E}_z(x_e,y_e,s) \tag{12}$$

$$\bar{J}_z(s) = -e\int_{-\infty}^{\infty}\bar{f}_1(s,p)\,v\,\mathrm{d}p, \tag{13}$$

$$\bar{E}_z(x_e,y_e,s) = \bar{a}_m(s)\mathcal{E}_{mz}(x_e,y_e) + \frac{1}{\mathrm{i}\omega\epsilon}\bar{J}_z(s), \tag{14}$$

$$(s-\mathrm{i}\beta_m)\bar{a}_m(s) - a_m(0) = \frac{A_e}{\mathcal{N}_m}\mathcal{E}_{mz}(x_e,y_e)\bar{J}_z(s), \tag{15}$$

where we assumed $f_1(0,p)=0$.

Eqs. (12) to (15) can then be solved for $\bar{a}_m(s)$

$$\bar{a}_m(s) = a_m(0) \tag{16}$$
$$\times\frac{1+\frac{1}{2}(k_D'^2/s^2)G'(\zeta)}{(s-\mathrm{i}\beta_m)[1+\frac{1}{2}(k_D'^2/s^2)G'(\zeta)] + \frac{1}{2}\mathrm{i}\kappa(k_D'^2/s^2)G'(\zeta)}$$

where

$$G(\zeta)\equiv\int_{-\infty}^{\infty}\frac{g(x)}{x-\zeta}\,\mathrm{d}x. \tag{17}$$

$g(x)$ is the normalized distribution function:

$$\frac{n_0}{p_{th}}\,g\!\left(\frac{p-p_0}{p_{th}}\right)\equiv f_0(p). \tag{18}$$

$p_0=\gamma_0 m v_0$ $(\gamma_0\equiv(1-v_0^2/c^2)^{-1/2})$ is the average electron momentum of the beam and p_{th} is the momentum spread of the electron distribution.

$$\zeta \equiv (\mathrm{i}\omega/s - v_o)/v'_{th}, \qquad k'^2_D \equiv 2\omega'^2_p/v'^2_{th}, \tag{19,20}$$

$$\omega'^2_p \equiv \omega^2_p/\gamma^3_o \equiv e^2 n_o/m\epsilon\gamma^3_o. \tag{21}$$

$$v'_{th} \equiv p_{th}/\gamma^3_o m, \qquad \kappa \equiv \omega A_e \epsilon |\mathcal{E}_{mz}|^2/N_m. \tag{22,23}$$

For a lossless mode (β_m real) $N_m = 4P_m$, where P_m is the power of the mode. We thus can express κ in terms of the interaction impedance [13] $K_m \equiv |\mathcal{E}_{mz}|^2/2\beta^2_m P_m$.

$$\kappa = \tfrac{1}{2}\omega A_e \epsilon \beta^2_m K_m. \tag{24}$$

In the case where instead of a dielectric waveguide (Cerenkov laser) a periodic waveguide is used, like in T.W.T. and Smith-Purcell laser, the expressions obtained hold just the same except that β_m stands for the wave number of the space harmonic with which the interaction takes places and $\mathcal{E}_{mz}$ in eq. (23) is the field of this space harmonic.

To find the amplitude of the mode $a_m(z)$ after interaction length l, one should perform the inverse Laplace transform of eq. (16). This would give

$$a_m(l) = \sum_j A_j \exp(s_j l), \tag{25}$$

where s_j are the nulls of the denominator of eq. (16) and A_j are found from the residues at s_j. Thus

$$(s - \mathrm{i}\beta_m)\left[1 + \frac{1}{2}\frac{k'^2_D}{s^2}G'(\zeta)\right] + \frac{\mathrm{i}}{2}\kappa\frac{k'^2_D}{s^2}G'(\zeta) = 0 \tag{26}$$

is the dispersion equation of the coupled electromagnetic wave and electron beam. As $\kappa \to 0$ (no interaction) it reduces to $s^{(0)} = \mathrm{i}\beta_m$ and $1 + \frac{1}{2}(k'^2_D/s^2)G'(\zeta) = 0$ which are the wavenumber of the unperturbed electromagnetic mode and the plasma dispersion equation of the unperturbed electron beam respectively.

In order to obtain explicit solutions for s_j we will examine eq. (16) at various limits. We first assume

$$|\zeta| \gg 1. \tag{27}$$

At this limit the plasma dispersion function $G(\zeta)$ can be substituted by its asymptotic approximation

$$G(\zeta) = -1/\zeta. \tag{28}$$

This and (19) can now be substituted in (16), giving

$$a_m(s) = \frac{[(s - \mathrm{i}\omega/v_o)^2 + \omega'^2_p/v^2_o]\, a_m(0)}{(s - \mathrm{i}\beta_m)[(s - \mathrm{i}\omega/v_o)^2 + \omega'^2_p/v^2_o] + \mathrm{i}\kappa(\omega'^2_p/v^2_o)} \tag{29}$$

Thus the dispersion equation in this case is a third degree equation

$$(s - \mathrm{i}\beta_m)[(s - \mathrm{i}\omega/v_o)^2 + \omega'^2_p/v^2_o] + \mathrm{i}\kappa(\omega'^2_p/v^2_o) = 0, \tag{30}$$

which can be related to the classical dispersion equation of T.W. amplifiers [13].

Near synchronism ($\beta_m \approx \omega/v_o$) and in the limit of strong interaction

$$\frac{\kappa}{\omega'_p/v_o} \gg 1 \tag{31}$$

it simplifies to

$$(s - \mathrm{i}\beta_m)^3 = -\mathrm{i}\kappa(\omega'^2_p/v^2_o). \tag{32}$$

This equation has three roots. The root which has a positive real part and corresponds to gain is

$$s = \mathrm{i}\beta_m + \tfrac{1}{2}(\kappa\,\omega'^2_p/v^2_o)^{1/3}(\mathrm{i} + \sqrt{3}) \tag{33}$$

so that the gain is exponential and proportional to $n_o^{1/3}$ as calculated by Walsh [14] and known for T.W. amplifiers.

When condition (31) does not apply and when out of synchronism, another approximation can be used. Assuming κ is small, then in zero order in κ the solutions of (30) are

$$s^{(0)}_1 = \mathrm{i}\beta_m, \qquad s^{(0)}_2 = \mathrm{i}\omega/v_o + \mathrm{i}\omega'_p/v_o,$$

$$s^{(0)}_3 = \mathrm{i}\omega/v_o - \mathrm{i}\omega'_p/v_o. \tag{34}$$

We further calculate the roots s_j and the residues A_j to first order in κ using eqs. (29), (30). These are substituted in (24) and after some tedious mathematical calculation the expression for the optical power gain is found to be

$$\frac{P(l)}{P(0)} = \frac{|a_m(l)|^2}{|a_m(0)|^2} = 1 + \kappa\frac{\omega'^2_p}{v_o} l^3 F(\theta, \theta_p), \tag{35}$$

where

$$F(\theta, \theta_p) \equiv \frac{1}{\theta_p}\left[\frac{1-\cos(\theta+\theta_p)}{(\theta+\theta_p)^2} - \frac{1-\cos(\theta-\theta_p)}{(\theta-\theta_p)^2}\right] \quad (36)$$

$$\theta_p \equiv \frac{\omega_p'}{v_0} l, \quad (37)$$

and θ is defined in (2). In deriving this expression we had to assume a short enough interaction length

$$\kappa l \ll |\theta|/\theta_p . \quad (38)$$

Similar nonexponential gain was derived earlier in [16] for the "Orotron", and in the limit $\theta_p \to 0$

$$F(\theta, 0) = F(\theta), \quad (39)$$

where $F(\theta)$, given in (1), is the gain curve which was derived independently in [6], [10] and [11] for bremsstrahlung and Smith-Purcell laser in the single electron interaction regime.

We now recall that in deriving (29) we had to assume inequality (27). Consider now the opposite regime

$$|\zeta| \lesssim 1. \quad (40)$$

In the limit of weak interaction ($\kappa \to 0$) or when $k_D'^2/\beta_m^2 \ll 1$ one can again solve the dispersion equation (26) by first order expansion of its roots in terms of κ. In the regime (40) it is possible to show that the plasma dispersion equation $1 - \frac{1}{2}(k_D'^2/\beta_m^2)G'(\zeta) = 0$ has only complex solutions for s. These solutions correspond to plasma waves which decay strongly by Landau damping. It is sufficient then to consider only the root of (26) which corresponds to the electromagnetic wave (β_m) coupled to the plasma. To first order in κ:

$$s^{(1)} = i\beta_m + i\kappa \frac{\frac{1}{2}(k_D'^2/\beta_m^2)G'(\zeta)}{1 - \frac{1}{2}(k_D'^2/\beta_m^2)G'(\zeta)}, \quad (41)$$

$$\zeta = (\omega/\beta_m - v_0)/v_{th}' . \quad (42)$$

In the limit $k_D'^2/\beta_m^2 \ll 1$ we find from (24) that the gain is exponential and

$$\ln[P(l)/P(0)] = 2\ln|a_m(l)/a_m(0)|$$

$$= \kappa l(k_D'^2/\beta_m^2)\,\mathrm{Im}\,G'(\zeta) = \kappa l(k_D'^2/\beta_m^2)\,\pi g'(\zeta) \quad (43)$$

so that the gain is proportional to the derivative of the electron distribution function in agreement with [4,5]. The right-hand side equality is valid as long as $|\mathrm{Im}\,\rho| \ll 1$.

In trying to develop optical Cerenkov or Smith-Purcell free electron laser it is desirable to examine

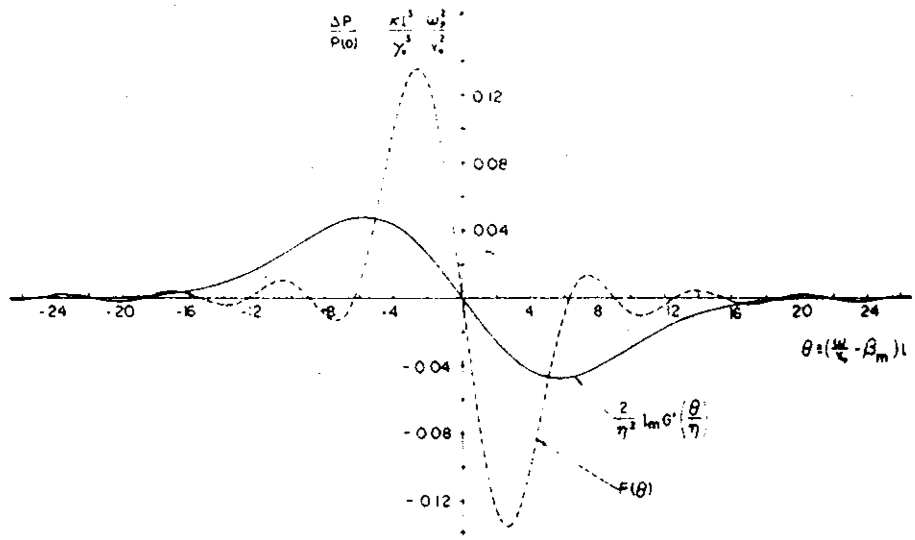

Fig. 1. The normalized gain dependence in the two regimes (46,47) for $g(x) = \pi^{-1/2}\exp(-x^2)$, $\eta = 5$ and low gain limit.

the gain expressions (35) and (43) comparatively and find the optimal regime for operation. Straightforward comparison is possible in the low gain limit where the exponential gain can be written as $\ln[P(l)/P(0)] \approx \Delta P/P(0) \ll 1$ where $\Delta P \equiv P(l) - P(0)$. We substitute in (43)

$$\zeta = \theta/\eta, \qquad \eta \equiv (v'_{th}/v_o)\beta_m l \tag{44,45}$$

then eqs. (35) and (43) can be both written in terms of the argument θ:

$$\frac{\Delta P}{P(0)} = \left[\frac{\kappa l^3}{\gamma_o^3}\frac{\omega_p^2}{v_o^2}\right] F(\theta). \qquad |\theta| \gg \eta. \tag{46}$$

$$\frac{\Delta P}{P(0)} \approx \ln\frac{P(l)}{P(0)} = \left[\frac{\kappa l^3}{\gamma_o^3}\frac{\omega_p^2}{v_o^2}\right]\frac{2}{\eta^2}\,\mathrm{Im}\,G'(\theta/\eta). \qquad |\theta| \lesssim \eta. \tag{47}$$

Eq. (47) corresponds to the case where the slow electromagnetic wave is synchronous with a non-negligible number of electrons in the electron distribution function which can then make radiative transitions and produce exponential gain or attenuation [5]. The other regime (46) is applicable only when the slow wave is out of synchronism with the electron beam distribution! It is essentially an interference effect which produces gain at finite interaction lengths only.

Fig. 1 shows on one scale both regimes of operation (46,47) in case $\eta > \pi$
A Maxwellian velocity distribution $g(x) = \pi^{-1/2} \exp(-x^2)$ is assumed in the calculation of (47). It is apparent that the gain is detennined predominantly by (47) since (46) starts holding only in a regime $\theta \gg \eta > \pi$ where the gain is low.

Note that eq. (43) holds only when $|\mathrm{Im}\zeta| \ll 1$, which corresponds to a low gain or attenuation limit

$$|\mathrm{Re}s| \ll \beta_m v'_{th}/v_o . \tag{48}$$

When $\eta \gg \pi$ (fig. 1) the right hand side of eq. (47) may hold for a number of e foldings of exponential gain before inequality (48) is violated. Thus in spite of the fact that (47) is attenuated by a factor $\eta^2 \gg \pi^2$ in comparison to (46), it still can give rise to higher gain than what was predicted from (46).

In conclusion we point out that the details of the electron beam distribution function are important in the design of a Cerenkov–Smith-Purcell laser, since the more relevant gain expression (47) depends on the electron distribution. In vacuum diodes the electron beam distribution is usually non-Maxwellian and has a steep slope on its low velocity side (corresponding to electrons leaving the cathode vicinity with zero thermal velocity [19]). Thus higher gain may be obtained with such beams in the regime where (43) or (47) apply.

References

[1] J.M. Madey, J. Appl. Phys. 42 (1971) 1906.
[2] L.R. Elias, W.M. Fairbank, J.M.J. Madey, H.A. Schwettman and T.I. Smith, Phys. Rev. Lett. 36 (1976) 717.
[3] D.A. Deacon, L.R. Elias, J.M. Madey, G.J. Ramian, H.A. Schwettman and T.I. Smith, Phys. Rev. Lett. 38 (1977) 892.
[4] A. Gover, A. Yariv and P. Yeh, VIII Intern. Quant. Electronics Conf. – Optics Comm. 18 (1976) 222; A. Gover, Wave interactions in periodic structures, Caltech Report (Feb. 1976).
[5] A. Gover and A. Yariv, Appl. Phys. 16 (1978) 121.
[6] F.A. Hopf, P. Meystre, M.O. Scully and W.H. Louisell, Phys. Rev. Lett. 37 (1976) 1215.
[7] I.B. Bernstein and J.L. Hirshfield, Phys. Rev. Lett. 40 (1978) 761.
[8] N.M. Kroll, The free electron laser as a travelling wave amplifier, Quantum Electronics Conference and Free Electron Laser Workshop, Telluride, Colorado (Aug. 1977).
[9] J.E. Walsh, Stimulated Cerenkov radiation, Quantum Electronics Conference and Free Electron Laser Workshop, Telluride, Colorado (Aug. 1977).
[10] A. Yariv and C.C. Shih, Optics Commun. 24 (1978) 233.
[11] J.M. Wachtell, Free electron lasers using the Smith-Purcell effect, unpublished.
[12] S.J. Smith and E.M. Purcell, Phys. Rev. 92 (1953) 1069.
[13] J.R. Pierce, Traveling wave tubes (Van Nostrand, Princeton 1950).
[14] J.E. Walsh, T.C. Marshall, M.R. Mross and S.P. Schlesinger, IEEE Transac. MIT-25, (1977) 561.

[15] F. Rusin and G.D. Bogomolov, Proc. of the IEEE 57 (1969) 720.
[16] F.S. Rusin, High Power Electronics, No. 5, p. 9 (1968) (russian).
[17] L.A. Vaynshtain, Electromagnetic waves (Sovietskoye Radio, Moskow (1957) (russian).
[18] A.A. Barybin and L.T. Ter-Martirosyan, Sov. Radio Engineering and Electronic Physics, 14 (1969) 237.
[19] A.H. Porusky: IRE Trans. PGED-2 (1953) 60.