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Electromagnetic instability supported by a rippled, magnetically focused relativistic electron beam

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The coupling of volume, long-wavelength TM electromagnetic and longitudinal space charge (electrostatic) waves by the rippling of magnetically focused electron beams is examined analytically. The dispersion relation is obtained and then solved for these types of wave. Instability, with growth rates proportional to the relative ripple amplitude of the beam, is found and discussed.

1. Introduction

The stability of rippled, magnetically focused, partially neutralized electron beams was recently investigated for the case of *electrostatic perturbations* (Cuperman & Petran 1982; Petran & Cuperman 1982; Cuperman & Gover 1983, communicated). It was found that such rippled beams are unstable for both long- and short-wavelength electrostatic perturbations. The physical mechanism responsible for this instability is as follows: When the wavenumber corresponding to the period of the modulation (rippling) of the beam phase matches the slow and the fast natural modes of the unrippled beam they beat together to form a resultant wave whose amplitude is modulated by a standing wave envelope. The ponderomotive force exerted by the resultant wave pushes electrons from low-density regions toward higher-density regions, thus further increasing the bunching already existing in the equilibrium state. This instability belongs to the class of parametric instabilities.

In this paper we investigate the stability of *relativistic*, rippled, magnetically focused electron beams against *coupled electromagnetic and electrostatic* perturbations which are phase-matched by the beam ripple.

The equilibrium configuration consists of a cold electron beam (radius b) which partially fills a conducting pipe (radius a) in the presence of an axially uniform magnetic field, $\mathbf{B}_z$ (see figure 1). The degree of neutralization of the electron beam (by positive ions supposed at rest) is arbitrary. The rippling is due to the imbalance of the forces acting on the fluid element of the beam at the exit from the gun.

In the stability analysis we concentrate on $\tilde{E}$-type (i.e. $\tilde{B}_z = 0$) volume waves and consider the limiting case of long-wavelength perturbation. In this, the static magnetic field is assumed to be relatively strong, so that only the perturbed longitudinal motion (along $\mathbf{B}_z$) is important. The resulting dispersion relation is solved for resonant mode coupling cases. Specifically we consider the resonant coupling of (i) fast TM mode and slow space charge mode and (ii) slow and fast space charge modes.

The paper is organized as follows. In §2 we present the basic equations used in the paper. §3 is concerned with the equilibrium and provides simple analytical expressions for the ripple characteristics and the corresponding particle density modulation. In §4 we carry out the stability analysis. A dispersion relation for coupled $\tilde{E}$-type wave modes is derived and then solved for resonant two-mode interactions for TM and electrostatic (e.s.) modes (§4.5.1) and e.s.–e.s. modes (§4.5.2). Simple expressions for growth rates of the instability are obtained and presented. A summary and discussion is provided in §5.

2. Basic equations

Consider a relativistic cold rippled magnetically focused cylindrical electron beam (radius b) which partially fills a conducting pipe (radius a) in the presence of an axially uniform magnetic field, $\mathbf{B}_z$ (see figure 1). The electron beam is partially neutralized by ions at rest which occupy the same space as the beam. (The beam is separated from the conducting pipe by vacuum). The fluid and Maxwell's equations describing such a system are

$$\frac{\partial n}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(nv_r r)+\frac{1}{r}\frac{\partial}{\partial \theta}(nv_\theta)+\frac{\partial}{\partial z}(n\hat{V}_z) = 0, \tag{1}$$

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\hat{V}_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r} = -\frac{e}{m\gamma}\left[E_r+\frac{1}{c}(\hat{\mathbf{V}}\times\mathbf{B})_r\right], \tag{2}$$

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\hat{V}_z\frac{\partial v_\theta}{\partial z}-\frac{v_r v_\theta}{r} = -\frac{e}{m\gamma}\left[E_\theta+\frac{1}{c}(\hat{\mathbf{V}}\times\mathbf{B})_\theta\right], \tag{3}$$

$$\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\hat{V}_z\frac{\partial v_z}{\partial z} = -\frac{e}{m\gamma^3}\left[E_z+\frac{1}{c}(\hat{\mathbf{V}}\times\mathbf{B})_z\right], \tag{4}$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \tag{5}$$

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}, \quad \mathbf{j} = \sum_\alpha e_\alpha n_\alpha \hat{\mathbf{V}}_\alpha, \quad (\alpha = e, i), \tag{6}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r)+\frac{1}{r}\frac{\partial E_\theta}{\partial \theta}+\frac{\partial E_z}{\partial z} = 4\pi\sum_\alpha n_\alpha e_\alpha \tag{7}$$

and

$$\nabla\cdot\mathbf{B} = 0. \tag{8}$$

Here $\hat{\mathbf{V}} = \mathbf{V}_z+\mathbf{v}$, $\hat{V}_z = V_z+v_z$, $\hat{V}_\theta \equiv v_\theta$ and $\hat{V}_r \equiv v_r$, with V_z being the constant axial streaming velocity in vacuum and v_z, v_θ, v_r the self-generated velocity components; $E_i\,(i = r, \theta, z)$ are the components of the electric field; $\mathbf{B}$ is the total magnetic

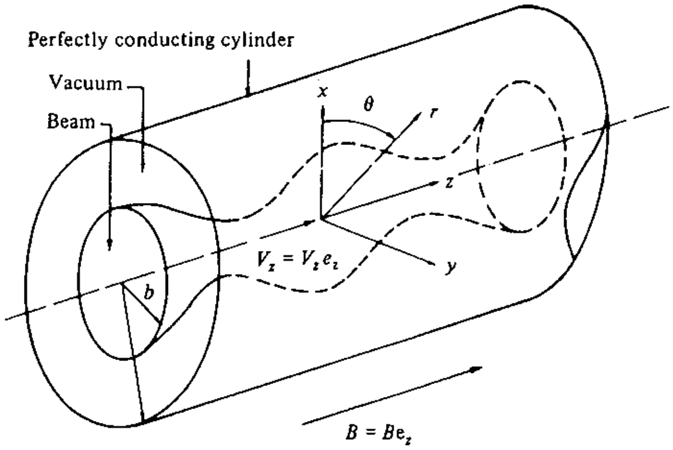

FIGURE 1. Equilibrium configuration and co-ordinate system.

field; n_α, e_α are the particle density and charge, respectively; $\gamma \equiv (1-V_z^2/c^2)^{-\frac{1}{2}}$ is the relativistic factor.

3. Equilibrium

For mathematical simplicity we here consider small induced azimuthal and axial motions and therefore neglect the self-magnetic equilibrium field, i.e. we take $\mathbf{B} \equiv \mathbf{B}_z$. Moreover, in order to obtain simple analytical solutions, we consider conditions consistent with the following assumptions: (i) constant axial streaming velocity, (ii) constant beam current ($nb^2V_z = \text{const.}$), (iii) radially uniform particle density and (iv) small relative ripple amplitude, $\bar{\delta} = (b-r_0)/r_0$. Under these conditions the equilibrium is determined by the motion of the outermost (edge) electrons described by the 'envelope' equation, as follows.

In a reference frame moving with the beam the radial motion equation reads

$$\frac{d^2r}{dt^2} = \frac{v_\theta^2}{r} + \frac{eE_r}{m\gamma} - \frac{v_\theta \omega_c}{\gamma}. \tag{9}$$

The three terms on the right-hand side of (9) represent, respectively, the centrifugal, electrostatic and magnetic accelerations. Unless these forces balance, a net force acts on the electrons and produces a periodic beam envelope. The radial space charge electric field appearing in (9) can be obtained by integration of (7) and is

$$E_r = \begin{cases} \dfrac{m}{e}\dfrac{\omega_{pb}^2(1-f)}{2\gamma} r, & 0 \leqslant r < b, \quad (10a) \\ \dfrac{m}{e}\dfrac{\omega_{pb}^2(1-f)}{2\gamma}\dfrac{b^2}{r}, & r \geqslant b. \quad (10b) \end{cases}$$

Here b is the rippled equilibrium beam radius, $\omega_{pb}^2 = 4\pi n_b e^2/m$ is the corresponding plasma frequency, $n_b(z)$ is the electron density, r is the radial distance from the axis and $f = n_{p0}/n_{e0}$ is the fractional neutralization of the beam. (In deriving (10) we used the assumption (ii) providing the relation $n_b b^2 = n_0 r_0^2$

and consequently $\omega_{pb}^2 = \omega_{p0}^2 (r_0/b)^2$, with ω_{p0} and r_0 being the unrippled electron plasma frequency and equilibrium radius, respectively).

The equilibrium angular frequency $\dot{\theta}$ ($\dot{\theta} \equiv v_\theta/r$) can be obtained from (2)–(5):

$$\dot{\theta}^{\pm}(b) = \frac{\omega_c}{2\gamma}\left\{1 \pm \left[1 - \frac{2\omega_{p0}^2}{\omega_c^2}(1-f)\right]^{\frac{1}{2}} \frac{r_0^2}{b^2}\right\}. \tag{11}$$

Notice that $\dot{\theta}$ is constant only in a given cross-section of the beam; it depends, however, on the radius of that cross-section, which varies along the beam (z direction). The $+$ and $-$ signs in (11) correspond to the fast and slow beam rotation modes, respectively. (The beam can be *either* in the slow mode of rotation, if the magnetic field at the cathode is in phase with the magnetic field in the pipe, *or* in the fast mode of rotation, if the two above mentioned fields are opposite).

Now the envelope equation describing the equilibrium radial motion of the outermost electrons can be obtained from (9), (10) and (11). After linearization ($|\delta|/r_0 \ll 1$), the following simple harmonic oscillator equation is found:

$$\ddot{\delta} + \omega_0^2 \delta = 0, \tag{12}$$

where ω_0, the linear proper frequency, is given by

$$\omega_0^2 = \frac{\omega_c^2}{\gamma^2}\left[1 - \frac{\omega_{p0}^2 \gamma}{\omega_c^2}(1-f)\right]. \tag{13}$$

The solution of (12), after converting to the rest frame, is

$$b = r_0 + \delta_0 \cos(k_S z + \phi). \tag{14}$$

Here

$$k_S \equiv \omega_0/V_z \tag{15}$$

is the ripple wavenumber (see figure 1) and ϕ depends on the state of the beam at the exit from the gun and entrance into the axial magnetic field region.

The modulation of the beam radius (cf. (14)) in conjunction with the condition (iii) leads to the axial modulation of the particle density (and therefore of the plasma frequency) namely

$$n(z) = n_0[1 + 2\bar{\delta}_0 \cos(k_S z + \phi)], \tag{16}$$

where

$$\bar{\delta}_0 \equiv \delta_0/r_0.$$

4. Stability analysis. Volume waves

4.1. *The wave equation*

From Maxwell's equations, we have the following general wave equation in a plasma system:

$$\nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t}. \tag{17}$$

The z component of (17) is ($\partial/\partial\theta = 0$)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{\partial^2 E_z}{\partial z^2} + \{\nabla(\nabla \cdot \mathbf{E})\}_z - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial j_z}{\partial t}. \tag{18}$$

4.2. *The linearized MHD equations*

For simplicity, we here consider the case of a relatively strong applied magnetic field, $\mathbf{B}_z$; that is, we neglect the *transverse* perturbed motion ($\tilde{v}_r = \tilde{v}_\theta = 0$). Then, the linearized MHD equations with axial symmetry ($\partial/\partial\theta = 0$) become

$$\frac{\partial \tilde{n}}{\partial t} + V_z \frac{\partial \tilde{n}}{\partial z} + v_r \frac{\partial \tilde{n}}{\partial r} + n \frac{\partial \tilde{v}_z}{\partial z} + \tilde{v}_z \frac{\partial n}{\partial z} = 0, \tag{1'}$$

$$\frac{\partial \tilde{v}_z}{\partial t} + V_z \frac{\partial \tilde{v}_z}{\partial z} = -\frac{e}{m\gamma^3} \tilde{E}_z, \tag{4'}$$

$$-\frac{\partial \tilde{E}_\theta}{\partial z} = \frac{1}{c}\frac{\partial \tilde{B}_r}{\partial t}, \tag{5a'}$$

$$\frac{\partial \tilde{E}_r}{\partial z} - \frac{\partial \tilde{E}_z}{\partial r} = -\frac{1}{c}\frac{\partial \tilde{B}_\theta}{\partial t}, \tag{5b'}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{E}_\theta) = -\frac{1}{c}\frac{\partial \tilde{B}_z}{\partial t}, \tag{5c'}$$

$$-\frac{\partial \tilde{B}_\theta}{\partial z} = \frac{1}{c}\frac{\partial \tilde{E}_r}{\partial t}, \tag{6a'}$$

$$\frac{\partial \tilde{B}_r}{\partial z} - \frac{\partial \tilde{B}_z}{\partial r} = \frac{1}{c}\frac{\partial \tilde{E}_\theta}{\partial t}, \tag{6b'}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{B}_\theta) = \frac{4\pi \tilde{j}_z}{c} + \frac{1}{c}\frac{\partial \tilde{E}_z}{\partial t}, \tag{6c'}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{E}_r) + \frac{\partial \tilde{E}_z}{\partial z} = 4\pi \tilde{n} e. \tag{7'}$$

4.3. *Details of the wave equation*

In order to describe the axially symmetric perturbations supported by the rippled beam equilibrium defined by (14) or (16), we can use the following Floquet expansion:

$$\tilde{s}(r,z,t) = \sum_n A_n(r)\, e^{i(\omega t - k_n z)}, \tag{19}$$

where

$$k_n = k_z \pm n k_S, \tag{20}$$

n is an integer and $\tilde{s} = \tilde{v}_z, \tilde{n}, \tilde{E}_z$.*

The explicit wave equation can be obtained from (1')–(7') and (18)–(20). First, by (19), equation (18) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \tilde{E}_{z,n}}{\partial r}\right) - k_n^2 \tilde{E}_{z,n} - 4\pi e i k_n \tilde{n}_n + \frac{\omega^2}{c^2}\tilde{E}_{z,n} = \frac{4\pi i \omega}{c^2}\tilde{j}_n. \tag{21}$$

In deriving (21) we used Poisson's equation providing

$$\nabla_z(\nabla \cdot \tilde{\mathbf{E}}) = \frac{\partial}{\partial z}(-4\pi e \tilde{n}). \tag{22}$$

* We recall that we here assume a model in which the main effect of the beam equilibrium periodicity (rippling) is to modulate the electron beam density according to (16); possible simultaneous modulation of the equilibrium velocity is not considered.

Thus, we should now find the expressions for the perturbed quantities appearing in (21).

From (4′) and (19) we find

$$\tilde{v}_{z,n} = \frac{ie}{m\Omega_n \gamma^3} \tilde{E}_{z,n}, \tag{23}$$

where

$$\Omega_n \equiv \omega - k_n V_z. \tag{24}$$

From (1′) and (19) we obtain ($\bar{\delta} \equiv \delta/r_0$)

$$\tilde{n}_n = \frac{k_n n_0}{\Omega_n} v_{z,n} + \frac{k_n n_0}{\Omega_n} \bar{\delta}(\tilde{v}_{z,n+1} + \tilde{v}_{z,n-1}). \tag{25}$$

In this derivation we (i) neglected the third term in (1′) as compared with the second*; (ii) used the relation (16) for n_0 with the + sign; (iii) expressed $\sin k_S z$ and $\cos k_S z$ in terms of exponential functions; and (iv) used the identities $k_{n+1} - k_S = k_{n-1} + k_S = k_n$.

From (21) and (23) we find the perturbed current, namely

$$\tilde{j}_z \equiv -e(\tilde{n}V_z + n\tilde{v}_z) = -\frac{ie^2}{m\gamma^3} \frac{n_0 \omega}{\Omega_n^2} \tilde{E}_{z,n} - \frac{ie}{m\gamma^3} \frac{n_0 \omega}{\Omega_n} \left(\frac{\tilde{E}_{z,n+1}}{\Omega_{n+1}} + \frac{\tilde{E}_{z,n-1}}{\Omega_{n-1}} \right). \tag{26}$$

Finally, upon substitution of the results (23)–(26) into (21), we find the following wave equation for the waves supported by the rippled beam configuration considered here (and illustrated in figure 1):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \tilde{E}_{z\,n}}{\partial r}\right) + \left(\frac{\omega^2}{c^2} - k_n^2\right)\left(1 - \frac{\omega_{p0}^2}{\gamma^3 \Omega_n^2}\right) \tilde{E}_{z,n} = \bar{\delta} \frac{\omega_{p0}^2}{\gamma^3 \Omega_n} \left(\frac{\omega^2}{c^2} - k_n^2\right) \left(\frac{\tilde{E}_{z\,n+1}}{\Omega_{n+1}} + \frac{\tilde{E}_{z\,n-1}}{\Omega_{n-1}} \right). \tag{27}$$

Equation (27) indicates that every three space harmonics namely n, $n+1$ and $n-1$ are coupled. Before considering the general case described by this equation, we consider the uncoupled case, $\bar{\delta} = 0$.

4.4. *The uncoupled case*

If $\bar{\delta} = 0$ we have the simple case of a uniform cross-section electron beam in a circular waveguide. Since the (equilibrium) periodic perturbation was removed, all the space harmonics in the expansion (19) vanish so that only the fundamental mode ($n = 0$) exists. Equation (27) becomes

$$\left.\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \tilde{E}_z}{\partial r}\right) - \tau^2 \tilde{E}_z = 0 \qquad r \leqslant r_0, \\ &\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \tilde{E}_z}{\partial r}\right) - \bar{\tau}^2 \tilde{E}_z = 0 \qquad r > r_0, \end{aligned}\right\} \tag{28}$$

where

$$\tau^2 = \bar{\tau}^2(1 - \omega_{p0}^2/\gamma^3 \Omega^2), \tag{29}$$

$$\bar{\tau}^2 = (k^2 - \omega^2/c^2) \tag{30}$$

* As will be shown later, in the long-wavelength limit, $\tilde{n} \sim J_0$, $\partial\tilde{n}/\partial z \sim k_n \tilde{n}$, $\partial\tilde{n}/\partial r \sim k_n J_1$ and consequently $\partial\tilde{n}/\partial r \ll \partial\tilde{n}/\partial z$ (J_0 and $J_1 \equiv J_0'$ are Bessel functions).

and

$$\Omega \equiv \omega - kV_z. \tag{31}$$

Equations (28) are Bessel type equations and their (fundamental) mode solutions are

$$E_z = AJ_0(\tau r), \qquad r \leqslant r_0, \tag{32}$$

$$E_z = B[I_0(\bar{\tau} r)\, K_0(\bar{\tau} a) - I_0(\bar{\tau} r)\, K_0(\bar{\tau} a)], \quad r > r_0, \tag{33}$$

where the solution (32) was chosen so as to remain finite at $r = 0$ while the solution (33) was chosen to satisfy the boundary condition at the perfectly conductive waveguide wall. To determine the constant B and the mode parameters k_z, τ and $\bar{\tau}$ one should match the tangential electric and magnetic field components at the beam–vacuum interface. This will be carried out explicitly in the long-wavelength limit.

At this point we choose to consider $\tilde{E}$-type waves, i.e. waves characterized by $\tilde{B}_z = 0$. They may include longitudinal (electrostatic) modes (only $\tilde{E}_z \neq 0$) and transverse magnetic (TM) electromagnetic modes (having all but the $\tilde{B}_z$ component non-zero). In the case of relatively strong magnetic field ($\tilde{j}_\perp = 0$), for azimuthally symmetric modes ($\partial/\partial\theta = 0$), Maxwell's equations reduce to*

$$-ik\tilde{E}_r - \frac{\partial \tilde{E}_z}{\partial r} = -\frac{i\omega}{c}\tilde{B}_\theta \tag{5b''}$$

and

$$ik\tilde{B}_\theta = \frac{i\omega}{c}\tilde{E}_r. \tag{6a''}$$

Solving these equations provides

$$\tilde{B}_\theta = \beta\frac{\partial \tilde{E}_z}{\partial r}, \quad \tilde{E}_r = \alpha\frac{\partial \tilde{E}_z}{\partial r} \tag{34}$$

where the constants α and β are given by

$$\alpha = \frac{i}{k}\frac{1}{1-\omega^2/k^2c^2}, \quad \beta = \frac{\omega}{kc}\alpha. \tag{35}$$

Thus the continuity of the tangential components $\tilde{E}_z$ and $\tilde{B}_\theta$ at $r = r_0$ requires (after division term by term of the two matching conditions)

$$\left(\frac{1}{\tilde{E}_z}\frac{\partial \tilde{E}_z}{\partial r}\right)^{\text{in}}_{r=r_0} = \left(\frac{1}{\tilde{E}_z}\frac{\partial \tilde{E}_z}{\partial r}\right)^{\text{out}}_{r=r_0}. \tag{36}$$

(Notice that in this case, because $\tilde{j}_\perp \simeq \tilde{v}_\perp = 0$, the transverse field component, $\tilde{E}_r$ is also continuous ($\tilde{E}_r \simeq \tilde{B}_\theta$). This is no longer valid in the case in which

* The reduction of Maxwell's equations (5′) and (6′) (in which $\partial/\partial\theta = 0$ and $j_\perp = 0$) to (5b″) and (6a″) proceeds as follows. Integration of (5c′) for TM modes ($\tilde{B}_z = 0$) gives $r\tilde{E}_\theta$ = const., i.e. $\tilde{E}_\theta$ = const./r; the physical requirement of $\tilde{E}_\theta$ being finite at $r = 0$ implies $\tilde{E}_\theta = 0$. Using $\tilde{E}_\theta = 0$ in (5a′) provides $\tilde{B}_r = 0$. Thus only $\tilde{E}_r$, $\tilde{E}_z$ and $\tilde{B}_\theta$ are non-zero.

$\tilde{j}_\perp \neq 0$.) Substitution of (32) and (33) into (36) results in the uncoupled modes dispersion relation, namely

$$\tau \frac{J_1(\tau r_0)}{J_0(\tau r_0)} = \bar{\tau} \frac{I_1(\bar{\tau} r_0)\, K_0(\bar{\tau} a) + I_0(\bar{\tau} a)\, K_1(\bar{\tau} r_0)}{I_0(\bar{\tau} r_0)\, K_0(\bar{\tau} a) - I_0(\bar{\tau} a)\, K_0(\bar{\tau} r_0)}. \tag{37}$$

After some mathematical manipulation, this equation can be simplified in the long-wavelength limit, $\tau r_0 \ll 1$, $\tau a \ll 1$, and becomes

$$\tau^2 \equiv (k_z^2 - \omega^2/c^2)\,(1 - \omega_{p0}^2/\gamma^3\Omega^2) = -2/r_0^2 \ln(a/r_0). \tag{38}$$

Here, two distinct cases may occur.

(i) If $\omega \gg k_z V_z$ one has $\Omega \equiv \omega - k_z V_z \simeq \omega$ and (38) reduces to

$$k_z^2 = \frac{\omega^2}{c^2} - \frac{2}{r_0^2 \ln(a/r_0)} \frac{1}{1 - \omega_{p0}^2/\Omega^2\gamma^3}. \tag{39}$$

Since $k_z < \omega/c$, (39) describes two electromagnetic (TM) waves: a forward and a backward propagating wave.

(ii) If $k_z \gtrsim \omega/c$, $\omega_{p0}^2/\gamma^3\Omega^2 \gg 1$, (38) can be written as

$$\Omega^2(1+R) = R\omega_{p0}^2/\gamma^3, \quad R \equiv \left(k^2 - \frac{\omega^2}{c^2}\right) \frac{r_0^2 \ln(a/r_0)}{2} \tag{40}$$

where $R(\ll 1)$ represents the 'reduction factor' due to the finiteness of the beam. To first order, the solution of (40) is

$$\Omega_s^f = \pm R^{\frac{1}{2}} \omega_{p0}/\gamma^{\frac{3}{2}} \tag{41}$$

or alternatively

$$\left.\begin{aligned} \omega_s^f &= kV_z \pm R^{\frac{1}{2}} \omega_{p0}/\gamma^{\frac{3}{2}}, \\ k_s^f &= \omega/V_z \mp R^{\frac{1}{2}} \omega_{p0}/V_z \gamma^{\frac{3}{2}}. \end{aligned}\right\} \tag{42}$$

Equations (41) or (42) represent the dispersion relation of two forward propagating space charge waves, namely, a fast wave and a slow wave.

4.5. *Non-zero coupling*

Equation (27) represents an infinite set of coupled differential equations and its explicit solution is rather difficult. For the case of weak coupling (small $\bar{\delta}$) a reasonable approximation is to assume that the transverse profile of the fundamental space harmonic, $n = 0$, is similar to that of the unrippled beam (i.e. uncoupled) case (cf. (32) and (33)). For the other space harmonics, $n \neq 0$, (27) can be regarded as inhomogeneous equations. Thus, for any space harmonic n, the solution will consist of a homogeneous solution together with an inhomogeneous solution which is forced by the coupling to the next lower order harmonic. In the present analysis we ignore the homogeneous solution and thus assume that all the space harmonics have the same transverse profile identical to the unperturbed beam profile inside the beam (equation (32)):

$$\tilde{E}_z = J_0(\tau r) \sum_n a_n \exp[i(\omega t - k_n z)]. \tag{43}$$

By (43), equation (27) turns into an infinite set of algebraic equations, namely

$$(\tau_n^2 - \tau^2)\, a_n = \bar{\delta} \frac{\omega_{p0}^2}{\gamma^3 \Omega_n^2} \left(\frac{\omega^2}{c^2} - k_n^2\right) \left(\frac{a_{n+1}}{\Omega_{n+1}} + \frac{a_{n-1}}{\Omega_{n-1}}\right), \tag{44}$$

where

$$\tau_n^2 \equiv \bar{\tau}_n^2(1-\omega_{p0}^2/\gamma^3\Omega_n^2), \tag{45}$$

and

$$\bar{\tau}_n^2 = k_n^2 - \omega^2/c^2. \tag{46}$$

The assumption that the different space harmonics have similar transverse field profiles ($\tau_n \simeq \tau$) results in $nk_S k_z \ll \tau^2$ which is a long-wavelength limit condition. In the long-wavelength limit we can use the approximation (38) and reduce (44) to

$$\left[\left(k_n^2-\frac{\omega^2}{c^2}\right)\left(1-\frac{\omega_{p0}^2}{\gamma^3\Omega_n^2}\right)+\frac{2}{r_0^2\ln(a/r_0)}\right]a_n = \bar{\delta}\frac{\omega_{p0}^2}{\gamma^3\Omega_n}\left(k_n^2-\frac{\omega^2}{c^2}\right)\left[\frac{a_{n+1}}{\Omega_{n+1}}+\frac{a_{n-1}}{\Omega_{n-1}}\right]. \tag{47}$$

In practice, the infinite set of equations (47) is terminated and only two or three space harmonics are kept. With appropriate choice of k_S, the electromagnetic mode k_z^{EM} (equation (39)) can be coupled resonantly to the slow or fast space charge waves (equation (42)) through the first-order space harmonic of the EM waves. Alternatively, the slow and fast space charge waves may be coupled to each other through the first-order space harmonic of the fast wave.

4.5.1. *Coupling of electromagnetic and space charge waves*

For $n = 0$ and $n = 1$, from (47) we obtain, respectively,

$$\left[\left(k_0^2-\frac{\omega^2}{c^2}\right)\left(1-\frac{\omega_{p0}^2}{\gamma^3\Omega_0^2}\right)+\frac{2}{r_0^2\ln(a/r_0)}\right]a_0 = \bar{\delta}\frac{\omega_{p0}^2}{\gamma^3\Omega_0}\left[\left(k_0^2-\frac{\omega^2}{c^2}\right)\left(\frac{a_1}{\Omega_1}+\frac{a_{-1}}{\Omega_{-1}}\right)\right] \tag{48}$$

and

$$\left[\left(k_1^2-\frac{\omega^2}{c^2}\right)\left(1-\frac{\omega_{p0}^2}{\gamma^3\Omega_1^2}\right)+\frac{2}{r_0^2\ln(a/r_0)}\right]a_1 = \bar{\delta}\frac{\omega_{p0}^2}{\gamma^3\Omega_1}\left[\left(k_1^2-\frac{\omega^2}{c^2}\right)\left(\frac{a_2}{\Omega_2}+\frac{a_0}{\Omega_0}\right)\right]. \tag{49}$$

We solve (48) and (49) in the resonant regime, i.e. we neglect the space harmonics $n = 2$ and $n = -1$. Thus, we consider the coupling of the fundamental electromagnetic (TM) wave (see (39)) and the slow first-harmonic space charge wave (see (42)). The resonance condition is

$$\omega \simeq \omega_{TM} \simeq \omega_{ES}. \tag{50}$$

Thus, for the TM mode ($k_0 < \omega/c$, $\omega < \Omega_0$), recalling that k_0^2 means $k_{n=0}^2 = (k_z \pm nk_S^2)_{n=0} = k_z^2$, (48) can be written as

$$\left[k_0^2-\frac{\omega^2}{c^2}+\frac{2}{r_0\ln(a/r_0)}\frac{\Omega_0^2}{\Omega_0^2-\omega_{p0}^2/\gamma^3}\right]a_0 = \bar{\delta}\frac{\omega_{p0}^2}{\gamma^3}\left[\left(k_0^2-\frac{\omega^2}{c^2}\right)\frac{\Omega_0}{\Omega_1}\frac{1}{\Omega_0^2-\omega_{p0}^2/\gamma^3}\right]a_1. \tag{51}$$

For the space charge wave ($k_1 \gg \omega/c$), (49) reads

$$[\Omega_1^2(1+R_1)-(\omega_{p0}^2/\gamma^3)R_1]a_1 = \bar{\delta}(\omega_{p0}^2/\gamma^3)[(\Omega_1/\Omega_0)R_1]a_0. \tag{52}$$

Equations (51) and (52) for a_1 and a_0 provide the dispersion relation

$$\left[k_0^2-\frac{\omega^2}{c^2}+\frac{2}{r_0^2\ln(a/r_0)}\frac{\Omega_0^2}{\Omega_0^2-\omega_{p0}^2/\gamma^3}\right]\left(\Omega_1^2-\frac{\omega_{p0}^2}{\gamma^3}R_1\right) = \bar{\delta}^2\frac{\omega_{p0}^4}{\gamma^6}\left(k_0^2-\frac{\omega^2}{c^2}\right)\frac{R_1}{\Omega_0^2-\omega_{p0}^2/\gamma^3}. \tag{53}$$

For convenience, we rewrite (53) as

$$AB = \bar{\delta}^2 C. \tag{53'}$$

To obtain a relatively simple analytical solution, we simplify (53) as follows.

(i) Since for electromagnetic waves $\omega \gg k_z V_z$, we approximate $\Omega_0/[\Omega_0 - \omega_{p0}^2/\gamma^3]$ in A by $\omega^2/(\omega^2 - \omega_{p0}^2/\gamma^3)$. (Recall that $\Omega_n \equiv \omega - k_n V_z$, $\Omega_0 \equiv \omega - k_z V_z$, $k_n = k_z \pm n k_S$.)

(ii) Comparing the last two terms in A with (39) (giving the TM-eigenmodes in the uncoupled case, k_n') we find that A can now be written as

$$k_0^2 - k_0'^2 = (k_0 - k_0')(k_0 + k') \simeq 2k_0'(k_0 - k_0').$$

(iii) In B, considering the slow space charge mode $\Omega_1 \simeq -\omega_{p0} R_1^{\frac{1}{2}}/\gamma^3$, (equation (42)), one may write

$$B \equiv \Omega_1^2 - \frac{\omega_{p0}^2}{\gamma^3} R_1 = \left(\Omega_1 - \frac{\omega_{p0}}{\gamma^{\frac{3}{2}}} R_1^{\frac{1}{2}}\right)\left(\Omega_1 - \frac{\omega_{p0}}{\gamma^{\frac{3}{2}}} R_1^{\frac{1}{2}}\right) \simeq -2\frac{\omega_{p0}}{\gamma^{\frac{3}{2}}} R_1^{\frac{1}{2}}\left(\Omega_1 + \frac{\omega_{p0}}{\gamma^{\frac{3}{2}}} R_1^{\frac{1}{2}}\right).$$

(iv) In C we use (from (39))

$$k_0^2 - \omega^2/c^2 \simeq -[2/r_0^2 \ln(a/r_0)][\omega^2/(\omega^2 - \omega_{p0}^2/\gamma^3)].$$

Then, by (i)–(iv), (53) reads (we now use the notation k_z for k_0 and $k_{0,z}$ for k_0')

$$(k_z - k_{z,0})(\Omega_1 + \omega_{p0} R_1^{\frac{1}{2}}/\gamma^{\frac{3}{2}}) = \bar{\delta}^2 \frac{\omega_{p0}^3 R_1^{\frac{1}{2}}}{2\gamma^{\frac{9}{2}}[r_0 \ln(a/r_0)]} \frac{1}{k_{z,0}^2} \frac{\omega^2}{(\omega^2 - \omega_{p0}^2/\gamma^3)^2} \tag{54}$$

with $\Omega_1 \equiv \omega - (k_z + k_S) V_z$. Solving this equation provides

$$\operatorname{Im} k_z = \pm i\bar{\delta} \frac{\omega_{p0}}{\gamma^{\frac{3}{2}}} \frac{\omega}{\omega^2 - \omega_{p0}^2/\gamma^3} \left[\frac{\omega_{p0} R_1^{\frac{1}{2}}}{2k_{z,0}\gamma^{\frac{1}{2}} V_z r_0^2 \ln(a/r_0)}\right]^{\frac{1}{2}}. \tag{55}$$

4.5.2. *Coupling of space charge waves*

In the limit $k_z \gg \omega/c$, the slow and fast space charge wave solutions (equation (42)) may be coupled through the first-order space harmonic of the fast space charge wave. Taking $k_0 = k_z = k^f$ we keep in (47) only the terms with $n = 0, 1$ and write it in the form

$$(\Omega_0^2 - \omega_{p0}^2 R_0/\gamma^3) a_0 = \bar{\delta}(\omega_{p0}^2 R_0/\gamma^3)(\Omega_0/\Omega_1) a_1 \tag{56a}$$

and

$$(\Omega_1^2 - \omega_{p0}^2 R_1/\gamma^3) a_1 = \bar{\delta}(\omega_{p0}^2 R_1/\gamma^3)(\Omega_1/\Omega_0) a_0 \tag{56b}$$

where, in accordance with (40), we define $R_n \equiv 0{\cdot}5(k_n^2 - \omega^2/c^2)(r_0^2 \ln(a/r_0))$.

Using (56a) in (56b) and solving the resulting dispersion relation at the resonance condition $k_S = (R_0^{\frac{1}{2}} + R_1^{\frac{1}{2}})\omega_{p0}^2/\gamma^3$ provides a complex solution with

$$\operatorname{Im} k = \pm i \frac{\bar{\delta}}{2} \frac{\omega_{p0}}{\gamma^{\frac{3}{2}} V_z} (R_0 R_1)^{\frac{1}{2}}.$$

Physically, however, this solution contradicts the assumption of strong magnetic field, $\omega_c \gg \omega_{p0}$ used in this paper. Indeed, the resonance conditions in (57) are

$$\omega - k_z V_z = +\omega_{p0} R_0^{\frac{1}{2}}/\gamma^{\frac{3}{2}} \quad (n = 0, \text{ fast mode})$$

and

$$\omega - (k_z + k_S) V_z = -\omega_{p0} R_1^{\frac{1}{2}}/\gamma^{\frac{3}{2}} \quad (n = 1, \text{ slow mode}).$$

These conditions imply

$$k_S V_z = \omega_{p0}(R_0^{\frac{1}{2}} + R_1^{\frac{1}{2}})/\gamma^{\frac{3}{2}}$$

which, by (15) and (13), reads

$$\omega_c^2 = \omega_{p0}^2[\gamma(1-f) + (R_0^{\frac{1}{2}} + R_1^{\frac{1}{2}})^2/\gamma].$$

Now since $\gamma \gtrsim 1$, $(1-f) \leqslant 1$, $R_0^{\frac{1}{2}} + R_1^{\frac{1}{2}} \leqslant 1$, the last condition implies $\omega_c \simeq \omega_{p0}$ which contradicts the assumption $\omega_c \gg \omega_{p0}$ indicated above.

However, short-wavelength *volume* or long-wavelength *surface* space charge perturbations are *unstable*. The non-relativistic treatment of these cases has been given in Cuperman & Petran (1982) and Petran & Cuperman (1982).

5. Summary and discussion

We investigated the stability against mixed electromagnetic and electrostatic perturbations of rippled, magnetically focused relativistic electron beams. Specifically, we considered the resonant coupling of a volume TM (fast) wave and a volume longitudinal space charge (slow) wave, in the long-wavelength limit. We found instability with the (convective) growth rate given by (55). Essentially, the growth rate increases with the increase of the relative ripple amplitude, $\bar{\delta}$, the plasma frequency, ω_{p0}, and the reduced plasma frequency, $\omega_{p0} R_1^{\frac{1}{2}}$; it decreases with the increase in the relativistic factor, γ.

For simplicity, in the solution of (53) we considered the specific resonant 'slow' space charge mode as given by (42) and thus reduced (53) to the (second-order) equation (54). This provided the result of (55).

Now if this 'educated guess' for the selection of the specific slow space charge mode (as represented by the step (iii) preceding (54)) is not made, a more general, third-order equation replacing (54) is obtained. This equation reads

$$(k - k_{z,0})(k_z + k_S - \omega/V_z + \omega_p'/V_z)(k_z + k_S - \omega/V_z - \omega_p'/V_z) + Q = 0 \tag{54'}$$

where

$$\omega_p' \equiv \omega_{p0} R_1^{\frac{1}{2}}/\gamma^{\frac{3}{2}} \tag{57}$$

and

$$Q \equiv -\frac{\bar{\delta}^2 \omega_{p0}^4 R_1}{2 V_z^2 k_{z,0} \gamma^6} \frac{k_z^2 - \omega^2/c^2}{\Omega_0^2 - \omega_{p0}^2/\gamma^3}. \tag{58}$$

Equation (54′) is similar to that occurring in travelling wave tubes (Pierce 1950), free electron lasers of various kinds (e.g. Kroll & McMullin 1978; Sprangle 1974; Gover & Sprangle 1981) as well as plasma instabilities (e.g. Bekefi & Shefer 1979). Notice, however, the presence in the (small) coupling term, Q (equation (58)) and also in $\Omega_0 \equiv \omega - k_z V_z$ of k_z. Thus, for simplicity, in the expression for Q we use $k_z^2 \simeq k_{z,0}^2$ and (see (39)) $k_{z,0}^2 - \omega^2/c^2 = -[2/r_0^2 \ln(a/r_0)][\Omega_0^2/(\Omega_0^2 - \omega_{p0}^2/\gamma^3)]$. Finally, we approximate $\Omega_0 \simeq \omega$ ($\Omega \gg k_z V_z$) and obtain for Q

$$Q = \frac{\bar{\delta}^2 \omega_{p0}^4 R_1}{V_z^2 k_{z,0} \gamma^6} \frac{1}{r_0^2 \ln(a/r_0)} \frac{\omega^2}{(\omega^2 - \omega_{p0}^2/\gamma^3)^2}. \tag{58'}$$

We are now in a position to discuss (54′) (with Q given by (58′)). Defining

$$\delta k \equiv k_z - k_{z,0}, \tag{59}$$

$$\theta_p \equiv \omega_p'/V_z, \tag{60}$$

and the 'detuning' parameter

$$\theta \equiv \omega/V_z - k_{z,0} - k_S, \tag{61}$$

(54′) can be written in the form

$$\delta k(\delta k - \theta - \theta_p)(\delta k - \theta + \theta_p) + Q = 0. \tag{62}$$

A detailed investigation of this third-order algebraic equation is given in Gover & Sprangle (1981). A number of useful analytic approximate solutions can be found corresponding to different gain regimes. Equation (62) has three roots and the condition that two of them be complex (conjugate) is

$$Q > \tfrac{2}{27}[9\theta\theta_p^2 - \theta^3 + (\theta^2 + 3\theta_p^2)^{\frac{3}{2}}]. \tag{63}$$

When this condition is satisfied, one of the roots has a positive imaginary ($\mathrm{Im}\,\delta k$) part. This root gives rise to an exponentially growing convective instability which can grow out of noise (or from a RF radiation signal coupled at the entry of the device, at $z = 0$). When $|\mathrm{Im}\,\delta k|z \gg 1$ (high gain regime) the exponentially growing term exceeds the other two terms which correspond to the other two roots (one oscillating and the other exponentially decaying). The only important term is then the exponentially growing term.

From (62) and (58′) two high gain approximate expressions can be readily derived. For $\theta \simeq -\theta_p$ and $|\delta k| \ll \theta_p$, (62) reduces to a second-order equation, $(\delta k)^2 = -Q/2\theta_p$. Upon substitution of expressions (60) and (58′) for θ_p and Q, respectively, one recovers the result (55). This gain regime is the collective gain regime. The synchronization condition $\theta \simeq -\theta_p$ can be written as

$$\frac{\omega}{V_z} + \frac{\omega_p'}{V_z} = k_{z,0} + k_S, \tag{64}$$

which means that the TM electromagnetic wave is phase matched to the slow space charge wave via the wavenumber of the periodic envelope perturbation k_S. We made this assumption in deriving the simple result (55). The validity range of this gain regime can be written as $Q^{\frac{1}{3}} \ll \theta_p$.

In the opposite limit, $\theta_p \gg |\delta k|$, $Q^{\frac{1}{3}} \gg \theta_p$ and $|\theta| \ll |\delta k|$, (62) reduces to a third-order equation, $(\delta k)^3 = -Q$ and the root corresponding to exponential growth is $\delta k = -Q^{\frac{1}{3}}(1 + 3^{\frac{1}{2}}i)/2$. The exponential growth parameter is

$$\mathrm{Im}\,k = \frac{3^{\frac{1}{2}}}{2} Q^{\frac{1}{3}} = \frac{3^{\frac{1}{2}}}{2\gamma^2}\left[\frac{\delta^2 \omega_{p0}^4 R_1}{V_z^2 k_{z,0}} \frac{1}{r_0^2 \ln(a/r_0)} \frac{\omega^2}{(\omega^2 - \omega_{p0}^2/\gamma^3)^2}\right]^{\frac{1}{3}}.$$

This is the strong pump regime.

Besides the stability against mixed electromagnetic and electrostatic perturbations (§5.1), we also investigated the stability of resonantly coupled slow and fast space charge waves (§5.2). Under the conditions considered here, namely, relatively strong applied magnetic field and long wavelength, $n = 0$ and $\bar{n} = 1$, volume space charge waves, no instability was found. (Instability occurs, however, when the magnetic field is relatively weak).

It should be mentioned that in this paper we treated only the coupling of waves due to (equilibrium) density modulation of magnetically focused electron beams; (for simplicity, we assumed constant axial streaming velocity, V_z). Thus, additional effects arising from simultaneously occurring equilibrium velocity modulation, or specially prepared solely velocity modulated beams, should also be considered. Furthermore, there is still scope for a more accurate three-dimensional analysis which will take into account the homogeneous solutions of (27) and allow for different forms of transverse field components for different space harmonics.

Finally, we point out that the instability due to beam envelope rippling in the case of magnetically focused electron beams should be of interest in the field of free electron lasers. It can be a source of laser instability when inappropriate beam focusing is used. It can also be considered as a possible mechanism for free electron lasers.

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