Laser line broadening due to classical and quantum noise and the free-electron-laser linewidth

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The problem of fundamental laser line broadening due to random spontaneous emission of radiation and amplification of thermal radiation noise is analyzed in terms of a classical fluctuating field phasor model. We derive a general expression for the intrinsic linewidth, given in terms of the spectral power of the radiation noise source, which can be classical or quantum mechanical in nature. In the case of a two-level atomic laser, we recover by the use of Einstein relations, the traditional linewidth formula of the Schalow-Townes form. In the case of the free-electron laser (FEL), using the explicit expression for the spontaneous emission, we present calculation of the laser linewidth by purely classical methods. The result agrees with the one obtained in the framework of a quantum-mechanical model. By using "extended Einstein relations" which are applicable to classical radiators, we show that a Schalow-Townes-type formula can also be obtained for the FEL. The theory predicts extremely narrow intrinsic linewidth (10^{-7} Hz) for cw FEL's with parameters similar to those of the FEL experiment of Elias *et al.*

I. INTRODUCTION

Perhaps the most significant property of the laser oscillator as a light source is its narrow spectral linewidth. Consequently the pioneering articles on the maser and laser development embarked extensively on the question of the intrinsic linewidths of these oscillators.[1,2]

There are many technical and environmental noise and instability factors that may substantially increase the linewidth of practical laser oscillators. The interest of the present article is only in the fundamental intrinsic linewidth of the laser oscillator, which is determined by the spontaneous emission noise source of the laser medium and the blackbody thermal radiation of the cavity. Considering only the thermal radiation noise source for the maser oscillator, Gordon, Zeiger, and Townes derived in their first paper on the maser the following expression for the fundamental linewidth:[1]

$$\Delta\nu_{\text{maser}}=2\pi kT\frac{(\Delta\nu_{\text{sp}})^2}{P_{\text{gen}}}\ , \tag{1}$$

where k is the Boltzmann constant, T is the temperature of the cavity walls, P_{gen} is the total power generated in the oscillator (which would be the total output power if internal dissipation losses were negligible), $\Delta\nu_{\text{sp}}$ is the FWHM (full width at half maximum) bandwidth of the molecular spontaneous emission line, and $\Delta\nu_{\text{maser}}$ is the FWHM maser oscillator linewidth.

In their first paper on the laser, Schawlow and Townes extended the maser linewidth formula (1) to atomic lasers arguing heuristically that in the optical regime the thermal agitation energy kT should be replaced by the much larger energy of one optical quantum $h\nu$ of the "zero point fluctuations noise"[2] resulting in

$$\Delta\nu_{\text{laser}}=2\pi h\nu\frac{(\Delta\nu_{\text{sp}})^2}{P_{\text{gen}}}\ , \tag{2}$$

where $\Delta\nu_{\text{sp}}$ is the FWHM spontaneous emission linewidth of the atomic laser, which was assumed to be much narrower than the cavity linewidth $\Delta\nu_{1/2}$. This linewidth can be written in various forms,

$$\Delta\nu_{1/2}=\frac{\nu}{Q}=\frac{1}{2\pi t_c}=\frac{1}{2\pi}\frac{c}{l}(1-\sqrt{R_1R_2})\ . \tag{3}$$

The parameter Q is the quality factor of the cavity (mostly useful for masers), t_c is the exponential decay time constant of stored radiation energy in the cavity. The last part of the equation refers specifically to a Fabry-Perot-type laser resonator, assumed to have a length l and mirror reflectivities R_1,R_2.[3]

It should be noted that in many laser devices the inequality $\Delta\nu_{\text{sp}}\ll\Delta\nu_{1/2}$ is not satisfied and therefore Eq. (2), known as the Schawlow-Townes formula, should be modified. Standard laser text books cite the following laser linewidth formula:[3]

$$\Delta\nu_{\text{laser}}=2\pi h\nu\frac{(\Delta\nu_{1/2})^2}{P_{\text{gen}}}\mu\ . \tag{4}$$

In addition to the substitution of $\Delta\nu_{1/2}$ for $\Delta\nu_{\text{sp}}$ in the Schawlow-Townes formula, Eq. (4) contains a level population enhancement factor $\mu=N_2/\Delta N>1$, where N_2 is the upper-level population and $\Delta N=N_2-N_1$ is the population inversion at lasing threshold.

Numerous analyses of the laser linewidth problem were published before, using both classical and quantum-mechanical approaches. Renewed interest in this problem has evolved recently mostly in connection to semiconductor lasers.[4] Though starting from general considerations, applicable to any laser oscillator, the main purpose of the present article is to derive a general expression for the free-electron-laser[5] (FEL) linewidth. This kind of laser was operated recently with long pulses (microseconds duration) using electrostatic accelerators, and recent experiments indicated that it may be operated at a single longitudinal mode.[6] It is expected that this kind of FEL may be operated in the future in cw mode, and with very high-power output. Consequently, as we shall show later on, such FEL may be endowed with an extremely narrow intrinsic linewidth.

Since the FEL is basically a classical device, it is expected that all its operating parameters, and in particular the laser oscillator linewidth, can be derived in terms of a classical theory. This is one of the goals of the present article. We will show that the results are consistent, in the common applicability regime with those derived recently by Becker *et al.*[7] using a quantum-mechanical approach.

We reiterate that the basic process of the laser oscillator line broadening is classical, regardless of the kind of laser considered and the nature of the emission process. The broadening results from the stochastic nature of the incoherent noise superimposed on the coherent radiation field, which was built inside the cavity by the stimulated emission process. This broadening mechanism can be described well by very simple classical formulas regardless of the nature of the noise, as long as we assume Gaussian noise statistical characteristics of the emission events.

The two fundamental noise sources in laser and maser oscillators are the spontaneous emission of the laser medium and the amplified thermal radiation of the cavity walls. These noise sources may be intrinsically classical or quantum mechanical in nature, depending on the circumstances.

The Schawlow-Townes formula [Eqs. (2) and (4)] seem explicitly quantum mechanical. They are often derived entirely within a quantum-mechanical model,[8] or semiclassically[3] (the noise emission process is described in terms of single optical quanta emission). We will show that the basic and more general expression for the linewidth is a simple, explicitly classical formula depending only on the incoherent noise spectral power and the stored energy in the cavity. However, a Schawlow-Townes-like formula similar to (4) may be restored from the basic equation by expressing the spontaneous emission noise source in terms of the stimulated emission power using Einstein relations, and then expressing the stimulated emission power at lasing threshold in terms of the cavity resonance linewidth $\Delta\nu_{1/2}$.

The extension to the FEL problem is straightforward. Since the basic linewidth formula is classical it can be applied directly to the FEL problem by substituting for the noise source power the known formula for the classical spontaneous emission in the FEL (which is the undulator synchrotron radiation[9]). Consequently, in this case one may calculate explicitly the line broadening without knowing the laser gain and without using the Einstein relations as needed for the Schawlow-Townes expression. Nevertheless, we show that also in this case a formula similar to the Schawlow-Townes expression can be obtained by going through a similar process as for the two-quantum-level laser, but using an extended form of the Einstein relations[10] (the standard relations do not apply directly to the classical FEL). When taking some simplifying limits, it turns out then that for the FEL, the place of the thermal agitation energy kT in (1) and the photon quantum energy ($h\nu$) in (2), (4) is taken by the energy extraction out of a single electron at laser saturation, which is a classical quantity (and usually of multiquanta magnitude).

For the sake of simplicity we use in the present article a very simple "field phasor" model in order to describe the line-broadening process by random noise generation. This intuitive derivation would be sufficient to obtain the expressions required in the present article. For a more rigorous analysis of the classical laser oscillator equations,[11,12] and the correspondence between classical and quantum-mechanical representations of the laser oscillation,[13] the interested reader is referred to an extensive series of papers by Lax *et al.* (of which we cited three). Lax *et al.* showed that to a good approximation the laser oscillator may be treated at all pumping levels as a rotating-wave van der Pol oscillator where the effect of the noise was treated by adding a driving "Langevin force" to the homogeneous equation.[11,12] The resulting stochastic equation requires a numerical solution in the general case. They showed that the linear approximations leading to the Schawlow-Townes formula are valid only well below lasing threshold. Well above threshold a more appropriate linearized analysis results in a linewidth, which is reduced by a factor of 2 relative to the Schawlow-Townes formula. This result relies on the assumption that in this limit the amplitude fluctuations due to noise are stabilized by the oscillator relaxation process. It is assumed that the phase and amplitude of oscillation are nearly independent variables, and thus the effect of the small amplitude fluctuations is only to add up a low wide spectrum background to the coherent signal field. Only the phase fluctuations contribute then to the broadening of the spectral line to a finite width. The oscillator line-broadening process can be described in this limit as a usual Brownian-motion process of the field vector phasor, producing random Gaussian phase noise modulation.

In the present article we assume a similar model, of an oscillator above threshold described by a randomly fluctuating phasor vector having its phase and amplitude variables *decoupled.* Consequently we expect to obtain expressions smaller by a factor of 2 relative to the Schawlow-Townes formula. We point out, though, that an additional enhancement factor may appear in the linewidth expression when the phase and amplitude are *coupled.*[11] This would happen when the oscillating frequency is a function of the oscillator operating point (nonlinear dispersion), which will cause spurious phase modulation noise associated with the relaxation oscillation of the amplitude fluctuations. This effect has been recently the subject of vigorous investigation and has a significant measure in the context of semiconductor lasers,[4,14–16] however, its consideration is kept out of the scope of the present paper.

II. GENERAL LINEWIDTH FORMULA

We consider a laser oscillator which supports a single cavity mode. An internal noise source produces electromagnetic radiation, which can be characterized as a train of randomly emitted wave packets feeding into the cavity mode. The random process may be quanta emission from excited atoms by spontaneous emission or from a surface of finite temperature (blackbody radiation). It can be also classical spontaneous radiation emission from electrons in an electronic tube or a free-electron laser (in this case the coherent wave packet may contain on the average multiple number of quanta or less than one quantum—see Appendix C). In either case we will assume that the wave-packet emission events are random, obeying Poisson statistics.

The simple phasor model depicted in Fig. 1 is sufficient to describe the phase noise modulation and the line broadening in a laser oscillator well above threshold. The phasor vector of length $\sqrt{\varepsilon_c}$ and phase Θ represents the coherent energy ε_c stored inside the cavity by a single cavity mode q. The phase and magnitude of the stored field and its representative vector change slightly every time a new wave packet of energy $\Delta\varepsilon_c$ is emitted with random phase ϕ relative to the stored field phase. It is assumed that the amplitude of the vector is restored back to its original value by the relaxation mechanism of the saturated oscillator. However, the phase of the vector does change at a random direction by an amount $\Delta\Theta=\sqrt{\Delta\varepsilon_c}\sin\phi$. When many emission events take place randomly, the phase of the vector drifts gradually in a Brownian-motion fashion giving rise to the frequency line broadening.

This model is a simple adaptation of the line-broadening intuitive model commonly used for conventional lasers,[3,4,15] except that we do not interpret the wave-packet emission vector (of length $\sqrt{\Delta\varepsilon_c}$) as representing a single quantum emission, but generalized it to represent any amount of energy emitted coherently in each event which is one out of a train of independent random emission events.

We assume that in general the radiative emission noise source is characterized by a spectral power $S_{q_t\omega}$, which is the total power per unit radian frequency fed into a single standing wave of transverse quantization indices q_t. $S_{q_t\omega}$ may be the spectral power of any of the classical or quantum noise emission processes mentioned in Sec. I. We assume that the power is generated by an average rate of K wave-packet emission events per second. Then the average energy fed into the cavity mode q (single transverse and longitudinal) per event is $\langle\Delta\varepsilon_q\rangle=S_{q_t\omega}\Delta\omega/K$, where $\Delta\omega$ is the radial frequency bandwidth subtended by the single cavity mode.

We can now state that the infinitesimal phase change after each emission event is $\Delta\Theta=(\Delta\varepsilon_q)^{1/2}\sin\phi/\sqrt{\varepsilon_c}$. After a time τ (in which $K\tau$ emission events take place) the mean square of the stored field phase is

$$\langle(\Delta\Theta)^2\rangle_\tau=\frac{\langle\Delta\varepsilon_q\rangle}{\varepsilon_c}\langle\sin^2\phi\rangle K\tau=\frac{1}{2}\frac{S_{q_t\omega}\Delta\omega}{\varepsilon_c}\tau\ . \qquad (5)$$

The line spectrum of the field stored in the cavity can now be found by the standard Fourier transformation of the field autocorrelation function (the Wiener-Khinchine relation). Neglecting the amplitude fluctuation, and assuming the linear time dependence (5) of the mean-squared phase random variable, results in a Lorentzian line-shape function of a FWHM width which is exactly

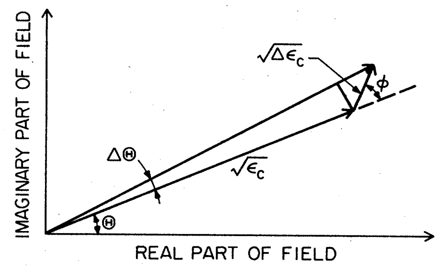

FIG. 1. Fluctuating phasor model. The large vector represents the stored electric field amplitude $\sqrt{\varepsilon_c}$ and phase Θ inside the cavity. The small vector corresponds to a random emission of a single radiation wave packet of energy $\Delta\varepsilon_c$ by a noise source. The wave packet is emitted at a random phase angle ϕ, relative to the stored field phasor vector, rendering a corresponding small angle (phase) rotation $\Delta\Theta$ of the vector.

equal to the diffusion coefficient in Eq. (5):[3,11,16]

$$\Delta\omega_{\text{laser}}=\frac{1}{2}\frac{S_{q_t\omega}\Delta\omega}{\varepsilon_c}\ . \qquad (6)$$

This expression can be interpreted as the frequency bandwidth associated with the time constant in which incoherent radiative energy equal to the stored energy ε_c, damps into the cavity mode.

A few notes on the results and the assumptions used are in order. We first point out that K canceled out in the final results [(5) and (6)], which means that the amount of energy carried by an individual wave packet is not relevant in the foregoing discussion (as long as the random emission events statistical model holds). it may correspond equally well to single quanta, multiquanta, or (statistically) fractional quantum emission at each radiation emission event. We note that Eq. (6) is explicitly classical and would become quantum only if the noise source is characterized by a quantum spectral power $S_{q_t\omega}$.

In (5) it is assumed that the wave-packet emission energy and phase may both be random variables but statistically independent. Quantum emitters have well-determined wave-packet energy: $\Delta\varepsilon_q=\langle\Delta\varepsilon_q\rangle=\hbar\omega$. For classical emitters the wave-packet energy is random when discussed in terms of the quantized field. The number of photons per emitted wave packet is a random number obeying Poisson statistics,[17] and may have large fluctuations when $\overline{\nu}_{sp}=\langle\Delta\varepsilon_q\rangle/\hbar\omega\leq 1$ (which is usually the case in FEL's—see Appendix C). However, the *second moment* of the phase fluctuations $\langle(\Delta\Theta)^2\rangle$ is related to the first moment of the wave-packet energy $\langle\Delta\varepsilon_q\rangle$ [Eq. (5)] which is a classical parameter, and not to the second moment which is $\hbar$ dependent. Thus, for a large number of electron radiators emitting during the oscillator coherence time $1/\Delta\nu_{\text{laser}}$, and a large number of photons in the oscillating mode stored energy, there will be no quantum effects that will modify the classical linewidth formula we derived for the FEL. Quantum analysis[18] results in a more involved scenario when considering the photon-number statistics of many-electron emission (Appendix C), but it still does not change the value of the average photon number.

There may be a question as to the appropriate mode bandwidth section $\Delta\omega$ to be used in (6). We will assume now that the noise source has a much wider spectrum than the frequency spacing between adjacent modes of the cavity. For this case it may be argued[19] that the spectral power is equally distributed among the modes, and one should substitute the intermode spacing for $\Delta\omega$. For a Fabry-Perot resonator structure (assuming a single transverse mode) the longitudinal standing-wave mode spacing is

$$\Delta\omega=\pi\frac{c}{l} . \tag{7}$$

[Alternatively we may have stated that the longitudinal mode density is $(\pi c/l)^{-1}$ and calculated the power per mode by dividing the spectral power into mode density.] The assumption of a wide noise spectrum is well satisfied for the thermal noise source and for the spontaneous emission noise in most atomic and solid-state lasers, and certainly in FEL's. In some atomic lasers and in masers the spontaneous emission spectral line may be narrow enough to invalidate our assumptions. In this case even the spontaneous emission will be modified by the presence of the cavity,[20] and cannot be calculated out of the free-space spectral power emission. Rather, the spontaneous energy emission $\Delta\varepsilon_q$ per cavity mode q should be calculated directly.

As for the spectral power $S_{q_t\omega}$, we emphasize that it is the spectral power fed into a *standing-wave* mode, hence for isotropic noise sources, like thermal noise and spontaneous emission in conventional lasers $S_{q_t\omega}$ is twice the spectral power $P_{q_t\omega}$ emitted into a single transverse *traveling* mode in one direction. In free-electron lasers the spontaneous emission (undulator synchrotron radiation) is unisotropic and emits predominantly in one direction. In this case we take $S_{q_t\omega}=P_{q_t\omega}$, where $P_{q_t\omega}$ is the forward spectral power emission into a single transverse traveling-wave mode.

The calculation of the spectral power into a transverse traveling-wave mode $P_{q_t\omega}$ can be carried out directly for either a quantum or classical spontaneous emission noise source. In particular the classical synchrotron undulator radiation into traveling-wave transverse modes was calculated specifically for rectangular waveguide modes,[21] for parallel-plate waveguide modes,[22] and for general modes (and specifically Hermite-Gaussian).[23] The spectral power of the spontaneous emission into a transverse traveling mode may also be calculated at certain conditions from the free-space spectral radiant intensity $d^2P/(d\Omega\, d\omega)$ using

$$P_{q_t\omega}=\frac{d^2P}{d\Omega\, d\omega}\Delta\Omega_{\text{mode}} , \tag{8}$$

where $\Delta\Omega_{\text{mode}}=\lambda^2/A_{\text{eff}}$ and A_{eff} is the effective cross-section area per mode. This procedure is correct only for a transversely over-moded cavity structure, and when the free-space angular spectrum of the noise source encompasses densely a large number of cavity transverse modes. There are numerous computations of $d^2P/(d\Omega\, d\omega)$—the nonisotropic spectral radiant intensity of the free-space synchrotron undulator radiation (which is the spontaneous emission source of FEL's).[9,24,25] In the case of conventional lasers, with isotropic spontaneous emission, note that the free-space spectral radiant intensity to be used in (8) is half the total (randomly polarized) spectral radiant intensity of the laser medium (in order to account for the well-defined polarization of a single mode of the cavity). It may be computed, for example, in terms of the dipole moment of the atomic transition using a simple semiclassical approach.[26]

III. TWO-LEVEL (ATOMIC) LASERS

In most conventional lasers lasing takes place between two distinct quantum levels, the emission process is isotropic, and the condition $\Delta\nu_{\text{sp}}\gg c/(2l)$ is satisfied. Hence we can use (7) in the linewidth formula (6). Expressing the noise spectral power as $S_{q_t\omega}=2P_{q_t\omega}$, where $P_{q_t\omega}$ is the sum of the unidirectional spontaneous emission and thermal noise amplification spectral powers per single transverse traveling mode, one obtains

$$\Delta\nu_{\text{laser}}=\frac{[(P_{q_t\omega})_{\text{sp}}+(\Delta P_{q_t\omega})_{\text{th}}]c/(2l)}{\epsilon_c} . \tag{9}$$

The thermal spectral power $(\Delta P_{q_t\omega})_{\text{th}}$ is the spectral power generated in one traversal by stimulated emission amplification of the circulating blackbody radiation in the cavity $(P_{q_t\omega})_{\text{th}}=(\hbar\omega/2\pi)(e^{\hbar\omega/kT}-1)^{-1}$. It is assumed that the circulating thermal radiation power is maintained at thermal equilibrium with the walls (the resonator mirrors), contributing a very low power-additive wide spectrum background to the highly monochromatic intense laser spectrum. Only the incremental incoherent power $(\Delta P_{q_t\omega})_{\text{th}}=(G-1)(P_{q_t\omega})_{\text{th}}$ contributes to the phasor drift process of line broadening.[17] Here $G-1=\Delta P_{\text{st}}/P_q$ is the single-path gain of the laser. The spontaneous emission spectral power per single transverse traveling mode can be usually calculated from the free-space radiant intensity of spontaneous emission[26] using (8).

We may express the cavity-stored energy in terms of the laser-generated power

$$\epsilon_c=\frac{P_{\text{gen}}}{2\pi\Delta\nu_{1/2}} \tag{10}$$

and thus obtain the following formula for the laser linewidth:

$$\Delta\nu_{\text{laser}}=\pi\frac{\left[(P_{q_t\omega})_{\text{sp}}+(G-1)\frac{\hbar\omega}{2\pi}(e^{\hbar\omega/kT}-1)^{-1}\right]c/l}{P_{\text{gen}}}\Delta\nu_{1/2} . \tag{11}$$

Though expressed in terms of the fundamental laser operating parameters, expression (11) is not very useful, mostly because the spontaneous emission spectral radiant intensity is sometimes difficult to measure or to calculate accurately. One may take advantage of the Einstein relations[28] in order to express the spontaneous emission and thermal noise spectral powers in terms of the stimulated emission (gain) parameter. To simplify the expression further we also use the relation of equality between gain and cavity losses at saturation,

$$G-1=\frac{(\Delta P_{q_t})_{st}}{P_{q_t}}\approx 1-\sqrt{R_1R_2}=2\pi\frac{l}{c}\Delta\nu_{1/2}\;. \tag{12}$$

Substituting the "Einstein relations" (A6) and (A7) directly into (9), using (10) and (12), the following linewidth expression results in

$$\Delta\nu_{\rm laser}=\pi[\mu+(e^{\hbar\omega/kT}-1)^{-1}]\hbar\omega\frac{(\Delta\nu_{1/2})^2}{P_{\rm gen}}\;. \tag{13}$$

This expression is of the Schawlow-Townes form. The second (thermal) term of (13) is identical with the corresponding term in (11) when (12) is used. Our expression is identical to the corresponding expressions of Yariv[3] except for a spurious factor of 2 which appears there. In the short-wavelength limit $\hbar\omega \gg kT$ the thermal noise term is negligible and one obtains an expression similar to the "Schawlow-Townes formula" (2). In the opposite limit (maser) $kT/(\hbar\omega)\gg\mu>1$ the spontaneous emission term is negligible, and one obtains an expression similar to the "Gordon-Zeiger-Townes formula" (1). These two expressions differ from (13) by a factor of 2, which should be expected in view of Lax's comment on the difference between linewidths above and below the threshold.[11] They also differ by having $\Delta\nu_{\rm sp}$—the spontaneous emission linewidth—replacing $\Delta\nu_{1/2}$, which seems to be the correct frequency bandwidth to be used[29] in the limit $\Delta\nu_{\rm sp}\ll\Delta\nu_{1/2}$, which is the opposite assumption to the one made in Sec. II. For complete correspondence of the expressions, one should also note that in (1) and (2) the assumption $\mu\approx 1$ $(N_2\gg N_1)$ was implicitly used.

IV. FREE-ELECTRON LASERS

Since in the FEL the spontaneous emission and gain are essentially unidirectional the spectral power that feeds into the single transverse standing-wave mode q_t equals the sum of the spontaneous emission and the amplified thermal noise emission of the FEL into a single transverse forward going traveling-wave mode q_t. Thus instead of (9) we obtain from Eq. (6)

$$\Delta\nu_{\rm laser}=\frac{1}{2}\frac{[(P_{q_t\omega})_{\rm sp}+(\Delta P_{q_t\omega})_{\rm th}]c/(2l)}{\varepsilon_c}\;. \tag{14}$$

Unlike atomic lasers, there are practical and reliable classical expressions for the FEL spontaneous emission and (the small-signal) gain. The forward spectral radiant power of spontaneous emission per single transverse traveling mode and the single-path gain (assuming in both cases a uniform cross-section area of the mode along the interaction length) are given by[23,24]

$$P_{q_t\omega}=R_{q_t}{\rm sinc}^2(\bar\theta/2)\;, \tag{15}$$

$$G-1=\bar Q_{q_t}\frac{d}{d\bar\theta}{\rm sinc}^2(\bar\theta/2)\;, \tag{16}$$

where ${\rm sinc}(x)\equiv\sin(x)/x$ and $\bar\theta$ is the detuning parameter. These expressions are valid for a large class of quasi-free-electron radiation devices (including Čerenkov and Smith-Purcell radiation and cyclotron resonance maser).[10,30,31] We will confine our discussion only to the magnetic bremsstrahlung FEL for which

$$\bar\theta=\left[\frac{\omega}{v_z}-k_z-k_w\right]L\;, \tag{17}$$

$$R_{q_t}=\frac{1}{8\pi}I_0e\left[\frac{\mu_0}{\varepsilon_0}\right]^{1/2}\frac{L_w^2}{A_{emq_t}}\frac{\bar a_w^2}{\gamma^2}\;, \tag{18}$$

$$\bar Q_{q_t}=\pi\frac{I_0e}{mc^2}\left[\frac{\mu_0}{\varepsilon_0}\right]^{1/2}\frac{L_w^3}{A_{emq_t}\lambda_w}\frac{\bar a_w^2}{\gamma^3}\;, \tag{19}$$

where k_z is the axial wave number of the radiation mode, which in (18) and (19) is assumed to satisfy $k_z\approx\omega/c$ (negligible transverse wave number). β_z is the average axial velocity of the monoenergetic electron beam propagating in the wiggler, $\gamma\equiv(1-\beta^2)^{-1/2}$, $\gamma_z\equiv(1-\beta_z^2)^{-1/2}$. L_w is the wiggle length and $k_w=2\pi/\lambda_w$ is the wiggler wave number, where the wiggler is assumed to be described by a periodic field function $\mathbf{B}(z)={\rm Re}(\tilde{\mathbf{B}}_w e^{-ik_wz})$. A_{emq_t} is the effective cross-section area of the mode,[23,31] I_0 is the electron beam current, and

$$\tilde{\mathbf{a}}_w=\frac{e\tilde B_w}{k_wmc} \tag{20}$$

is the normalized transverse momentum of the electron in the wiggler. This formulation applies to arbitrary wiggler polarization, where for a planar wiggler $|\tilde B_w|$ is equal to the maximum amplitude of the periodic wiggler, while in the helical wiggler it is larger than the amplitude by a factor of $\sqrt2$. In both cases $|\tilde{\mathbf{B}}_w|=\sqrt2\bar B_w$, where $\bar B_w$ is the rms value of the periodic magnetic field and correspondingly $|\tilde a_w|=\sqrt2\bar a_w$.

Substituting (15), (16), and (10) in (14) we obtain the following expressions for the FEL linewidth:

$$\begin{aligned}\Delta\nu_{\rm laser}&=\frac{\pi}{2}\frac{\left[R_{q_t}{\rm sinc}^2(\bar\theta/2)+(G-1)\frac{\hbar\omega}{2\pi}(e^{\hbar\omega/kT}-1)^{-1}\right]c/l}{P_{\rm gen}}\Delta\nu_{1/2}\\&=0.27\pi\frac{\left[R_{q_t}+\bar Q_q\frac{\hbar\omega}{4\pi}(e^{\hbar\omega/kT}-1)^{-1}\right]c/l}{P_{\rm gen}}\Delta\nu_{1/2}\;.\end{aligned} \tag{21}$$

In the second part of (21) we assumed the practical case of operating near the maximum gain point $\overline{\theta}=-2.6$, for which $d\,\mathrm{sinc}^2(\overline{\theta}_m/2)/d\,\overline{\theta}_m=0.27$ and $\mathrm{sinc}^2(\overline{\theta}_m/2)\approx 0.55$. In some cases it is more convenient not to substitute (16) for $(G-1)$ in the second part of (21), and obtain a slightly different result by expressing the gain in terms of the cavity-loss parameters,

$$G-1=\frac{(\Delta P_q)_{\mathrm{st}}}{P_q}\approx 2(1-\sqrt{R_1R_2})=4\pi\frac{l}{c}\Delta\nu_{1/2}\ . \qquad (22)$$

This relation is a modification of (12) for the case of unidirectional gain.

The following example, based on the parameters of the FEL experiments of Elias *et al.*,[6] demonstrates the use of the line-shape formula, and indicates the parameter values to be expected with FEL devices of this kind. Using the parameters of Table I in (20) we find $|\tilde{\mathbf{a}}_w|=0.17$ and consequently $\gamma_z=\gamma/(1+|\tilde{\mathbf{a}}_w|^2/2)^{1/2}=6.92$. This results in the synchrotron undulator radiation peak frequency $\nu=2\gamma_z^2c/\lambda_w=800$ GHz ($\lambda=375$ μm) and the spontaneous emission linewidth $\Delta\nu_{\mathrm{sp}}=\nu/N_w=5$ GHz (the gain bandwidth is about half this value). The longitudinal mode spacing is $c/(2l)=21.1$ MHz, and the cavity resonance bandwidth (3) is $\Delta\nu_{1/2}=335$ KHz. We note that a large number of modes are densely distributed within the homogeneously broadened emission linewidth, in consistence with the conditions required in the derivation of our linewidth formula.

The resonator cavity in the FEL is waveguiding in one (vertical) dimension and free in the other (horizontal) dimension. For this structure the effective cross-section area in the waist of the fundamental sinusoidal-Hermit Gaussian transverse mode is $A_{\mathrm{em}\,0,1}=w_0b\sqrt{\pi/8}$, where b is the vertical waveguide dimension, and w_0 is the horizontal (free-propagation) beam waist half-width (defined relative to the $1/e$ fall off points of the field amplitude). Using these parameters in (18) we find

$$R_{q_t}=1500\ \mathrm{eV}\ (P_{q_t\omega}=825\ \mathrm{eV})\ .$$

For estimating the thermal noise generation we first note that for mirrors kept at room temperature ($kT\approx 0.025$ eV), the condition $\hbar\omega \ll kT$ reads $\lambda \gg 50$ μm. This condition is well satisfied in the present example, and therefore the thermal noise spectral power is

$$(\Delta P_{q_t\omega})_{\mathrm{th}}=(G-1)\frac{kT}{2\pi}=2(1-\sqrt{R_1R_2})\frac{kT}{2\pi}$$
$$=4\times 10^{-4}\ \mathrm{eV}$$
$$\ll (P_{q_t\omega})_{\mathrm{sp}}\ .$$

Even though we operate at the long-wavelength (far-infrared) regime, where the cavity mode is thermally excited to contain many photons, still the thermal noise spectral power turned out to be negligible relative to the spontaneous emission.

TABLE I. FEL beam parameters (Ref. 6).

$w_0=1.65$ cm	$\lvert\tilde{\mathbf{B}}\rvert=0.05$ T	$\gamma_0=6.97$
$b=1.92$ cm	$\lambda_w=3.6$ cm	$I_0=2$ A
$l=712$ cm	$L_w=576$ cm	$1-\sqrt{R_1R_2}\approx 0.05$

Estimating now the power generation in the FEL from the saturation power extraction formula[31,32] $P_{\mathrm{sat}}=IV/(2N_{\omega})=19$ kW, we obtain from (21) $\Delta\nu_{\mathrm{laser}}=1.5\times 10^{-7}$ Hz, which is an exceedingly narrow intrinsic linewidth.

Following the lead of the two-level-laser derivation we derive now an alternative expression for the FEL linewidth using the "extended Einstein relations." The Einstein relations between spontaneous and stimulated emission coefficients (see Appendix A) are stated in terms of transitions between two distinct quantum levels. However, the magnetic bremsstrahlung FEL in the practically applicable classical regime is equivalent to a multilevel (continuous levels) system in which multiphoton emission and absorption are possible. The homogeneously broadened emission and absorption lines of the FEL are highly degenerate and are displaced in frequency only by a small "quantum recoil" parameter.[10,33,34] The net stimulated emission rate is proportional to the difference between the slightly displaced emission and absorption line-shape functions, which at the classical limit becomes the derivative of the spontaneous emission line-shape function. Consequently, the spontaneous emission and the net gain are proportional to different line-shape functions, and the Einstein relations do not apply.

Based on Ref. 10 the extended Einstein relations [(B8) and (B9)] are derived in Appendix B in analogy with the Einstein relations [(A6) and (A7)]. We note that similar relations were derived before classically by Madey, and can be derived from Bekefi's inhomogeneous broadening generalization of Einstein relations.[35,36] Our extended Einstein relations, are more general in the inclusion of the thermal noise emission and the formulation in terms of general single transverse modes. They are applicable to the magnetic bremsstrahlung FEL as well as to a large number of other free-electron radiation emitters.[10] Substituting (B8), (B9), and (10) in (14) one directly obtains the following expression for the FEL linewidth:

$$\Delta\nu_{\mathrm{laser}}=\pi\{\xi+\hbar\omega[\tfrac{1}{2}+(e^{\hbar\omega/kT}-1)^{-1}]\}\frac{(\Delta\nu_{1/2})^2}{P_{\mathrm{gen}}}\ , \qquad (23)$$

where

$$\xi=\frac{\beta_z^3\gamma_z^2\gamma mc^2}{2\pi}\frac{\lambda}{L_w}\frac{\mathrm{sinc}^2(\overline{\theta}/2)}{\dfrac{d}{d\overline{\theta}}[\mathrm{sinc}^2(\theta/2)]}$$
$$\approx\frac{\beta_z^3\gamma_z^2\gamma mc^2}{\pi}\frac{\lambda}{L_\omega}\approx\frac{\gamma mc^2}{\pi}\frac{1}{2N_w}\ , \qquad (24)$$

where in the second part of (24) we used the practical case assumption of operating near the maximum gain point (B7), and in the last part we took the highly relativistic limit, and used the synchronism condition $\lambda\approx\lambda_w/(2\gamma_z^2)$.

The third and second parts of Eq. (24) may be identified to have the interpretations as roughly one third of the energy extracted by each electron at saturation in the highly relativistic[32] and general[31] cases, respectively. Thus Eq. (23) displays striking similarity to the atomic laser linewidth expression (13), where ξ, the energy emitted by a single electron in the FEL (weighed by a factor

$1/\pi$), is analogous to the term $\hbar\omega\mu$ in the atomic laser, which is the energy emitted by a single excited atom $\hbar\omega$ (weighed by the population factor μ). The thermal noise term in both expressions is the same except for the spurious $\hbar\omega/2$ term in (23) that results from the spontaneous emission.

In the FEL each electron emits a large number of photons (about 10^5 with the parameters of Table I). The "thermal" term in (23) corresponds to half a photon energy in the optical limit $\hbar\omega \ll kT$, and to a relatively small number of photons $kT/(\hbar\omega)$ (eight photons for the parameters of Table I) in the submillimeter-microwave regime 10^{-4} eV $< \hbar\omega \ll kT$. We therefore conclude that the thermal noise contribution is negligible in the present example and in most practical FEL examples.

Neglecting the thermal noise term, Eq. (23) may assume an even simpler more compact form. Noting that the power generation P_{gen} may be written as the average energy lost per electron ($\pi\xi$) times the number of electrons flowing in the beam per unit time (I_0/e) we obtain

$$\Delta\nu_{\text{laser}} = \frac{(\Delta\nu_{1/2})^2}{I_0/e} . \qquad (25)$$

With the parameters of the FEL experiment of Ref. 6 given in Table I, Eq. (25) results in an estimate of $\Delta\nu_{\text{laser}} = 9\times 10^{-9}$ Hz which is an order of magnitude lower than the previous estimate. This discrepancy is discussed in Sec. V.

V. CONCLUSION

We demonstrated with the aid of the simple classical model of a fluctuating field phase vector the computation of the line-broadening mechanism in a laser oscillator above threshold. The field fluctuation source may be a quantum-mechanical noise (in which a single quantum is emitted at each radiation emission event) or classical noise (in which a multiphoton or fractional photon coherent wave packet is generated in each radiation emission event). This was used to derive linewidth expressions for both two-quantum-level (e.g., atomic) lasers and classical (e.g., FEL) radiation devices.

The final atomic laser linewidth expressions, in the appropriate limits, are consistent with previously known standard expressions. The FEL linewidth expressions are consistent in the limits of common applicability with the recent quantum mechanically derived expressions of Becker *et al.*[7]

The general linewidth expressions for both atomic and FEL lasers are simple functions of the spectral power of the noise sources which dominate the line-broadening mechanism. They both may be expressed in the alternative more conventional (Schawlow-Townes-like) form in terms of the cavity parameters and the radiative energy emitted at each random fundamental emission event. To obtain these results one should make use of the Einstein relations in the atomic laser case and an extended form of Einstein relations in the FEL case.

Both the noise-power-expressed and the single-electron emission-energy-expressed FEL linewidth formulas were applied to estimate the FEL linewidth in a numerical example corresponding to the FEL experiment.[6] They both resulted in exceedingly low linewidth values of the order of 10^{-8}–10^{-7} Hz. Even though the fundamental radiation emission energy per electron is much larger in the FEL (ξ) than in the atomic laser ($\hbar\omega$) and in the atomic maser (kT), one may expect for FEL's in general to have very narrow intrinsic linewidths because of the very narrow cavity linewidth $\Delta\nu_{1/2}$ and the large generated power P_{gen} which are typical to these devices. Since single-mode operation of FEL's seems to be realizable, one could expect such devices to operate with a very high degree of coherence. In practice, however, one will have to consider the limitations imposed by technical noise and spectral measurements capabilities (in particular, the length of measurement time required—see Appendix C).

The thermal radiation emission noise source was explicitly included in our analysis of both two-level lasers and FEL's. While in two-level lasers (masers) the thermal noise may dominate the linewidth formula in the long-wavelength limit $\hbar\omega \ll kT$, we concluded that for FEL's the thermal emission contribution would be negligible. Even though in the long-wavelength limit (as in the experiment) there is multiphoton thermal population of each cavity mode, the number of stimulated photons emitted at FEL saturation by each electron exceeds this number considerably ($\xi/\hbar\omega \gg kT/\hbar\omega$), making the thermal term in (23) negligible for all practical cases.

One may argue that our use of the FEL small-signal gain formula in the second part of (21) is not consistent with our other assumption of laser operation at saturation conditions. Similar questions may be asked about the validity, at saturation of the extended Einstein relations which were used in deriving the Schawlow-Townes-like formula (23). These difficulties are the source for the one order of magnitude discrepancy between the values obtained for $\Delta\nu_{\text{laser}}$ using the two different formulas [(21) and (25)]. The difference occurs in the present example because the small signal FEL gain exceeds the cavity losses, so that Eq. (22) cannot be used with the gain derived from the extended Einstein relation. Consequently the lower value derived from (25) is expected to be less accurate than the first estimate. One would expect that the two formulas would provide mutually consistent and correct numerical estimates of the genuine intrinsic linewidth when the FEL oscillator is designed to operate near oscillation threshold conditions.

As mentioned in the Introduction, additional effects, like coupling between amplitude and phase fluctuations and other elaborations, may modify the results of the present FEL linewidth simple model derivation, presumably to cause further broadening. While there is certainly room for further elaborations, it is suggested that the present results provide a useful lower bound and a correct order of magnitude estimate of the FEL oscillator intrinsic linewidth.

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APPENDIX A: EINSTEIN RELATIONS FOR A TWO-LEVEL LASER

It is straightforward to derive from basic quantum electrodynamical principles the relations between spontaneous emission, thermal noise generation, and stimulated emission. It might be noted that the original Einstein coefficients relations[28] were derived from a quite different point of view (preceding the development of modern quantum electrodynamics) and are somewhat less general.

The photon growth rate of a single cavity mode in a two-quantum-level laser is

$$\frac{d\nu_q}{dt}=\Gamma_{sp}\left[\left[\nu_q+\frac{1}{e^{\hbar\omega/kT-1}}\right](N_2-N_1)+N_2\right]=\left[\frac{d\nu_q}{dt}\right]_{st}+\left[\frac{d\nu_q}{dt}\right]_{th}+\left[\frac{d\nu_q}{dt}\right]_{sp}, \tag{A1}$$

where ν_q is the number of photon is the cold cavity mode, $(e^{\hbar\omega/kT}-1)^{-1}$ is the Bose-Einstein statistics thermal equilibrium photon occupation number of the mode when the cavity walls are at temperature T, and N_2, N_1 are the population of the upper and lower quantum levels, respectively. The three terms contributing to the growth of photons in the cavity are the stimulated emission due to amplification of the coherent circulating power, the stimulated emission due to amplification of the incoherent thermally emitted photons and the spontaneous emission.

The photon number in the cavity mode ν_q can be related to the coherent circulating power in the cavity

$$P_q=\tfrac{1}{2}\hbar\omega\nu_q\frac{c}{l}\ . \tag{A2}$$

Using this relation and the proportionality between the photon emission rate and the power generation, the following relations are exposed:

$$(\Delta P_q)_{st}:(\Delta P_q)_{th}:(P_q)_{sp}=\frac{P_q}{\frac{1}{2}\hbar\omega c/l}:\frac{1}{e^{\hbar\omega/kT}-1}:\mu\ , \tag{A3}$$

where $\mu\equiv N_2/(N_2-N_1)$. Due to their proportionality, the incremental power parameters in (A3) can be interpreted either as the *cavity mode* energy growth rate or the *unidirectional* power growth along the laser. These relations can also be written in the form

$$(P_q)_{sp}=\frac{(\Delta P_q)_{st}}{P_q}\frac{1}{2}\hbar\omega\frac{c}{l}\mu\ , \tag{A4}$$

$$(\Delta P_q)_{th}=\frac{(\Delta P_q)_{st}}{P_q}\frac{1}{2}\hbar\omega\frac{c}{l}\frac{1}{e^{\hbar\omega/kT}-1}\ . \tag{A5}$$

The spectral powers may be obtained by dividing into the cavity intermode frequency spacing $\Delta\omega=\pi c/l$ [in consistence with the cavity mode quantization assumed in (A2)]. Marking the single path incremental power parameters with subscripts t, and defining the single path gain $G-1\equiv(\Delta P_{q_t})_{sp}/P_q$, one obtains

$$(P_{q_t\omega})_{sp}=(G-1)\frac{\hbar\omega}{2\pi}\mu\ , \tag{A6}$$

$$(\Delta P_{q_t\omega})_{th}=(G-1)\frac{\hbar\omega}{2\pi}(e^{\hbar\omega/kT}-1)^{-1}\ . \tag{A7}$$

APPENDIX B: EXTENDED EINSTEIN RELATIONS FOR QUASI-FREE-ELECTRON RADIATION

The extended Einstein relations can be derived entirely classically. Here we show their derivation starting from an expression for the photon growth rate which can be derived by taking the electron-classical limit (negligible recoil) of the quantum-mechanical photon growth rate of the FEL.[34] The more general expression we cite here[10] applies to a large number of quasi-free-electron radiation devices (like Čerenkov-Smith-Purcell radiation, cyclotron resonance maser, etc.), and not only to the bremsstrahlung FEL. It also includes the thermal noise emission term, and applies in general to any cavity single radiation mode of uniform cross section along the interaction region,

$$\frac{d\nu_q}{dt}=\Gamma_{sp}\left\{\mathrm{sinc}^2(\bar\theta/2)+[\tfrac{1}{2}+(e^{\hbar\omega/kT}-1)^{-1}+\nu_q]\epsilon\frac{d}{d\bar\theta}\mathrm{sinc}^2(\bar\theta/2)\right\}=\left[\frac{d\nu_q}{dt}\right]_{sp}+\left[\frac{d\nu_q}{dt}\right]_{th}+\left[\frac{d\nu_q}{dt}\right]_{st}, \tag{B1}$$

where $\mathrm{sinc}(x)\equiv\sin(x)/x$, $\bar\theta$ is the normalized detuning parameter measuring the phase lag of the monoenergetic electron beam relative to the interacting longitudinal wave, and ϵ is the quantum recoil parameter.

For a bremsstrahlung FEL,

$$\bar\theta=\left[\frac{\omega}{v_z}-k_z-k_w\right]L_w\ , \tag{B2}$$

$$\epsilon=\frac{\hbar}{m\gamma\gamma_z^2v_z}\left[\frac{\omega}{v_z}\right]^2L_w\ , \tag{B3}$$

where v_z is the average axial velocity of the electron in the wiggler, $k_w=2\pi/\lambda_w$ is the wiggler wave number, L_w is the length of the wiggler, k_z is the axial wave number of the radiation mode and $\gamma\equiv(1-\beta^2)^{-1/2}$, $\gamma_z\equiv(1-\beta_z^2)^{-1/2}$. The classical limit condition $\epsilon\ll 2\pi$ is usually satisfied for any practical FEL parameters.

The three terms identified in (B1) are the spontaneous emission, the stimulated emission of amplified thermal radiation noise and the stimulated emission of amplified cavity-circulating coherent radiation. The term $\frac{1}{2}$ in the square brackets is included for convenience inside the thermal radiation emission term, however, it is actually a descendent of the quantum-mechanical spontaneous emission. It is the only pure quantum-electrodynamical term which remains in the electron-classical limit $\epsilon\ll 2\pi$ (as long as the field quantization is still kept).

Using the proportionality relation between the photon emission rate and the power generation, the following relations can be deduced from (B1):

$$(\Delta P_q)_{st}:(\Delta P_q)_{th}:(P_q)_{sp}=\nu_q\epsilon:[\tfrac{1}{2}+(e^{\hbar\omega/kT}-1)^{-1}]\epsilon:\text{sinc}^2(\bar{\theta}/2)\Big/\frac{d}{d\bar{\theta}}\text{sinc}^2(\bar{\theta}/2)\ . \tag{B4}$$

These relations can also be written in the form

$$(P_q)_{sp}=\frac{(\Delta P_q)_{st}}{P_q}\frac{\beta_z^3\gamma_z^2\gamma mc^2}{4\pi}\frac{\lambda}{L_w}\frac{c}{l}\frac{\text{sinc}^2(\bar{\theta}/2)}{\frac{d}{d\bar{\theta}}\text{sinc}^2(\bar{\theta}/2)}\ , \tag{B5}$$

$$(\Delta P_q)_{th}=\frac{(\Delta P_q)_{st}}{P_q}[\tfrac{1}{2}+(e^{\hbar\omega/kT}-1)^{-1}]\tfrac{1}{2}\hbar\omega\frac{c}{l}\ . \tag{B6}$$

The spectral powers can be obtained now by dividing into the cavity intermode frequency spacing $\Delta\omega=\pi c/l$. Also for simplicity we assume the practical case assumption that the oscillating mode operates near the maximum gain detuning point $\bar{\theta}=\bar{\theta}_m=-2.6$ for which

$$\text{sinc}^2(\bar{\theta}/2)/\frac{d}{d\bar{\theta}}\text{sinc}^2(\bar{\theta}/2)\approx 2\ . \tag{B7}$$

We then obtain for the unidirectional incremental power parameters

$$(P_{q\omega})_{sp}=(G-1)\frac{\beta_z^2\gamma_z^2\gamma mc^2}{2\pi^2}\frac{\lambda}{L_\omega}\ , \tag{B8}$$

$$(\Delta P_{q\omega})_{th}=(G-1)[\tfrac{1}{2}+(e^{\hbar\omega/kT}-1)^{-1}]\frac{\hbar\omega}{2\pi}\ . \tag{B9}$$

Equations (B8) and (B9) will be referred to as the extended Einstein relations.

APPENDIX C: NUMBER OF PHOTONS EMITTED SPONTANEOUSLY INTO A SINGLE MODE BY AN ELECTRON

To calculate the *average* number of photons emitted by a single electron into a single cavity mode, we multiply the spectral power per mode [(15) and (18)] by the longitudinal mode spacing $\Delta\omega=\pi c/l$, (7), and divide by the electron's injection rate I_0/e and the photon energy $\hbar\omega$. This results in

$$\nu_{sp}=\frac{1}{4}\alpha\frac{\bar{a}_w^2}{1+\bar{a}_w^2}N_w\frac{\lambda^2}{A_{em}}\frac{L_w}{l}\text{sinc}^2(\bar{\theta}/2)\ , \tag{C1}$$

where $\alpha\equiv e^2/(4\pi\varepsilon_0 mc^2)=\frac{1}{137}$ is the fine-structure constant (expressed in mks units).

It is interesting to compare this expression with the expression for the total average number of spontaneously emitted photons by an undulating electron in unbound space (which can be directly computed from the angle integrated dipole antenna radiation formulas[37])

$$(\nu_{tot})_{\substack{\text{free}\\ \text{space}}}=\frac{4\pi}{3}\alpha\frac{\bar{a}_w^2}{1+\bar{a}_w^2}N_w\ . \tag{C2}$$

In the over-moded waveguide limit Eq. (C1) may be interpreted as the total average number of emitted photons multiplied by "filling factors" corresponding to the solid angle subtended by a single transverse mode and the frequency regime section subtended by a single longitudinal mode in the spontaneous emission bandwidth. Note, however, that when the waveguide is not overmoded, the total average number of photons is not given any more by (C2).

Inspection of Eq. (C1) indicates that the average number of photons emitted into a single cavity mode per electron is for most practical cases smaller than one, though multiphoton emission is conceivable with very long wigglers. For the parameters of the experiment of Ref. 6 (Table I) at $\bar{\theta}=-2.6$ we obtain

$$\nu_{sp}=1.3\times 10^{-6}\ .$$

This is a very small number, and obviously refers only to the *probability* of photon emission.

The adequate quantum field description of the radiation from a classical current like an undulating electron, is the Glauber coherence state. This corresponds to a Poissonian quantum statistical distribution of the photon number as a random variable.[17] This means that if one measures a system parameter (like the linewidth) which depends on the average number of photon emission, and the measurement is carried out with a small number of electron radiators, there will be a great uncertainty in the measurement. Furthermore, since the number of emitting electrons is also a random (Poisson statistics) variable, the uncertainty in the number of photons (and the measured system parameter) is even larger, corresponding to thermal statistics of the total photon number.[18] This may introduce greater uncertainty in the value of the measured parameter, but still keeps its average value (over an ensemble of experiments) the same. Note that for measuring linewidth, a very long measuring time—at least the reciprocal of the oscillator linewidth $(\Delta\nu_{laser})^{-1}$—is necessary. During such a time the total average number of spontaneous emission photons is equal to the average number of photons stored in the cavity $\varepsilon_c/\hbar\omega$ [as we noted in the interpretation of Eq. (6) in Sec. II]. This is in practice always a very large number.

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[1]J. P. Gordon, H. J. Zeiger, and C. H. Townes, Phys. Rev. **99**, 1264 (1955).
[2]A. L. Schawlow and C. H. Townes, Phys. Rev. **112**, 1490 (1958).
[3]A. Yariv, *Optical Electronics*, 3rd ed. (Holt, Reinhart, and Winston, New York, 1985).
[4]A. Mooradian, Phys. Today **38**(5), 43 (1985).
[5]L. R. Elias, W. Fairbank, J. Madey, H. A. Schwettman, and T. Smith, Phys. Rev. Lett. **36**, 717 (1976).
[6]L. R. Elias, G. Ramian, J. Hu, and A. Amir, Phys. Rev. Lett. **57**, 424 (1986).
[7]W. Becker, J. Gea-Banacloche, and M. O. Scully, Phys. Rev. A

33, 2174 (1986).
[8]M. Sargent III, M. O. Scully, and W. E. Lamb, *Laser Physics* (Addison-Wesley, Reading, Mass., 1976).
[9]H. Motz, J. Appl. Phys. **22**, 527 (1951).
[10]A. Gover, A. Friedman, G. Kuritzki, S. Ruschin, and A. Yariv (unpublished).
[11]M. Lax, Phys. Rev. **160**, 290 (1967).
[12]R. D. Hempstead and M. Lax, Phys. Rev. **161**, 350 (1967).
[13]M. Lax and W. H. Louisell, J. Quantum Electron. **3**, 47 (1967).
[14]D. Welford and A. Mooradian, Appl. Phys. Lett. **40**, 56 (1982).
[15]C. H. Henry, IEEE J. Quantum Electron. **18**, 259 (1982).
[16]K. Vahala and A. Yariv, IEEE J. Quantum Electron. **19**, 1096 (1983).
[17]D. Marcuse, *Principles of Quantum Electronics* (Academic, New York, 1980), p. 254.
[18]W. Becker and J. K. McIver, Phys. Rev. A **28**, 1838 (1983); J. C. Banacloche, *ibid.* **31**, 1607 (1985).
[19]D. Marcuse, IEEE J. Quantum Electron. **19**, 1228 (1983).
[20]R. G. Halet, E. S. Hilfer, and D. Kleppner, Phys. Rev. Lett. **65**, 2137 (1985).
[21]H. Haus and M. Islam, J. Appl. Phys. **54**, 4784 (1983).
[22]A. Amir, I. Boscolo, and L. R. Elias, Phys. Rev. A 32, 2864 (1985).
[23]A. Gover, A. Friedman, and I. Shraga (unpublished).
[24]A. Gover, H. Freund, V. L. Granatstein, J. H. McAdoo, and C. M. Tang, *Infrared and Millimeter Waves,* edited by K. Button (Academic, New York, 1984), Vol. 11, p. 291.
[25]See, for example, B. M. Kincaid, J. Appl. Phys. **48**, 2684 (1977); W. B. Colson, Ph.D. thesis (Stanford University); H. Freund, Phys. Rev. A **24**, 1967 (1979).
[26]See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1971).
[27]J. M. J. Madey, Nuovo Cimento **50B**, 64 (1979).
[28]A. Einstein, Phys. Z **18**, 121 (1917).
[29]C. Townes (private communication).
[30]A. Gover and A. Yariv, Appl. Phys. **16**, 121 (1978).
[31]A. Gover and P. Sprangle, IEEE Trans. Quantum Electron. **17**, 1196 (1981).
[32]P. Sprangle and R. Smith, Phys. Rev. A **21**, 293 (1980).
[33]J. M. Madey, J. Appl. Phys. **42**, 1906 (1971).
[34]G. Dattoli and A. Renieri, J. Phys. (Paris) Colloq. Suppl. 2, **44**, C1-125 (1983).
[35]G. Bekefi, *Radiation Processes in Plasma* (Wiley, New York, 1966).
[36]R. C. Davidson and W. A. McMullin, Phys. Rev. A **26**, 1997 (1982).
[37]A. Gover, *Free-Electron Generators of Extreme Ultraviolet Coherent Radiation,* AIP Conf. Proc. No. 118, edited by J. M. J. Madey and C. Pellegrini (AIP, New York, 1984), p. 144.