A generalized formulation of free-electron lasers in the low-gain regime including transverse velocity spread and wiggler incoherence [a)]

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It is shown that in the small-signal, low-gain regime, an analytic expression for the gain can be derived which is valid for a general electron distribution function and general longitudinal variation of the transverse magnetic field pump. This expression is used to evaluate the effects on the gain curve due to transverse momentum spread in the beam as well as inaccuracies (incoherence) in the magnetic pump phase and amplitude. The restricting criteria for the neglect of these effects are derived.

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The linear (small-signal) gain regimes of the free-electron laser have been analyzed extensively by various techniques.[1–10] The present paper relies on the formulation of Ref. 4, expanding the analysis to the case of a transverse magnetic wiggler (pump) with arbitrary axial variation and an arbitrary three-dimensional velocity distribution function of the electron beam. In the low-gain regime $\Delta P/P(0) << 1$ (which is of interest mainly for the application of oscillators) the formulation results in an explicit generalized gain expression which can be used to find the effect of random perturbations in an ideally periodic helical wiggler field, as well as the effect of the beam transverse velocity spread, on the gain of the free-electron laser.

The basic structure of the free-electron laser assumed in most analyses[1–10] is that of an amplifier device composed of a transversely uniform periodoic (helical) transverse magnetic ("pump" or wiggler) field and an electron beam which propagates along the magnet axis (z axis). The laser amplifies a TEM optical wave which is propagated along the z axis.

The analysis in Ref. 4 is based on the solution of the linearlized Vlasov equation

$$\frac{\partial f^{(1)}}{\partial t} + v_z \frac{\partial f^{(1)}}{\partial z} - \frac{e}{c}[\mathbf{v}\times\mathbf{B}_0(z)]\cdot\frac{\partial f^{(1)}}{\partial \mathbf{p}} = e\left(\mathbf{E}(z,t) + \frac{\mathbf{v}\times\mathbf{B}(z,t)}{c}\right)\cdot\frac{\partial f^{(0)}}{\partial \mathbf{p}}, \quad (1)$$

where the perturbation on the electron beam distribution $f^{(1)}(\mathbf{p},z,t)$ is due to the optical fields $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$, and $f^{(0)}(\mathbf{p},z)$ is the electron beam momentum distribution function in the pump field $\mathbf{B}_0(z)$.

The magnetic pump field $\mathbf{B}_0(z)$ can be described from the vector potential $\mathbf{A}_0(z)$:

$$\mathbf{B}_0(z) = \nabla\times\mathbf{A}_0(z), \quad (2)$$

$$\mathbf{A}_0(z) = A_{0x}(z)\,\hat{e}_x + A_{0y}(z)e_y \quad (3)$$

and following Ref. 4 we find by solving the electron force equations the three constants of motion of the electrons in the static field $\mathbf{A}_0(z)$:

$$\alpha(p_x,z) = p_x - (e/c)\,A_{0x}(z), \quad (4)$$

$$\beta(p_y,z) = p_y - (e/c)\,A_{0y}(z), \quad (5)$$

$$u(p_x,p_y,p_z) = (p_x^2 + p_y^2 + p_z^2)^{1/2}, \quad (6)$$

where $e = |e|$, α and β are the $\mathbf{x}$ and $\mathbf{y}$ components of the *canonical momentum*, and u is the total mechanical momentum of the electron. These variables are independent of z. Substituting the momentum variables (p_x,p_y,p_z) in terms of the new variables (α,β,u) simplifies Eq. (1) into[4]

$$\frac{\partial g^{(1)}}{\partial t} + v_z(\alpha,\beta,u,z)\frac{\partial g^{(1)}}{\partial z} = e\left(\mathbf{E}(\mathbf{r},t) + \frac{1}{c}\mathbf{v}(\alpha,\beta,u,z)\times\mathbf{B}(\mathbf{r},t)\right)\cdot\left(\frac{\partial g^{(0)}}{\partial\alpha}\hat{e}_x + \frac{\partial g^{(0)}}{\partial\beta}\hat{e}_y + \frac{\mathbf{p}(\alpha,\beta,u,z)}{u}\frac{\partial g^{(0)}}{\partial u}\right), \quad (7)$$

where

$$g^{(0)}(\alpha,\beta,u) = f^{(0)}(\mathbf{p},z),$$
$$g^{(1)}(\alpha,\beta,u,z,t) = f^{(1)}(\mathbf{p},z,t),$$
$$\mathbf{v}(\alpha,\beta,u,z) = \mathbf{p}(\alpha,\beta,u,z)/[m_0\gamma_0(u)].$$

The transformation of Eqs. (4)–(7) corresponds to the exact solution of the linearized Vlasov equation in terms of the pump magnetic field. This formulation allows one to solve the electron equation to first order (small signal) in the electromagnetic wave $[\mathbf{E}(\mathbf{r},t), \mathbf{B}(\mathbf{r},t)]$. It is however, exact to all orders in the pump field $\mathbf{B}_0$. The distribution function $g^{(0)}(\alpha,\beta,u)$ is the distribution in terms of the transverse canonical momentum α,β and the total momentum u of the electrons in the pump field. If the pump is turned on adiaba-

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tically it is also equal to the initial momentum distribution of the electron beam before entering the interaction region.

The analysis in Ref. 4 proceeds with the solution of Eq. (7) together with the Maxwell equations (including Poisson's equation) for the electromagnetic field and the space-charge field generated in the beam. In order to derive the free-electron-laser dispersion equation and explicit gain expressions in the different gain regimes, a specific **z** dependence of the pump field (3) was assumed (helical field):

$$\mathbf{A}_0(z) = A_0 \cos k_0 z\, \hat{e}_x + A_0 \sin k_0 z\, \hat{e}_y. \tag{8}$$

The tranverse momentum spread of the electron beam was neglected, and the equilibrium distribution function was taken to be

$$g^{(0)}(\alpha,\beta,u) = n_0 \delta(\alpha)\delta(\beta) g_0(u), \tag{9}$$

where n_0 is the average electron beam density.

With these assumptions the analysis of Ref. 4 produced with no further approximations the dispersion equation of the free-electron laser with a helical magnetic field pump. The dispersion equation was then solved and expressions for the laser gain were derived by using analytic approximations in different regimes. Gain regimes which were identified include the thermal-beam regime (considering the effect of momentum spread in the total momentum—**u** only) the high-gain, cold-beam, weak-pump regime (collective regime), the high-gain, cold-beam, strong-pump regime, and the low-gain, tenuous-cold-beam regime.

In the present paper we focus our attention on the low-gain regime $\Delta P/P(0) << 1$ for both cold and thermal beams, excluding collective effects. We find in this case that an explicit gain expression can be derived without resorting to the particular field dependence (8) or electron momentum distribution function (9). Since the resulting linear gain expression would be "exact" in terms of the pump field and electron distribution function, we examine as a special case the effects of deviations of the fields and the distribution function from the ideal forms given by Eqs. (8) and (9).

The detailed analysis in[4] indicated that the contribution of the first two terms in the second set of large parentheses in Eq. (7) is negligible in comparison to the contribution of the third term. Hence for the sake of simplicity we will neglect them from now on (this approximation is implicitly used in all the other analyses.[1–10]

We assume that the electromagnetic fields are transversely uniform across the electron beam (plane wave). The electromagnetic fields are expressed in terms of right-hand and left-hand circularly polarized transverse vector potential components $\tilde{\mathbf{A}}_+(z)$ and $\tilde{\mathbf{A}}_-(z)$:

$$\mathbf{E}(z,t) = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{i\omega}{c}[\tilde{A}_+(z)\hat{e}_+ + \tilde{A}_-(z)\hat{e}_-]\, e^{-i\omega t} + \text{c.c.}, \tag{10}$$

$$\mathbf{B}(z,t) = \nabla\times\mathbf{A} = i\left(-\frac{d}{dz}\tilde{A}_+(z)\hat{e}_+ + \frac{d}{dz}\tilde{A}_-(z)\hat{e}_-\right)e^{-i\omega t} + \text{c.c.}, \tag{11}$$

where

$\hat{e}_+ = \frac{1}{2}(\hat{e}_x + i\hat{e}_y)$ and $\hat{e}_- = \frac{1}{2}(\hat{e}_x - i\hat{e}_y)$ are respectively the right- and left-hand circular polarization unit vectors, and $\tilde{A}_+ = \tilde{A}_x - i\tilde{A}_y$, $\tilde{A}_- = \tilde{A}_x + i\tilde{A}_y$.

With all these assumptions, Eq. (7) now simplifies into

$$-i\omega\tilde{g}^{(1)}(\alpha,\beta,u,z) + v_z(\alpha,\beta,u,z)\frac{\partial}{\partial z}\tilde{g}^{(1)}(\alpha,\beta,u,z) = \frac{ie\omega}{2c}[p_-(\alpha,\beta,u,z)\tilde{A}_+(z) + p_+(\alpha,\beta,u,z)\tilde{A}_-(z)]\frac{1}{u}\frac{\partial}{\partial u}g^{(0)}(\alpha,\beta,u), \tag{12}$$

where

$g^{(1)}(\alpha,\beta,u,z,t) \equiv \tilde{g}^{(1)}(\alpha,\beta,u,z)e^{-i\omega t} + \text{c.c}$

and $p_\pm = p_x \mp ip_y$.

The perturbation in the distribution function gives rise to a transverse current

$$\mathbf{J}(z,t) = [\tilde{J}_x(z)\hat{e}_x + \tilde{J}_y(z)\hat{e}_y]e^{-i\omega t} + \text{c.c.} = [\tilde{J}_+(z)\hat{e}_+ + \tilde{J}_-(z)\hat{e}_-]e^{-i\omega t} + \text{c.c.}, \tag{13}$$

where

$$\tilde{J}_\pm(z) = -e\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{u\, d\alpha\, d\beta\, du}{p_z(\alpha,\beta,u,z)}v_\pm(\alpha,\beta,u,z) \times\tilde{g}^{(1)}(\alpha,\beta,u,z), \tag{14}$$

$$v_\pm = p_\pm/\gamma(u)m, \tag{15}$$

$$p_\pm = p_x \mp ip_y = \alpha \mp i\beta - \frac{e}{c}(A_{0x} \mp iA_{0y}), \tag{16}$$

$$p_z = \left[u^2 - \left(\alpha + \frac{e}{c}A_{0z}\right)^2 - \left(\beta + \frac{e}{c}A_{0y}\right)^2\right]^{1/2} \tag{17}$$

The exact solution of Eq. (12) is

$$g^{(1)}(\alpha,\beta,u,z) = \exp\left(i\int_0^z \frac{\omega}{v_z(\alpha,\beta,u,\bar{z})}d\bar{z}\right)\int_0^z \exp\left(-i\int_0^{z''}\frac{\omega}{v_z(\alpha,\beta,u,\bar{z}'')}d\bar{z}''\right)\frac{H(\alpha,\beta,u,z'')}{v_z(\alpha,\beta,u,z'')}dz'', \tag{18}$$

where $H(\alpha,\beta,u,z)$ stands for the right-hand side of Eq. (12) and we assume $g^{(1)}(\alpha,\beta,u,0) = 0$ (with no prebunching of the electron beam). Substitution of (18) in (14) gives

$$\tilde{J}_\pm(z) = -i\frac{e^2}{2c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{z} d\alpha\, d\beta\, du\, dz''\left[\exp\left(i\int_0^z \frac{\omega}{v_z(\alpha,\beta,u,\bar{z})}d\bar{z}\right)\ p_z(\alpha,\beta,u,z)\right]$$
$$\times\left[\exp\left(-i\int_0^{z''}\frac{\omega}{v_z(\alpha,\beta,u,\bar{z}'')}d\bar{z}''\right)\ p_z(\alpha,\beta,u,z'')\right]p_\pm(\alpha,\beta,u,z)[p_-(\alpha,\beta,u,z'')\tilde{A}_+(z'')$$
$$+ p_+(\alpha,\beta,u,z'')\tilde{A}_-(z'')]\frac{\partial}{\partial u}g^0(\alpha,\beta,u). \tag{19}$$

We now turn to Maxwell's equations to obtain the electromagnetic field which is induced by the current (19). The equation for the electromagnetic wave-vector potential is

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\omega^2}{c^2}\right)\widetilde{A}_\pm(z)=-\frac{4\pi}{c}\widetilde{J}_\pm(z). \tag{20}$$

This second-order equation can be reduced to a first-order equation if we assume that there is only a forward-going wave propagating with a wave number $\sim\omega/c$, and that there is no coupling to a backward-going electromagnetic wave. We can then reduce (20) to

$$\left(\frac{\partial}{\partial z}-i\frac{\omega}{c}\right)\widetilde{A}_\pm(z)=\frac{2\pi i}{\omega}\widetilde{J}_\pm(z), \tag{21}$$

which can be readily transformed into an integral equation

$$\widetilde{A}_\pm(z)=\widetilde{A}_\pm(0)e^{i(\omega/c)z}+\frac{2\pi i}{\omega}e^{i(\omega/c)z}\times\int_0^z e^{-i(\omega/c)z'}\widetilde{J}_\pm(z')dz'. \tag{22}$$

Equations (19) and (22) are a set of two coupled-integral equations which can be used to solve for the linear evolution of the electromagnetic field. These equations can be solved by an iterative process. To zero order (no interaction) we have $\widetilde{A}_\pm(z)=\widetilde{A}_\pm(0)e^{i\omega z/c}$. We substitute the zeroth-order interation in (19) which is in turn substituted into (22) to yield the expression for the electromagnetic field to first order in $\widetilde{A}_\pm(0)$. The iterative process may be continued in this way to higher orders.

At present we are interested only in the linear response (small-signal) limit and therefore calculate $\widetilde{A}_\pm(z)$ to first order:

$$\widetilde{A}_\pm(z)=\Bigg(\widetilde{A}_\pm(o)+\frac{\pi e^2}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^z d\alpha\, d\beta\, du\, dz'\int_0^{z'}dz''\frac{\partial}{\partial u}g^{(0)}(\alpha,\beta,u)\left\{\exp\left[i\int_0^{z'}\left(\frac{\omega}{v_z'}-\frac{\omega}{c}\right)d\bar{z}'\right]\Big/p_z'\right\}$$
$$\times\left\{\exp\left[-i\int_0^{z''}\left(\frac{\omega}{v_z''}-\frac{\omega}{c}\right)d\bar{z}''\right]\Big/p_z''\right\}p'_\pm\,[p''_-\,\widetilde{A}_+(0)+p''_+\,\widetilde{A}_-(0)]\Bigg)e^{i(\omega/c)z}. \tag{23}$$

The primed and double-primed parameters $p_z, p_\pm$ are all dependent on α, β, u, and z' or z'', respectively; $v_z'=v_z(\alpha,\beta,u,\bar{z}')$, $v_z''=v_z(\alpha,\beta,u,\bar{z}'')$. The Poynting vector power density of the fields in Eqs. (10) and (11) is found to be given by

$$\mathbf{S}=\hat{e}_z\frac{i}{8}\frac{\omega}{c}\left[\widetilde{A}_+(z)\frac{d}{dz}A^*_+(z)+\widetilde{A}_-(z)\frac{d}{dz}A^*_-(z)\right]+\text{c.c.} \tag{24}$$

We assume that in (23) the expression in large bold parentheses varies very slowly with z in comparison to $\exp(i\omega z/c)$, hence instead of (24) we can write

$$\mathbf{S}=\hat{e}_z\tfrac{1}{4}(\omega^2/c)\,[|\widetilde{A}_+(z)|^2+|\widetilde{A}_-(z)|^2]. \tag{25}$$

Let us now assume that the incident electromagnetic wave is right-hand circularly polarized $[\widetilde{A}_-(0)=0]$. The power gain of this wave $\widetilde{A}_+(z)$ after an interaction distance l is

$$\frac{\Delta P_+(l)}{P_+(0)}=\left|\frac{A_+(l)}{A_+(0)}\right|^2-1\simeq\frac{\pi e^2}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^l d\alpha\, d\beta\, du\, dz'\int_0^{z'}dz''\frac{\partial}{\partial \mathrm{u}}\mathrm{g}^{(0)}(\alpha,\beta,\mathrm{u})\frac{\mathrm{p}'_+}{p_z'}\exp\left[i\int_0^{z'}\left(\frac{\omega}{v_z'}-\frac{\omega}{c}\right)d\bar{z}\right]$$
$$\times\frac{\mathrm{p}''_-}{p_z''}\exp\left[-i\int_0^{z''}\left(\frac{\omega}{v_z''}-\frac{\omega}{c}\right)d\bar{z}''\right]+\text{c.c.} \tag{26}$$

This is a general expression for the linear gain in the low-gain limit which within the limits of the assumptions used during the derivation is applicable for arbitrary three-dimensional beam velocity distribution and arbitrary transverse field pump, which can be written in the form (3).

We may now apply Eq. (26) to the case of the Stanford free-electron laser experiment,[10] where we assume a pure right-hand helical magnetic field (8).

We also assume that the electron beam has negligible transverse velocity spread, and zero transverse canonical momentum (9).

In this case we get from (16) and (17)

$$p_\pm=-(e/c)A_0e^{\pm ik_0z}, \tag{27}$$

$$p_z=\{u^2-[(e/c)A_0]^2\}^{1/2}, \tag{28}$$

and we see that p_z and v_z are no longer functions of z. Equation (26) simplifies into

$$\frac{\Delta P_+(l)}{P_+(0)}=\frac{\omega_p^2}{2}\frac{m}{c}\left(\frac{eA_0}{c}\right)^2\int_0^{\infty}\int_0^l\frac{du}{p_z^2}dz'\int_0^{z'}d\tau\frac{dg_0(u)}{du}\cos\left(\frac{\omega}{v_z}-\frac{\omega}{c}-k_0\right)\tau$$
$$=\frac{\omega_p^2}{4}\frac{m}{c}\left(\frac{eA_0}{c}\right)^2\int_0^{\infty}\frac{du}{p_z^2}\frac{dg_0(u)}{du}\frac{\sin^2\frac{1}{2}(\omega/v_z-\omega/c-k_0)l}{[\frac{1}{2}(\omega/v_z-\omega/c-k_0)]^2}. \tag{29}$$

In the cold-beam limit

$$v_{zth}/v_{z0}<<\beta_{z0}\lambda/l, \tag{30}$$

where v_{zth} is the beam velocity spread in the z direction and $\beta_{z0}=v_{z0}/c$, we may substitute $g_0(u)=\delta(u-u_0)$ into (29).

Equation (29) then reduces to the conventional gain expression in the low-gain, tenuous-cold-beam regime[1-10]:

$$\frac{\Delta P_+(l)}{P_+(0)} = \frac{1}{8}\frac{1}{\gamma_0\gamma_{z0}^2}\left(\frac{\omega_p}{v_{z0}}\right)^2\left(\frac{P_{\perp 0}}{P_{z0}}\right)^2\frac{u_0}{p_{z0}}\frac{\omega}{c}l^3 \times\frac{d}{d\theta_0}\left(\frac{\sin\theta_0}{\theta_0}\right)^2, \tag{31}$$

where

$$\theta_0 \equiv \tfrac{1}{2}(\omega/v_{z0} - \omega/c - k_0)l, \tag{32}$$

$$p_{\perp 0} \equiv (e/c)A_0 = (e/c)B_0/k_0, \tag{33}$$

$$p_{z0} \equiv (u_0^2 - p_{\perp 0}^2)^{1/2} \tag{34}$$

$$\gamma_{z0} \equiv (1 - v_{z0}^2/c^2)^{-1/2} \tag{35}$$

The gain curve (31) obtains its maximum value at $\theta_0 = -1.3$.

It is worth noting that the simplified form of Eq. (29) is due to the fact that in the limit of an ideal helical pump field and $\alpha = \beta = 0$, the longitudinal momentum p_z [Eq. (28)] is a motion constant and is independent of z. If, for instance, the pump field was not helical but linearly polarized [$A_0(z) = A_0 \cos k_0 z \hat{e}_x$] instead of (8)], then p_z and v_z in Eq. (26) would be still dependent on z in a periodic way [see Eq. (17)]. The integration over z' z'' is then not immediate. We will not elaborate on this case beyond noting that the gain spectrum for this pump will be rich with harmonics.

We now will examine the effect on the gain of transverse momentum spread of the electron beam and of irregularities in the helical magnetic pump field. Of course, if the magnetic pump field $\mathbf{B}_0(z)$ is known exactly (e.g., measured directly in a specific experiment), then it is possible to find $A_0(z)$ by integrating Eq. (2). If the beam velocity distribution $g_0(\alpha,\beta,u)$ is also known explicitly, then the gain can be calculated "exactly" using the general gain expression (26). In practice we are interested in estimating the effects on the gain due to a transverse momentum spread and irregularities in the helical pump field even if we do not know exactly the beam velocity distribution and the exact form for the pump field. We will therefore find the conditions in which the effects on the gain of transverse momentum spread and of irregularities in the pump field are significant enough so that (8) and (9) and consequently (29) and (31) are not valid.

Irregularities in the helical field can be described as amplitude and phase modulation of the ideal helical field, so that

$$\mathbf{A}_0(z) = A_0(z)\cos[k_0 z + \phi(z)] + A_0(z)\sin[k_0 z + \phi(z)]\hat{e}_y \tag{36}$$

instead of (8). The amplitude $A_0(z)$ is randomly modulated around an average value $\langle A_0(z)\rangle = A_0$. The term $\phi(z)$ is a random-phase deviation of the *vector potential field* $\mathbf{A}_0(z)$. It should be pointed out here that $\phi(z)$ and $A_0(z)$ are not related in a simple way to the phase and amplitude modulation of the *magnetic field* $\mathbf{B}_0(z)$. They can be evaluated only after integrating Eq. (2).

With Eq. (36) and without assuming (9) we have from (16)

$$p_\pm = \alpha \mp i\beta - (e/c)A_0(z)e^{\mp i[k_0 z + \phi(z)]}, \tag{37}$$

and by expanding (17) to first order in $\delta A_0(z)$, α, and β we obtain

$$\frac{1}{p_z(z)} \simeq \frac{1}{p_z(u)}\left(1 + \frac{p_{\perp 0}}{p_z^2}\{\delta p_{\perp 0}(z) + \alpha\cos[k_0 z + \phi(z)] + \beta\sin[k_0 z + \phi(z)]\}\right), \tag{38}$$

$$\frac{1}{v_z(z)} \simeq \frac{1}{v_z(u)}\left(1 + \frac{p_{\perp 0}}{p_z^2}\{\delta p_{\perp 0}(z) + \alpha\cos[k_0 z + \phi(z)] + \beta\sin[k_0 z + \phi(z)]\}\right), \tag{39}$$

where

$$\delta p_{\perp 0}(z) \equiv (e/c)\delta A_0(z), \tag{40}$$

$$A_0(z) \equiv A_0 + \delta A_0(z). \tag{41}$$

It is a very good approximation to set $\alpha = \beta = 0$ in Eq. (37) and substitute in (26)

$$p_\pm \simeq -(e/c)A_0(z)e^{\mp i[k_0 z + \phi(z)]}. \tag{42}$$

This is true for two reasons: (i) The transverse momentum spread α_{th} is assumed to satisfy $\alpha_{th} << p_{\perp 0}$ and therefore the additive contribution of $\alpha \mp i\beta$ to (37) and in turn to (26) is neglibible; (ii) α and β do not have periodic $\mathbf{z}$ dependence like $A_\pm(z)$; therefore they will not contribute a synchronous term in the integrand of (26) and thus will have a negligible contribution. On the other hand we cannot set $\alpha = \beta = 0$ in (39) without careful examination, because $v_z(z)$ appears in the exponents of Eq. (26), and small deviations in $v_z(z)$ may have a significant effect on the integral and spoil the synchronism condition which led to the "resonant" expression in Eq. (29) or (31). As for $p_z(z)$ [Eq. (38)], we can (to zero order) approximate it by $p_z(z) = p_z(u)$ when used in (26). Again the reason is that the contribution of the first-order correction is additive and small.

We then get from (26)

$$\frac{\Delta P_+(l)}{P_+(0)} = 2\frac{\pi l^2}{c}\left(\frac{e}{c}A_0\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^l d\alpha\, d\beta\, du\, dz'\int_0^{z'} dz''\frac{1}{p_z^2(u)}\frac{\partial}{\partial u}g^{(0)}(\alpha,\beta,u) \times\cos\left[\left(\frac{\omega}{v_z(u)} - \frac{\omega}{c} - k_0\right)(z' - z'') + \psi(z') - \psi(z'')\right], \tag{43}$$

where

$$\psi(z) = \frac{\omega}{v_z(u)}\frac{p_{\perp 0}}{p_z^2}\int_0^z\{\delta p_{\perp 0}(\bar{z}) + \alpha\cos[k_0\bar{z} + \phi(\bar{z})] + \beta\sin[k_0\bar{z} + \phi(\bar{z})]\}d\bar{z} - \phi(z). \tag{44}$$

Using these expressions, we will first examine the effect of transverse velocity spread on the gain. For the sake of simplicity we assume now an ideal helical field $A_0(z) = A_0$, $\phi(z) = 0$ and a Maxwellian transverse momentum spread of the electron beam;

$$\exp\left[-\left(\frac{\omega}{v_z k_0}\frac{p_{\perp 0}}{p_z}\frac{\alpha_{th}}{p_z}\sin\tfrac{1}{2}k_0\tau\right)^2\right]\cos\left(\frac{\omega}{v_z} - \omega^c - k_0\right)\tau. \tag{45}$$

$$\frac{\Delta P_+(l)}{P_+(0)} = \frac{\omega_p^2}{2}\frac{m}{c}\left(\frac{eA_0}{c}\right)^2\int_0^\infty\int_0^l\frac{du}{p_z^2}dz'\int_0^{z'}d\tau\exp\left[-\left(\frac{\omega}{v_z k_0}\frac{p_{\perp 0}}{p_z}\frac{\alpha_{th}}{p_z}\sin\tfrac{1}{2}k_0\tau\right)^2\right]\cos\left(\frac{\omega}{v_z} - \frac{\omega}{c} - k_0\right)\tau. \tag{46}$$

Further investigation of the integral in (46) will not be carried out at this time. We only note that besides reducing the gain, the transverse momentum also generates harmonic frequencies in the gain spectrum. We may also find from (46) that the condition at which the neglect of the transverse momentum spread is allowed is

$$\frac{\omega}{v_{z0}k_0}\frac{p_{\perp 0}}{p_{z0}}\frac{\alpha_{th}}{p_{z0}} < < 1.$$

Using the synchronism condition $\omega/c \simeq \omega/v_{z0} - k_0$, this can be written as

$$\alpha_{th}/p_{z0} < < 1/[(1+\beta_{z0})\gamma_{z0}^2\beta_{\perp 0}], \tag{47}$$

where

$$\beta_{\perp 0} \equiv p_{\perp 0}/p_{z0} = eB_0/(\gamma_0 mcv_{z0}k_0).$$

In the relativistic limit ($\gamma_0 > > 1$) this condition is

$$\mathscr{E}_{th}/\mathscr{E}_0 < < 1/(2\gamma_{z0}^2\beta_{\perp 0}). \tag{48}$$

When (47) or (48) is well satisfied we may set in (46) $\alpha_{th} = 0$ and recover (29).

It is interesting to compare conditions (47) and (48) with condition (30) for the longitudinal momentum spread, which can be written in the forms

$$p_{zth}/p_{z0} < < \beta_{z0}\gamma_{z0}^2\lambda/l, \tag{49}$$

or in the relativistic limit

$$\mathscr{E}_{zth}/\mathscr{E}_0 < < \gamma_{z0}^2\lambda/l. \tag{50}$$

In the regime

$$\frac{l}{\lambda}\frac{1}{(1+\beta_{z0})\beta_{z0}\beta_{\perp 0}\gamma_{0z}^4} < 1\left(\frac{l}{\lambda}\frac{l}{2\beta_{\perp 0}\gamma_{0z}^4} < 1\right),$$

the second being the relativistic expression, we find that the condition on the transverse momentum spread (47) is more stringent than the condition on the longitudinal spread. With high-energy beams this may easily be the case and conditions (47) and (48) may be of practical concern. It is easy to verify, that condition (48) and (50) are reasonably well satisfied for the parameters of the Stanford experiment.[10]

We now consider the effect of irregularities in the helical pump field, and this time we assume for simplicity $\alpha_{th} = 0$ [Eq. (9)]. Equation (43) then simplifies into

$$\frac{\Delta P_+(l)}{P_+(0)} = \omega_p^2\frac{m}{c}\int_0^\infty\int_0^l du\,dz'\int_0^{z'}dz''\frac{dg_0(u)}{du}\frac{p_{\perp 0}(z')}{p_z(z')}\frac{p_{\perp 0}(z'')}{p_z(z'')}\cos\left[\left(\frac{\omega}{v_z} - \frac{\omega}{c} - k_0\right)(z'-z'') + \psi(z') - \psi(z'')\right], \tag{51}$$

where

$$\psi(z) = \frac{\omega}{v_z}\left(\frac{e}{c}\right)^2\frac{A_0}{p_z^2}\int_0^z\delta A_0(\bar{z})d\bar{z} - \phi(z). \tag{52}$$

Notice that the phase deviation $\psi(z)$ is affected not only by the phase deviation of the helical field vector potential $\phi(z)$ but also by the amplitude deviation $\delta A_0(z)$.

The condition that Eq. (51) be reduced to (29) and the effect of imperfections in the helical field be neglected is

$$|\psi(z)| < < \pi \tag{53}$$

for any $0 < z < l$. In this condition it is possible to expand the integrand in (51) to first order in $\psi(z)$ and the first-order term will be small compared to the zeroth-order term (29). Condition (53) may be quite a stringent condition when the interaction length is long. For example, to appreciate the effect of changes in the magnetic field amplitude assume that the magnetic field amplitude changes in a linear way along the axis while the helical phase stays constant:

$$\mathbf{B}_0(z) = -[B_0 + \delta B_0(z)][\cos k_0 z\,\hat{e}_x + \sin k_0 z\,\hat{e}_y], \tag{54}$$

$$\frac{\delta B_0(z)}{B_0} = \frac{\delta B_0(l)}{B_0}\frac{z}{l}. \tag{55}$$

Integrating (2) using (54) and (55), and comparing to (36) and (40), we find that

$$\frac{\delta A_0(z)}{A_0} = 2\frac{\delta B_0(l)}{B_0}\frac{z}{l}, \tag{56}$$

and the resultant phase shift $\phi(l)$ is negligible. Applying inequality (53) at $z = l$ gives

$$\left|\frac{\delta B_0(l)}{B_0}\right| < < \frac{\beta_{z0}}{\beta_{\perp 0}^2}\frac{\lambda}{2l}. \tag{57}$$

For $\lambda = 10\,\mu$m, $l = 4$ m, $\beta_{\perp 0} = 0.05$, and $\beta_{z0} = 1$ this requires that $|\delta B_0(l)/B_0| < < 5\times 10^{-4}$, which is a quite stringent condition.

We can now appreciate the significance of inaccuracies in the periodicity and amplitude of the helical field. Small local deviations in the field period will not cause major change in the gain expression (29) as long as there is no cumulative phase deviation throughout the interaction length l which violates (53). In other words, the long-range coherence of the periodic structure is important. We may add that similar conditions would apply also to the problem of stimulated Compton and Raman scattering of an electromagnetic wave from electron beams,[11–13] where the coherent length of the scattering wave may limit the obtainable gain. This may be an important consideration especially when the scattering (pump) wave is produced by a low-coherence source like a high-power laser or a wide-band microwave source.

In the case where condition (53) is not satisfied the free-electron laser gain curve may differ substantially from (31). For any specific laser structure with pump field which deviates from the ideal helical dependence (8), the integration of (51) would result different gain curve. A statistical analysis may be carried out to estimate the quantitative effect of random perturbations of the pump field on the laser gain.[14] Such an analysis can yield expressions for the average laser gain, which for a static pump is essentially an ensemble average over many randomly perturbed free-electron lasers. For an electromagnetic pump (stimulated Compton scattering) the statistical analysis would yield a time average of the laser gain.

Detailed statistical analysis of the laser gain in the regime where (53) does not hold is beyond the scope of the present article. We will only point out that the gain in this case depends basically on two parameters of the random perturbation of the pump field—the phase modulation index $<\psi^2(z)>$ and the pump coherence length l_c. Whenever $<\psi^2(z)> \ll 1$ or $l/l_c \ll 1$ the average gain curve is close to (31). In the limit of short coherence length the average gain curve changes substantially and is reduced by a factor of about $(l_c/l)^2$ relative to the unperturbed free-electron laser gain.

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