Three-dimensional theory of free-electron lasers in the collective regime

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(Received 20 October 1986)

We present an analysis of a free-electron laser within a three-dimensional model, in which the finite transverse dimensions of the electron beam are taken into account together with the confinement of the electromagnetic modes by a guiding structure. The analysis is based on the expansion of the beam's self-potential into an infinite set of modes that interact simultaneously with the wiggler and signal fields. Explicit gain-dispersion equations are developed for several cases including a waveguide tube completely filled by an electron beam of uniform density, and a uniform-density electron beam partially filling a waveguide (or in free space). This is carried out in both the magnetized beam limit and the general case which includes surface currents. The results bear significant effect on free-electron-laser gain operating parameters in the collective regimes and on the threshold of absolute instability oscillation.

I. INTRODUCTION

The main features of the magnetic bremsstrahlung free-electron laser (FEL) may be well described by a one-dimensional model. The finite transverse dimensions of the electron beam in a practical embodiment are usually taken into account by means of a "filling factor" equal to the cross-sectional areas ratio of the *e* beam relative to the electromagnetic wave. The coupling between the radiation wave and the beam is assumed to be reduced by this factor relative to the coupling in the ideal one-dimensional transversely infinite FEL model.[1]

Recently there has been substantial interest in and numerous publications on three-dimensional analysis of FEL. Various aspects of three-dimensionality have been considered; most of them are out of the scope of the present paper. The main three-dimensional aspect considered so far is the radiation field diffraction. The generated stimulated-emission radiation is emitted along the FEL interaction length from a source (or "aperture") equal to the dimensions of the electron beam. If this beam is narrower than the input electromagnetic wave, then the generated radiation diffracts differently (more) than the input radiation wave, and a one-dimensional model will not describe the electromagnetic wave properly (in particular, in the high-gain regime).[2]

A significant simplification of the three-dimensional FEL problem is possible in a system which is translationally invariant and transversely confined. In this case it is possible to expand the electromagnetic wave in terms of axially uniform transverse modes. The frequency-domain Maxwell equations can then be reduced into one-dimensional linear equations (in terms of the *z*-axial coordinate) with constant coefficients.[3–5]

Such a model of axially uniform radiation modes is well suited to describe a waveguide FEL, but also is a good approximation for free-space mode propagation when the interaction region is shorter than a Rayleigh length. This model allows derivation of dispersion relations for the individual modes which are valid not only in the low-gain regime (where a diffracting mode model can be used[6]) but also in the high-gain regimes.

The dispersion equations of the radiation modes are in general coupled to each other, but in most cases the coupling is neglected and, furthermore, it is assumed that only one transverse radiation mode propagates in the system. The single-mode dispersion equation is similar to the one-dimensional model equation and the three-dimensional effects are taken into consideration only through a cross-sectional overlap integral of the radiation mode with the *e*-beam current and wiggler field, which modifies appropriately the one-dimensional FEL coupling coefficient.[7] We note that unless the FEL gain is exceedingly high, the single-mode approximation is quite practical because it is possible to control quite well the laser transverse modes. In an amplifier this is done by inserting an appropriate field distribution, and, in an oscillator, by the resonator mirrors' apertures. If the laser is based on a waveguide, the transmission properties of the waveguide (cutoff, mode selective loses) can be used to control the FEL transverse modes.

While three-dimensional *radiation* effects are usually taken care of by mode expansion of the radiation field or by more elaborate models, little attention was paid so far to the three-dimensional aspects of the electron-beam plasma-wave propagation. These aspects are expected to be significant only in collective (Raman) regime FEL's[8] where the radiation wave exchanges energy predominantly with the collective plasma waves of the beam. In such FEL's *e*-beam three-dimensional effects are expected to be significant because the narrow *e* beam cannot be modeled by a transversely infinite plasma, and transverse boundary conditions have a substantial effect. Furthermore, the finite beam plasma is known to have an infinite number of modes and contrary to the radiation modes their external control is virtually impossible. Consequently, a true three-dimensional analysis of the FEL plasma waves is an important problem to address. This interest is also derived by recent experimental progress in high-current long-wiggler FEL's and in particular the demonstration of high-gain induction-linac-driven FEL's[9] and proposals for high-current long-wiggler storage-ring-driven x-ray and uv FEL's.[10] Due to the feasibility of high-current long-wiggler accelerators, collective effects may appear even in FEL's operating at short (optical) wavelengths.

The goal of the present work is to investigate the e-beam plasma three-dimensional (3D) effects on the FEL by means of analytical techniques. We find it again helpful to use a model of translationally invariant electron-beam and electromagnetic wave structures. This allows us to obtain relatively simple analytical expressions for the dispersion relation which show similarity to the well-investigated[11] one-dimensional dispersion relation, and reduce to it in the appropriate limits. The analytical technique is based on expansion of the electron-beam current in terms of the finite-beam infinite set of plasma modes. The FEL gain-dispersion relation is obtained with all the modes included. Only in illustrative examples do we consider the special case of coupling to a single transverse plasma mode where the dispersion equation simplifies to a one-dimensional-like form. In this case, similarly to the radiation mode expansion problem, the operating parameters' definitions include overlap integrals with the transverse profile of the particular transverse plasma mode.

Previous work on e-beam three-dimensional effects was reported recently for specific cases of a helical wiggler FEL with a cylindrical waveguide[12,13] and a linear wiggler[14] (though without consideration of transverse boundary conditions for the beam fields). In the present work the analysis is carried out in the framework of a general model which permits arbitrary transverse geometry of the waveguide (slab and cylindrical waveguides are considered as illustrative examples). The general parametric interaction model used[15] admits both static and electromagnetic wigglers (pumps) and an additional axial guide field. The wiggler and signal electromagnetic modes may have arbitrary profile and polarization. We avoid taking the extreme relativistic limit $\gamma \gg 1$ since many experimentally investigated collective FEL's are only moderately relativistic.

As mentioned before, the thrust of this paper is the e-beam plasma-wave-related 3D effects, and many other 3D effects of the FEL problem are not treated in the present model. In particular, electron-beam energy and angular spread are ignored to permit the use of a laminar flow fluid charge model for the beam equations. The interesting and timely 3D problem of optical guiding by the FEL e beam[16] is not considered here, though it turns out to be mathematically related. The e-beam guiding effect results from substantial modification of the radiation modes (confinement) due to strong interaction with the e beam in the high-gain regime. In the present model we assume that the transverse profile of the radiation modes are determined predominantly by the external structure (waveguide) of free-space propagation. Solving for possible e-beam-guided radiation modes in the collective regime FEL would require further development of the present model.

As in any self-consistent theory of the FEL problem, our work here will be based on the coupling of two kinds of equations: (a) the field equations, which describe the excitation of the waveguide electromagnetic modes by a given current source, and (b) the e-beam equation which describes the excitation of the charge-density modulation on the e beam by the electromagnetic modes of the waveguide. The first equation was treated in full generality by many authors[3,5] and will be introduced in Sec. II. In Sec. III we develop the excitation equation of cold-electron-beam self-fields driven by the electromagnetic waveguide modes. These self-fields submitted to the boundary conditions on the waveguide walls are expressed in terms of an *infinite set of beam modes* which are determined by the homogeneous solutions of the same equation. A simultaneous solution of these two equations yields the FEL's gain-dispersion relation expressed in terms of multimode parametric interaction between a waveguide electromagnetic mode and the beam modes. These equations are derived in Sec. IV.

II. THE FIELD EQUATION

We mark the wiggler (pump) and signal fields by indexes w and s, respectively. The signal radiation field propagates in the same direction as the e beam $(+z)$, and the wiggler field propagates in counter direction to the e beam $(-z)$. The fields are

$$\begin{aligned} \mathbf{E}_s(\mathbf{r},t) &= \mathrm{Re}[C_s(z)\tilde{\mathcal{E}}_s(x,y)e^{ik_s z - i\omega_s t}] , \\ \mathbf{B}_s(\mathbf{r},t) &= \mathrm{Re}[C_s(z)\tilde{\mathcal{B}}_s(x,y)e^{ik_s z - i\omega_s t}] , \\ \mathbf{E}_w(\mathbf{r},t) &= \mathrm{Re}[C_w\tilde{\mathcal{E}}_w(x,y)e^{-ik_w z - i\omega_w t}] , \\ \mathbf{B}_w(\mathbf{r},t) &= \mathrm{Re}[C_w\tilde{\mathcal{B}}_w(x,y)e^{-ik_w z - i\omega_w t}] , \end{aligned} \tag{1}$$

where $\{\tilde{\mathcal{E}}_{s,w}\tilde{\mathcal{B}}_{s,w}\}e^{\pm ik_{s,w}z - i\omega_{s,w}t}$ are the fields of the empty waveguide modes, $C_{s,w}(z)$ are the amplitudes of the waves in the presence of the beam, and C_w is assumed to be constant (nondepleted pump approximation). In the common case of a magnetostatic wiggler $\omega_w = 0 = \tilde{\mathcal{E}}_w$, and the assumption $C_w = \text{const}$ is not an approximation. The excitation equation for the amplified-wave amplitude $C_s(z)$ is derived directly from Maxwell equations:[4,5]

$$\frac{d}{dz}C_s(z) = -\frac{1}{4|\mathcal{P}_s|}e^{-ik_s z} \times \int\int \tilde{\mathbf{J}}_{Ts}(x,y,z)\cdot\tilde{\mathcal{E}}^*_{Ts}(x,y)dx\,dy , \tag{2}$$

where $\tilde{\mathbf{J}}_{Ts}(x,y,z)$ is the current component at the frequency of the signal wave (ω_s) and T denotes the transverse component. Here we assumed that the excited current at frequency ω_s is purely transverse, which is an exact statement in the normal case of a transverse static wiggler and when the signal wave is TE or TEM. We exclude here the case of a longitudinal wiggler[17] and neglect the longitudinal modulation in the case of TM signal and wiggler modes. This is valid as long as the longitudinal electric field is small compared to the transverse one, which is the case far from cutoff, i.e., when $\omega_{s,w}/C \gg K_c$ where K_c is the cutoff wave number of the waveguide. $\mathcal{P}_s$ is the normalization power of the electromagnetic (e.m.) signal:

$$\mathcal{P}_s=\tfrac{1}{2}\mathrm{Re}\int\int(\mathcal{E}_s\times\mathcal{H}_s^*)\cdot\hat{\mathbf{i}}_z\,dx\,dy\ . \tag{3}$$

We define now the beat frequency $\omega_i=\omega_s-\omega_w$, and assume the presence of a density (idler) wave at the beat frequency

$$n_1=\mathrm{Re}[\tilde{n}_1(\mathbf{r})e^{-i\omega_i t}]\ . \tag{4}$$

The first-order response of the transverse electron velocities to the electromagnetic fields is

$$\mathbf{V}_T(\mathbf{r},t)=\tfrac{1}{2}[\tilde{\mathbf{V}}_{Ts}(\mathbf{r})e^{-i\omega_s t}+\tilde{\mathbf{V}}_{Tw}(\mathbf{r})e^{-i\omega_w t}]+\text{c.c.} \tag{5}$$

Thus the transverse current components are obtained by substituting Eqs. (4) and (5) into the expression

$$\mathbf{J}_T=-en_1(\mathbf{r},t)\mathbf{V}_T(\mathbf{r},t)\ .$$

Neglecting terms which are not oscillating with frequency ω_s, one finds that

$$\tilde{\mathbf{J}}_{Ts}=-\tfrac{1}{2}e\tilde{n}_1\tilde{\mathbf{V}}_{Tw}^*\ , \tag{6}$$

where $\tilde{n}_1$ oscillates with frequency ω_i and $\tilde{\mathbf{V}}_{Tw}$ is a phasor of the velocity field vector with frequency ω_w and wave number $-k_w$. For the sake of simplicity, in the proceeding analysis we express $\tilde{\mathbf{V}}_{Tw}$ in terms of the wiggler amplitude C_w and the transverse wiggler velocity profile $\mathcal{V}_{Tw}(x,y)$:

$$\tilde{\mathbf{V}}_{Tw}(x,y,z)=C_w e^{-ik_w z}\mathcal{V}_{Tw}(x,y)\ . \tag{7}$$

Analytic expressions for the first-order transverse velocities of the electrons were developed in many other papers for various configurations (Refs. 14 and 15); thus we shall not discuss them here, beyond listing the expressions for $\mathcal{V}_{Tw},\mathcal{V}_{Ts}$ in Table I. The parameters of Table I are specified for the particular example of linearly polarized fields (though the analysis so far admits arbitrary field polarization). The given parameters also include axial guide magnetic field effects.

Substituting Eqs. (6) and (7) in Eq. (2), and Laplace-transforming the resulting equation in the z direction, results in

$$s\overline{C}_s(s)-C_s(0)=\frac{C_w}{8\,|\mathcal{P}_s|}\int\int\overline{n}_1(x,y,s+iK)\times\mathcal{V}_{Tw}(x,y)\cdot\mathcal{E}_{Ts}^*(x,y)\,dx\,dy\ , \tag{8}$$

where $K=k_s+k_w$ and

$$\overline{A}(s)\equiv\int_{-\infty}^{\infty}e^{-sz}A(z)dz\ .$$

Eq. (8) is our excitation equation of the e.m. signal field in the waveguide. We turn now to develop the second equation, which describes the excitation of the density modulation n_1.

III. THE BEAM EQUATION

In this section we develop the beam self-field equation; this equation, submitted to the appropriate boundary conditions, yields the beam self-fields, expressed as an infinite sum of the *beam modes*. The beam first-order density modulation n_1 (4) will be determined directly from the

TABLE I. First-order transverse velocities of the electrons in the beam due to the electromagnetic modes of the waveguide. (a) Beams without guiding magnetic field, (b) beams confined by an axial magnetic field; these expressions are valid far from the cyclotron resonance (see Ref. 3). Only components with frequency $\omega_{s,w}$ and wave number $k_{s,w}$ are given.

	Signal	Wiggler
	(a)	
Fields	$\mathbf{E}_s(\mathbf{r},t)=\mathrm{Re}[C_s(z)\mathcal{E}_s(\mathbf{r}_T)e^{ik_sz-i\omega_st}]$	$\mathbf{E}_w(\mathbf{r},t)=\mathrm{Re}[C_w\mathcal{E}_w(\mathbf{r}_T)e^{-ik_wz-i\omega_wt}]$
	$\mathbf{B}_s(\mathbf{r},t)=\mathrm{Re}[C_s(z)\mathcal{B}_s(\mathbf{r}_T)e^{ik_sz-i\omega_st}]$	$\mathbf{B}_w(\mathbf{r},t)=\mathrm{Re}[C_w\mathcal{B}_w(\mathbf{r}_T)e^{-ik_wz-i\omega_wt}]$
$\tilde{\mathbf{V}}_{Ts,w}$	$\tilde{\mathbf{V}}_{Ts}=C_s(z)e^{ik_sz}\mathcal{V}_{Ts}$	$\tilde{\mathbf{V}}_{Tw}=C_we^{-ik_wz}\mathcal{V}_{Tw}$
	$\mathcal{V}_{Ts}=-\dfrac{ie/\gamma_0 m}{\omega_s-k_sV_{0z}}[\mathcal{E}_{Ts}(\mathbf{r}_T)+\hat{\mathbf{i}}_zV_{0z}\times\mathcal{B}_{Ts}(\mathbf{r}_T)]$	$\mathcal{V}_{Tw}=-\dfrac{ie/\gamma_{0m}}{\omega_w+k_wV_{0z}}[\mathcal{E}_{Tw}(\mathbf{r}_T)+\hat{\mathbf{i}}_zV_{0z}\times\mathcal{B}_{Tw}(\mathbf{r}_T)]$
	(b)	
Fields	$\mathbf{E}_s(\mathbf{r},t)=\mathrm{Re}[C_s(z)\mathcal{E}_s(\mathbf{r}_T)e^{ik_sz-i\omega_st}]$	$\mathbf{E}_w=0$
	$\mathcal{E}_s(\mathbf{r}_T)=\hat{\mathbf{i}}_y\tilde{\mathcal{E}}_s(\mathbf{r}_T)$	
	$\mathbf{B}_s(\mathbf{r},t)=\mathrm{Re}[C_s(z)\mathcal{B}(\mathbf{r}_T)e^{ik_sz-i\omega_st}]$	$\mathbf{B}=\mathrm{Re}(C_w\hat{\mathbf{i}}_x\mathcal{B}_we^{-ik_wz})+\hat{\mathbf{i}}_zB_\parallel$
	$\mathcal{B}_s(\mathbf{r}_T)=-\hat{\mathbf{i}}_x\dfrac{\omega_s}{c^2k_s}\tilde{\mathcal{E}}_s(\mathbf{r}_T)$	
$\tilde{\mathbf{V}}_{Ts,w}$	$\tilde{\mathbf{V}}_{Ts}=C_s(z)e^{ik_sz}\mathcal{V}_{Ts}$	$\tilde{\mathbf{V}}_{Tw}=C_we^{ik_wz}\mathcal{V}_{Tw}$
	$\mathcal{V}_{Ts}=-\hat{\mathbf{i}}_x\dfrac{\Omega_\parallel/\Omega}{1-(\Omega_\parallel/\Omega)^2}\dfrac{e}{\gamma_0m\omega_s}\tilde{\mathcal{E}}_s(\mathbf{r}_T)$	$\mathcal{V}_{Tw}=\hat{\mathbf{i}}_x\dfrac{\Omega\Omega_\parallel V_{0z}}{\Omega_\parallel^2-(k_wV_{0z})^2}$
	$-\hat{\mathbf{i}}_y\dfrac{i}{1-(\Omega_\parallel/\Omega)^2}\dfrac{e}{\gamma_0m\omega_s}\tilde{\mathcal{E}}_s(\mathbf{r}_T)$	$+\hat{\mathbf{i}}_yi\dfrac{\Omega k_wV_{0z}^2}{\Omega_\parallel^2-(k_wV_{0z})^2}$
	$\Omega_\parallel=eB_\parallel/\gamma_0m\ ;\ \Omega=eB_w/\gamma_0m$	

beam self-fields, thus taking into consideration the influence of the waveguide walls and the finiteness of the beam on the excitation of the density wave n_1.

Two assumptions are made throughout the deviations: (a) Transverse variations of the beam modes are small compared to the longitudinal ones: $\partial/\partial x$, $\partial/\partial y \ll \partial/\partial z$. (b) The electromagnetic waveguide modes are asynchronous with the electrons or with the plasma modes. Only the ponderomotive wave (which is the result of the beat between the wiggler and signal fields through the nonlinear Lorentz force) is synchronous with the beam or with one of its modes, and can interact with them.

The total beam electron density $n(\mathbf{r},t)$ is given by

$$n(\mathbf{r},t)=n_0(\mathbf{r},t)+n_1(\mathbf{r},t),\quad n_1 \ll n_0 \tag{9}$$

where $n(\mathbf{r},t)$ is the total electron density, $\mathbf{r}_T=(x,y)$ is the transverse coordinate, and $\mathbf{r}=x,y,z$ is a 3D vector. Also, we define the total velocity field as

$$\mathbf{V}=V_{0z}\hat{\mathbf{i}}_z+V_{1z}\hat{\mathbf{i}}_z+\mathbf{V}_T,\quad |V_{1z}|,|\mathbf{V}_T| \ll V_{0z}\ . \tag{10}$$

V_{0z} is the unperturbed beam velocity; V_{1z} and $\mathbf{V}_T$ are the perturbed longitudinal and transverse beam velocities, respectively.

We concentrate our attention on the perturbed quantities which oscillate with frequency $\omega_i=\omega_s-\omega_w$; thus

$$V_{1z}(\mathbf{r},t)=\mathrm{Re}[\tilde{V}_{1z}(\mathbf{r})e^{-i\omega_i t}]\ , \tag{11}$$

$$n_1(\mathbf{r},t)=\mathrm{Re}[\tilde{n}_1(\mathbf{r})e^{-i\omega_i t}]\ . \tag{12}$$

The transverse velocity field $\mathbf{V}_T$ is composed of three phasor components: the first oscillates with frequency ω_i and arises due to the transverse components of the beam self-fields; the other two components oscillate with frequencies ω_s,ω_w and arise due to the transverse fields of the waveguide modes. Thus

$$\mathbf{V}_T(\mathbf{r},t)=\mathrm{Re}[\tilde{\mathbf{V}}_T^{\omega_i}(\mathbf{r})e^{-i\omega_i t}+C_w\boldsymbol{\mathcal{V}}_{Tw}(\mathbf{r}_T)e^{-ik_w z-i\omega_w t}+C_s(z)\boldsymbol{\mathcal{V}}_{Ts}(\mathbf{r}_T)e^{ik_s z-i\omega_s t}]\ . \tag{13}$$

Expressions for $\boldsymbol{\mathcal{V}}_{Ts},\boldsymbol{\mathcal{V}}_{Tw}$ are summarized in Table I.

We start our analysis from the small-signal fluid plasma equations:

$$\nabla\cdot\mathbf{J}=\frac{\partial}{\partial t}n_1 e\ , \tag{14}$$

$$\mathbf{J}\simeq -e(V_{1z}n_0+V_{0z}n_1)\hat{\mathbf{i}}_z-en_0\mathbf{V}_T\ , \tag{15}$$

$$\left[\frac{\partial}{\partial t}+V_{0z}\frac{\partial}{\partial z}\right]V_{1z}=-\frac{e}{\gamma_0\gamma_z^2 m}[E_z+(\mathbf{V}\times\mathbf{B})_z^{\omega_i}]\ , \tag{16}$$

where E_z represents the longitudinal self-electric-fields of the electron beam which oscillates with frequency ω_i and $(\mathbf{V}\times\mathbf{B})_z^{\omega_i}$ is the well-known ponderomotive force which oscillates with frequency ω_i.

Using the phasor notation of Eqs. (11)–(13) and Laplace-transforming Eqs. (14)–(16) we find

$$\bar{V}_{1z}=-\frac{e}{V_{0z}s-i\omega_i}\frac{1}{\gamma_0\gamma_z^2 m}(\bar{E}_{\text{pond}}^{\omega_i}+\bar{E}_z)\ , \tag{17a}$$

$$n_0 s\bar{V}_{1z}+V_{0z}s\bar{n}_1+\nabla_T\cdot(n_0\bar{\mathbf{V}}_T^{\omega_i})=i\omega_i\bar{n}_1\ , \tag{17b}$$

where $\bar{A}=\int_{-\infty}^{\infty}e^{-sz}A(z)dz$ and the pondermotive wave phasor which oscillates with frequency ω_i is given by[15]

$$\begin{aligned}\tilde{E}_{\text{pond}}^{\omega_i}&=\tfrac{1}{2}(\tilde{\mathbf{V}}_{Ts}\times\tilde{\mathbf{B}}_{Tw}^{*}+\tilde{\mathbf{V}}_{Tw}^{*}\times\tilde{\mathbf{B}}_{Ts})\cdot\hat{\mathbf{i}}_z\\&=C_w^{*}C_s(z)\tilde{\mathcal{E}}_{\text{pond}}^{\omega_i}e^{i(k_s+k_w)z}\ ,\\ \tilde{\mathcal{E}}_{\text{pond}}^{\omega_i}&=\tfrac{1}{2}(\boldsymbol{\mathcal{V}}_{Ts}\times\boldsymbol{\mathcal{B}}_{Tw}^{*}+\boldsymbol{\mathcal{V}}_{Tw}^{*}\times\boldsymbol{\mathcal{B}}_{Ts})\cdot\hat{\mathbf{i}}_z\ .\end{aligned} \tag{18}$$

The transverse derivative on the left-hand side of Eq. (17b), $\nabla\cdot(n_0\tilde{\mathbf{V}}_{Tw})$, will be neglected from now on due to assumption (a) at the beginning of this section.

We turn to express $\bar{E}_z$ in Eq. (17a) in terms of the self-scalar-potential of the beam, Φ, which under the Lorentz gauge condition obeys the scalar Helmholtz equation:

$$\nabla^2\Phi-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi=\frac{n_1 e}{\epsilon_0}\ . \tag{19}$$

The source term in the right-hand side oscillates with frequency ω_i; thus the excited self-potential can be expressed in a phasor notation as follows:

$$\Phi=\Phi^{\omega_i}=\mathrm{Re}[\tilde{\Phi}^{\omega_i}(\mathbf{r})e^{-i\omega_i t}]\ .$$

Using Lorentz gauge condition, we find [under assumption (a)]

$$\frac{\partial}{\partial z}A_z^{\omega_i}\simeq-\frac{1}{c^2}\frac{\partial}{\partial t}\Phi^{\omega_i}\ , \tag{20}$$

where $A_z^{\omega_i}=\mathrm{Re}[\tilde{A}_z^{\omega_i}(\mathbf{r})e^{-i\omega_i t}]$ is the self-vector-potential of the beam. Thus the self-longitudinal electric field of the beam is given by

$$E_z^{\omega_i}=-\frac{\partial}{\partial z}\Phi^{\omega_i}-\frac{\partial}{\partial t}A_z^{\omega_i}\ . \tag{21}$$

Writing Eqs. (19)–(21) in phasor notation, and Laplace-transforming the resulting equations, we find

$$\bar{E}_z^{\omega_i}(\mathbf{r}_T,s)=-\left[s+\frac{\omega_i^2}{c^2 s}\right]\bar{\Phi}^{\omega_i}(\mathbf{r}_T,s)\ , \tag{22a}$$

$$\bar{n}_1(\mathbf{r}_T,s)=\frac{\epsilon_0}{e}\left[\nabla_T^2+s^2+\frac{\omega_i^2}{c^2}\right]\bar{\Phi}^{\omega_i}(\mathbf{r}_T,s)\ , \tag{22b}$$

where

$$\nabla_T^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\ .$$

By inserting Eqs. (22) and (17a) in (17b), one finds

$$\nabla_T^2\bar{\Phi}^{\omega_i}(\mathbf{r}_T,s)+\left[s^2+\frac{\omega_i^2}{c^2}\right]\left[1+\frac{\omega_p'^2(\mathbf{r}_T)}{(V_{0z}s-i\omega_i)^2}\right]\bar{\Phi}^{\omega_i}(\mathbf{r}_T,s)=\frac{\omega_p'(\mathbf{r}_T)s}{(V_{0z}s-i\omega_i)^2}\bar{E}_{\text{pond}}^{\omega_i}(\mathbf{r}_T,s)\ , \tag{23}$$

where $\omega_p'^2(\mathbf{r}_T)=e^2n_0(\mathbf{r}_T)/\epsilon_0 m\gamma_0\gamma_z^2$, and $\bar{\Phi}^{\omega_i}(\mathbf{r}_T,s)$ is submitted to homogeneous boundary conditions on the waveguide walls:

$$\bar{\Phi}^{\omega_i}(\mathbf{r}_T,s)\,|_{\text{walls}}=0\ . \tag{24}$$

Equation (23), which is of the Sturm-Liouville type, is the "beam's longitudinal waves equation." It describes the excitation of the longitudinal plasma waves (space-charge waves) in the beam. Upon substituting the formal solution of this equation back in Eq. (22b), we find the perturbed electron density of the beam. From now on we shall call Eq. (23) "the beam equation."

IV. INTERACTION OF WAVEGUIDE RADIATION MODES WITH BEAM MODES

A. Uniform beam completely filling the waveguide

We start with the simplest case in which a homogeneous beam fills the waveguide completely. In such a case ω_p' is constant across the waveguide cross section, and the beam equation can be written in the following way:

$$\nabla_T^2\bar{\Phi}^{\omega_i}+\xi^2(s)\bar{\Phi}^{\omega_i}=\frac{\omega_p'^2 s}{(V_{0z}s-i\omega_i)^2}\bar{E}^{\omega_i}_{\text{pond}}\ , \tag{25}$$

$$\xi^2(s)=\left[s^2+\frac{\omega_i^2}{c^2}\right]\left[1+\frac{\omega_p'^2}{(V_{0z}s-i\omega_i)^2}\right]\ , \tag{26}$$

$$\bar{\Phi}^{\omega_i}\,|_{\text{boundary}}=0\ . \tag{27}$$

We shall first solve the homogeneous equation

$$\nabla_T^2\Phi+k_T^2\Phi=0\ .$$

with the same homogeneous boundary conditions of Eq. (27). The solutions are given by a set of eigenfunctions $\{\hat{\Phi}_n\}$ and eigenvalues $\{k_{Tn}\}$:

$$\nabla_T^2\hat{\Phi}_n(\mathbf{r}_T;k_{Tn})+k_{Tn}^2\hat{\Phi}_n(\mathbf{r}_T;k_{Tn})=0\ ,\qquad n=1,2,3,\ldots\ . \tag{28}$$

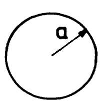
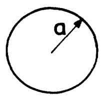
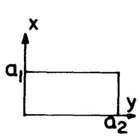
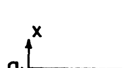
It is straightforward to find the solutions $\{\hat{\Phi}_n\}$ and $\{k_{Tn}\}$. Such solutions for various geometries are summarized in Table II. The dispersion relation of the beam modes in the absence of the ponderomotive force is obviously

$$\xi^2(s)=k_{Tn}^2,\quad n=1,2,3,\ldots\ . \tag{29}$$

or

$$\left[s^2+\frac{\omega_i^2}{c^2}\right]\left[1+\frac{\omega_p'^2}{(V_{0z}s-i\omega_i)^2}\right]=k_{Tn}^2\ .$$

TABLE II. Eigenmodes and eigenvalues of the self-fields of the beam for uniformly loaded geometries.

Waveguide geometry	$\hat{\Phi}_n$	k_{Tn}
 $n\equiv l,m$	$\left[\frac{2}{\pi}\right]^{1/2}\frac{1}{aJ_l'(p_{lm})}Jl\left[\frac{p_{lm}r}{a}\right]\cos(l\varphi)$	$\frac{p_{lm}}{a}$
	J_l is a Bessel function of order l	p_{lm} is the mth root of J_l
 Azimuthal symmetry	$\frac{1}{aJ_1(p_{0n})\sqrt{\pi}}J_0\left[\frac{p_{0n}}{a}r\right]$	$\frac{p_{0n}}{a}$
 $n\equiv l,m$	$\frac{2}{\sqrt{a_1a_2}}\sin\left[\frac{l\pi}{a_1}x\right]\sin\left[\frac{m\pi}{a_2}y\right]$	$\left[\left[\frac{l\pi}{a_1}\right]^2+\left[\frac{m\pi}{b}\right]^2\right]^{1/2}$
 slab ($a_2\rightarrow\infty$)	$\left[\frac{2}{a}\right]^{1/2}\sin\left[\frac{n\pi}{a_1}x\right]$	$\frac{n\pi}{a_1}$

It is convenient to express the Laplace variable by a wave-number notation $s=ik$. After some algebra the dispersion equation reduces to the normalized form:

$$(Y^2-X_n^2)\left[1-\frac{1}{(Y-\beta_z X_n)^2}\right]=\alpha_n^2 , \qquad (30)$$

where $\beta_z=V_{0z}/c$ and

$$Y=\frac{\omega_i}{\omega_p'} ,$$
$$X_n=\frac{k_n c}{\omega_p'} , \qquad (31)$$
$$\alpha_n=\frac{c}{\omega_p'}k_{Tn} .$$

Equation (30) was solved numerically for an example of $\beta_z=0.5$, $\alpha_n=1.57n$ ($n=1,2,3,\ldots$) which corresponds to the specific case of a slab waveguide (see Table II) with width $a=2(c/\omega_p')$. The dispersion curves of the four fundamental modes ($n=1,2,3,4$) of the beam are shown in Fig. 1. The beam modes differ from the well-known modes of the infinite plasma in a number of aspects:

(a) In the beam problem the solution is characterized by an infinite number of modes, while in the infinite plasma model only two longitudinal space-charge modes appear: the slow and the fast space-charge waves. (In the limit of infinite plasma $k_{Tn}=0$, Eq. (30) collapses and yields only these two space-charge waves: $Y-\beta_z X=\pm 1$).

(b) All the branches of the beam modes bend and pass through the origin of the k,ω plane (or X,Y plane) in such a way that the phase and group velocities of the modes vary, but never exceed the velocity of light. By contrast the modes of the infinite plasma can propagate with phase velocity greater than c, and their group velocities are constant and equal to the beam velocity.

It is important to notice that as long as k_{Tn} approaches zero but is not identically zero, the slope of the dispersion curves near and at the origin will not exceed the velocity of light. When k_{Tn} takes very small values (i.e., a very wide, but finite, beam) the dispersion curves tend asymptotically towards the lines

$$Y=\pm X; \quad Y=\beta_z X\pm 1 .$$

The solutions of Eq. (28) $\hat{\Phi}_n$ are a complete and orthogonal set,[18] satisfying:

$$\int\int \hat{\Phi}_n(\mathbf{r}_T;k_{Tn})\hat{\Phi}_m(\mathbf{r}_T;k_{Tm})dx\,dy=\delta_{nm} . \qquad (32)$$

The general solution of Eq. (25) can be expressed as an infinite sum of $\hat{\Phi}_n$. The characteristic Green's function of Eq. (25) is given by the sum:[18]

$$G(\mathbf{r}_T,\mathbf{r}_T',s)=\sum_m \frac{\hat{\Phi}_m(\mathbf{r}_T;k_{Tm})\hat{\Phi}_m(\mathbf{r}_T';k_{Tm})}{\xi^2(s)-k_{Tm}^2} \qquad (33)$$

and the full solution for $\overline{\Phi}^{\omega_i}(\mathbf{r}_T,s)$ can be expressed in terms of it as

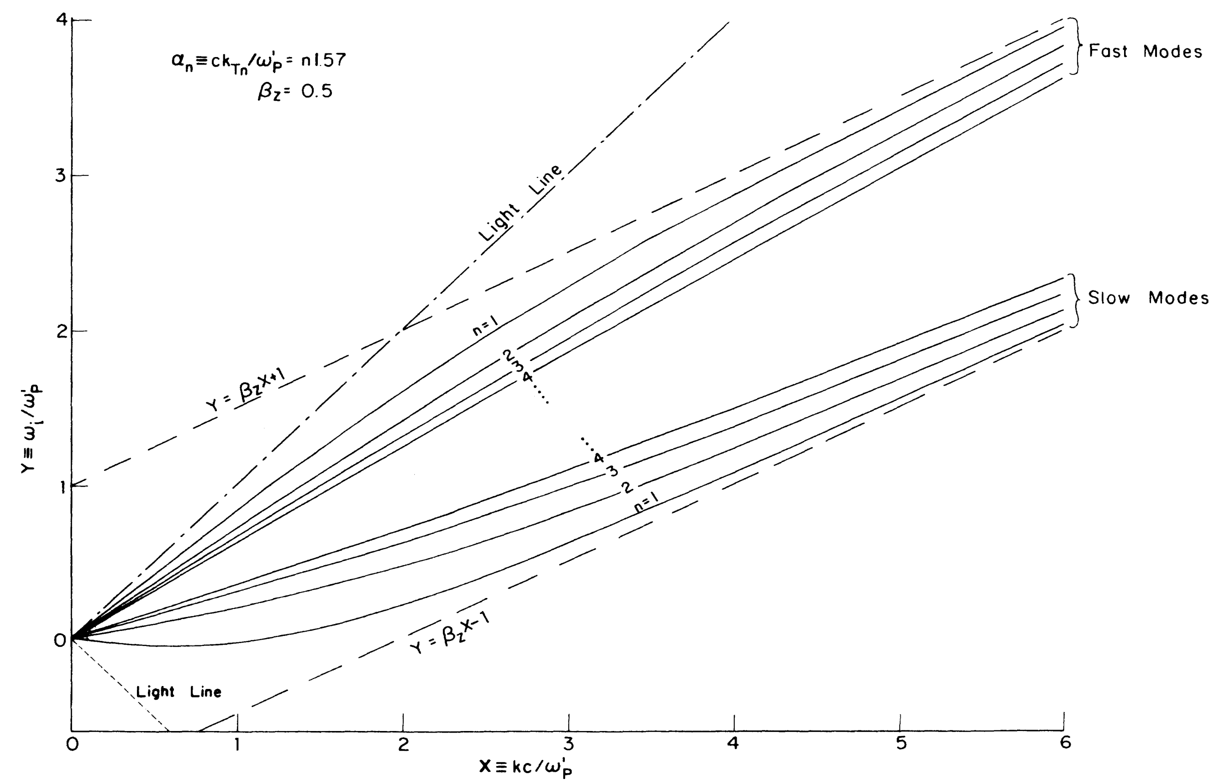


FIG. 1. The dispersion curves of the first four fundamental slow and fast beam modes of a uniform beam which completely fills the waveguide, for $\alpha_n=1.57n$ and $\beta_z=0.5$. The light lines denote the dispersion curves $Y=\pm X$.

$$\overline{\Phi}^{\,\omega_i}(\mathbf{r}_T,s)=\int\int G(\mathbf{r}_T,\mathbf{r}'_T,s)\frac{\omega_p'^2 s}{(V_{0z}s-i\omega_i)^2}\overline{E}^{\,\omega_i}_{\rm pond}(\mathbf{r}'_T,s)dx'\,dy' \ . \tag{34}$$

This defines explicitly the eigenmode expansion of the general solution and the expansion coefficients:

$$\overline{\Phi}^{\,\omega_i}(\mathbf{r}_T,s)=\sum_n \overline{A}_n(s)\hat{\Phi}_n(\mathbf{r}_T;k_{Tn}) \ , \tag{35}$$

$$\overline{A}_n(s)=\frac{1}{\xi^2(s)-k_{Tn}^2}\frac{\omega_p'^2}{(V_{0z}s-i\omega_i)^2}\int\int \hat{\Phi}_n(\mathbf{r}_T;k_{Tn})\overline{E}^{\,\omega_i}_{\rm pond}(\mathbf{r}_T,s)dx\,dy \ . \tag{36}$$

Equations (35) and (36) give a full solution for $\overline{\Phi}^{\,\omega_i}$ which is excited by the ponderomotive wave $\overline{E}^{\,\omega_i}_{\rm pond}$. The ponderomotive wave excites an infinite number of plasma modes in the beam; each has its own amplitude $\overline{A}_n(s)$ and thus its own growth rate along the interaction region.

The perturbed density wave $\bar{n}_1$ can be determined directly from $\overline{\Phi}^{\,\omega_i}$ with the aid of Eq. (22b). Inserting $\overline{\Phi}^{\,\omega_i}$ into this equation and using Eq. (28) we find

$$\bar{n}_1(\mathbf{r}_T,s)=\frac{\epsilon_0}{e}\sum_m \overline{A}_m(s)\left[s^2+\frac{\omega_i^2}{c^2}-k_{Tm}^2\right]\hat{\Phi}_m(\mathbf{r}_T;k_{Tm}) \ , \tag{37}$$

where $\overline{A}_m(s)$ is given in Eq. (36).

In order to get the gain-dispersion equation of the electromagnetic mode coupled to the plasma modes, we substitute Eq. (37) in (8), and after some algebra we find

$$\frac{\overline{C}_s(s)}{C_s(0)}=\left[s-\frac{|C_w|^2\epsilon_0\theta_p^2}{8|\mathcal{P}_s|}\sum_n\frac{(s+iK)\left[(s+iK)^2+\left[\frac{\omega_i}{c}\right]^2-k_{Tn}^2\right]}{\left[(s+iK)^2+\left[\frac{\omega_i}{c}\right]^2\right][(i\theta-s)^2+\theta_p^2]-k_{Tn}^2(i\theta-s)^2}G_n\eta_n\right]^{-1} \ , \tag{38}$$

where

$$\begin{aligned}
&K=k_s+k_w \ ,\\
&\theta=\frac{\omega_i}{V_{0z}}-K \ ,\\
&\theta_p=\frac{\omega_p'}{V_{0z}} \ ,\\
&G_n=\int\int \hat{\Phi}_n(\mathbf{r}_T;k_{Tn})\tilde{\mathcal{E}}^{\,\omega_i}_{\rm pond}(\mathbf{r}_T)dx\,dy \ ,\\
&\eta_n=\int\int \Phi_n(\mathbf{r}_T;k_{Tn})\mathcal{V}_{Tw}(\mathbf{r}_T)\cdot\tilde{\mathcal{E}}^{\,*}_{Ts}(\mathbf{r}_T)dx\,dy \ ,
\end{aligned} \tag{39}$$

and the transverse velocities profiles $\mathcal{V}_T$ are summarized in Table I. Equation (38) is the general gain-dispersion equation of the FEL which describes the interaction of a single electromagnetic waveguide mode (the signal) with all the modes of the beam.

Assuming that only the fundamental plasma mode is dominant ($G_n\eta_n\to 0$ for every n except $n=1$) and making the small growth rate approximation $|s|\ll K$, we find the following simple gain-dispersion equation:

$$\frac{\overline{C}_s(s)}{C_s(0)}=\frac{(i\theta-s)^2+\theta_{pr}^2}{s[i\theta-s)^2+\theta_{pr}^2]-i\theta_p^2\kappa_r} \ , \tag{40}$$

where

$$\begin{aligned}
&\kappa_r=|C_w|^2\frac{\epsilon_0 K}{8|\mathcal{P}_s|}G_1\eta_1 \ ,\\
&\theta_{pr}^2=\theta_p^2\left[1+\frac{k_{T1}^2}{K^2-\left[\frac{\omega_i}{c}\right]^2}\right]^{-1}=\theta_p^2R^2 \ .
\end{aligned} \tag{41}$$

It is of interest to compare Eq. (40) with the well-known gain-dispersion equation of the conventional transversely infinite FEL. The latter equation is known to be[5,15,19]

$$\begin{aligned}
&\frac{\overline{C}_s(s)}{C_s(0)}=\frac{(i\theta-s)^2+\theta_p^2}{s[(i\theta-s)^2+\theta_p^2]-i\theta_p^2\kappa} \ ,\\
&\theta_p=\frac{\omega_p'}{V_{0z}} \ ,\\
&\theta=\frac{\omega_i}{V_{0z}}-K \ ,\\
&\kappa=|C_w|^2\frac{\epsilon_0 K}{8|\mathcal{P}_s|}A_e\tilde{\mathcal{E}}^{\,*}_{Ts}\cdot\mathcal{V}_{Tw}\tilde{\mathcal{E}}^{\,\omega_i}_{\rm pond} \ ,
\end{aligned} \tag{42}$$

where A_e is the effective cross-sectional area of the electron beam.

Equation (40) differs from the one-dimensional model expression (42) only in the G_n and η_n "overlap integrals" with the beam modes which appear in the definition of κ_r and in the space-charge parameter reduction factor

$$R=\left[1+\frac{k_{T1}^2}{K^2-\left[\frac{\omega_i}{c}\right]^2}\right]^{-1/2}$$

in (41) which originates from the transverse finiteness of the beam and the waveguide. This reduction factor is closely related to the plasma reduction factors which were discovered and investigated a few decades ago in connection with microwave tubes.[20]

In the limit of transverse infinite dimensions one would expect the 3D expression (40) to reduce to the 1D from (42). Indeed this happens, since in this limit $k_{T1}\to 0$ and $\theta_{pr}\to\theta_p$. Note, however, that the definition of the coupling parameter κ_r still includes overlap integrals G_n,η_n (39) with the fundamental plasma mode.

The 1D limit condition

$$k_{T1}^2/\left[K^2-\left[\frac{\omega_i}{c}\right]^2\right]\ll 1$$

can be simplified by substituting $\omega_i/k=V_{\text{pond}}$ (the phase velocity of the ponderomotive wave). The space-charge parameter reduction factor then tends to unity when the following simple condition is satisfied:

$$k_T\ll K/\gamma_{\text{pond}}=K' ,$$

where K' is the ponderomotive wave number in the pondermotive potential rest frame. This simply means that in this moving frame the transverse dimensions of the waveguide should be much larger than the ponderomotive wavelength. For near synchronism condition $\gamma_{\text{pond}}\simeq\gamma_{0z}\gg 1$, and taking for example a circular cylindrical waveguide (Table II), the 1D limit condition becomes

$$2a\gg\frac{p_{01}}{\pi}\lambda\gamma_z\simeq\frac{p_{01}}{2\pi}\frac{\lambda_w}{\gamma_z}=0.38\frac{\lambda_w}{\gamma_z} ,$$

a condition which is not always well satisfied with typical low-energy Raman regime FEL's.

Another limit that is worth checking for examining the consistency of the presently derived dispersion equation with the conventional one is the limit $\theta_p\ll\pi$. In this regime (the tenous beam or "Compton" regime) space-charge plasma waves should not play a role at all in the FEL interaction and already the multiplasma modes dispersion equation (38) should reduce exactly to (42) with $\theta_p\to 0$ (while $\kappa_r\theta_p^2$ stays finite). This is shown in Appendix B for the more general case of a nonuniform beam.

B. Beam partially filling the waveguide

In FEL systems which contain a nonuniform beam, or a uniform beam which partially fills the waveguide, the analysis gets to be somewhat more involved. In such a case we write the beam equation in the following way:

$$\nabla_T^2\overline{\Phi}^{\omega_i}+Q^2\left[\frac{g(\mathbf{r}_T)}{H^2}-1\right]\overline{\Phi}^{\omega_i}=\frac{g(\mathbf{r}_T)s}{H^2}\overline{E}^{\omega_i}_{\text{pond}} , \quad (43)$$

where

$$Q^2=-\left[\frac{\omega_i^2}{c^2}+s^2\right] ,$$

$$H^2=-\frac{(V_{0z}s-i\omega_i)^2}{\omega_p'^2} , \quad (44)$$

and where ω_p' is evaluated at the center of the beam (where the electron density reaches the highest value) and $g(\mathbf{r}_T)$ is the transverse profile of the electron-beam density (in the previous example of a uniform beam which fills completely the waveguide, $g(\mathbf{r}_T)=1$ for any $\mathbf{r}_T$ inside the waveguide).

We look for eigenvalues $s_n=ik_n$ for which the homogeneous part of Eq. (43) together with appropriate boundary conditions have eigensolutions $\hat{\mathbf{\Phi}}_n(\mathbf{r}_T)$. Contrary to the previous case of a uniform beam, in general these functions are not orthogonal in the simple sense of Eq. (32), and the orthogonality relation takes on the form (see Appendix A):

$$\int\int\left[Q_n^2\left[\frac{g(\mathbf{r}_T)}{H_n^2}-1\right]-Q_m^2\left[\frac{g(\mathbf{r}_T)}{H_m^2}-1\right]\right]\times\hat{\mathbf{\Phi}}_n(\mathbf{r}_T)\hat{\mathbf{\Phi}}_m(\mathbf{r}_T)dx\,dy=0 ,$$

$$Q_n^2=k_n^2-\frac{\omega_i^2}{c^2} , \quad (45)$$

$$H_n^2=\frac{(V_{0z}k_n-\omega_i)^2}{\omega_p'^2} .$$

1. *Beam modes in the magnetized beam model*

To escape the problem of boundary condition definitions at the beam boundaries, we assume first a magnetized (longitudinal) beam model. In this model a strong axial magnetic field suppresses the rf rippling of the beam boundary so that its surface can be considered smooth. For the FEL problem (contrary to traveling wave tubes, for example) one should not take the extreme magnetized beam ($B_\parallel\to\infty$) limit literally, since then the transverse electron quiver which produces the FEL interaction is damped. The magnetized plasma model should be then understood as an approximation which has greater validity at strong but limited axial magnetic fields.] The beam density profile is now assumed to be z independent and uniform within the beam boundaries:

$$g(\mathbf{r}_T)=\begin{cases}1, & \text{inside the beam}\\ 0, & \text{outside the beam}\end{cases} \quad (46)$$

and the scalar potential $\overline{\Phi}^{\omega_i}$ and its first derivative are continuous at the boundaries of the beam.

In such a case the beam modes obey the following wave equations:

$$\nabla_T^2\hat{\mathbf{\Phi}}^{(1)}+Q^2\left[\frac{1}{H^2}-1\right]\hat{\mathbf{\Phi}}^{(1)}=0 \quad (47a)$$

inside the beam, and

$$\nabla_T^2\hat{\mathbf{\Phi}}^{(2)}-Q^2\hat{\mathbf{\Phi}}^{(2)}=0 \quad (47b)$$

between the beam boundaries and the waveguide walls. In addition, the following boundary conditions must be satisfied:

$$\hat{\Phi}^{(1)}(\mathbf{r}_T)\,|_a=\hat{\Phi}^{(2)}(\mathbf{r}_T)\,|_a\;, \tag{48a}$$

$$\hat{\Phi}^{(2)}(\mathbf{r}_T)\,|_b=0\;, \tag{48b}$$

$$\nabla_T\hat{\Phi}^{(1)}(\mathbf{r}_T)\,|_a\cdot\hat{\mathbf{e}}_n=\nabla_T\hat{\Phi}^{(2)}(\mathbf{r}_T)\,|_a\cdot\hat{\mathbf{e}}_n\;, \tag{48c}$$

where a, b, and $\hat{\mathbf{e}}_n$ represent the beam boundaries, the waveguide walls, and a unit vector normal to the beam surface, respectively.

As an example consider the slab geometry depicted in Fig. 2: An electron beam of width $2a$ traverses along a parallel-plate waveguide of width $2b$. The system is assumed to be infinite in the y direction.

Two kinds of solutions exist in such a system: even and odd. The first kind (even solutions) is given by

$$\hat{\Phi}^{(1)}=C^{(1)}\cos(QPx),\quad |x|<a \tag{49a}$$

$$\hat{\Phi}^{(2)}=C^{(2)}e^{-Qx}+C^{(3)}e^{Qx},\quad a<|x|<b \tag{49b}$$

where $P^2=1/H^2-1$. The odd solutions are obtained by replacing $\cos(QPx)$ in Eq. (49a) with $\sin(QPx)$. Substituting Eqs. (49) in the boundary conditions (48) and looking for the nontrivial solution ($C^{(i)}\neq 0$) we find after some algebra the following transcendental equations:

$$P\tan(QPa)=\coth[Q(b-a)] \tag{50}$$

for the even solutions, and

$$P\cot(QPa)=-\coth[Q(b-a)] \tag{51}$$

for the odd solutions, where coth is the hyperbolic cotangence function, and

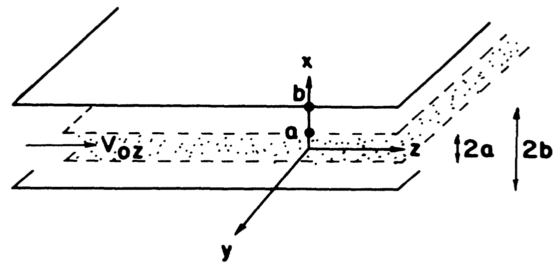


FIG. 2. Schematic of the slab geometry used in Sec. IV B.

$$\begin{aligned} Q^2&=(X^2-Y^2)\left[\frac{\omega_p'}{c}\right]^2,\\ P^2&=(Y-\beta_zX)^{-2}-1\;,\\ X&=\frac{kc}{\omega_p'}\;,\\ Y&=\frac{\omega_i}{\omega_p'}\;. \end{aligned} \tag{52}$$

Dispersion curves of the four fundamental even modes in a magnetized beam with slab geometry and $b\gg a$ are shown in Fig. 3 for various values of the beam width and velocity (the width parameters of Fig. 3(a) and 3(b) correspond to very wide systems, in order to emphasize the asymptotic behavior of the space-charge dispersion curves).

Following the above analysis, it is straightforward to develop the mode dispersion equations and eigenmode solutions for other geometries. In Table III we summarize the solutions and transcendental equations of slab and cylindrical geometries.

TABLE III. Eigenmodes and transcendental dispersion equations for various configurations with a uniform beam which partially fills the waveguide.

	$\hat{\Phi}_n$	Dispersion equation
Slab geometry	even $\begin{cases}\hat{\Phi}^{(1)}=C^{(1)}\cos(QPx)\\ \hat{\Phi}^{(2)}=C^{(2)}e^{-Qx}+C^{(3)}e^{Qx}\end{cases}$	even: $\Lambda^2P\tan(QPa)=\coth[Q(b-a)]$
	odd $\begin{cases}\hat{\Phi}^{(1)}=C^{(1)}\sin(QPx)\\ \hat{\Phi}^{(2)}=C^{(2)}e^{-Qx}+C^{(3)}e^{Qx}\end{cases}$	odd: $\Lambda^2P\cot(QPa)=-\coth[Q(b-a)]$
Cylindrical geometry	$\hat{\Phi}^{(1)}=C^{(1)}J_0(QPr)$ $\hat{\Phi}^{(2)}=C^{(2)}I_0(Qr)+C^{(3)}K_0(Qr)$	$\Lambda^2P\dfrac{J_1(QPa)}{J_0(QPa)}=\dfrac{I_0(Qb)K_1(Qa)+K_0(Qb)I_1(Qa)}{I_0(Qb)K_0(Qa)-K_0(Qb)I_0(Qa)}$

$$Q^2=(X^2-Y^2)\left[\frac{\omega_p'}{c}\right]^2;P^2=(Y-\beta_zX)^{-2}-1$$

$$\Lambda^2=\begin{cases}1, & \text{confined beam}\\ 1+\dfrac{\gamma_z^2}{X}\dfrac{X-\beta_zY}{(Y-\beta_zX)^2}, & B_\parallel=0\end{cases}$$

$$X=\frac{kc}{\omega_p'};\quad Y=\frac{\omega_i}{\omega_p'}$$

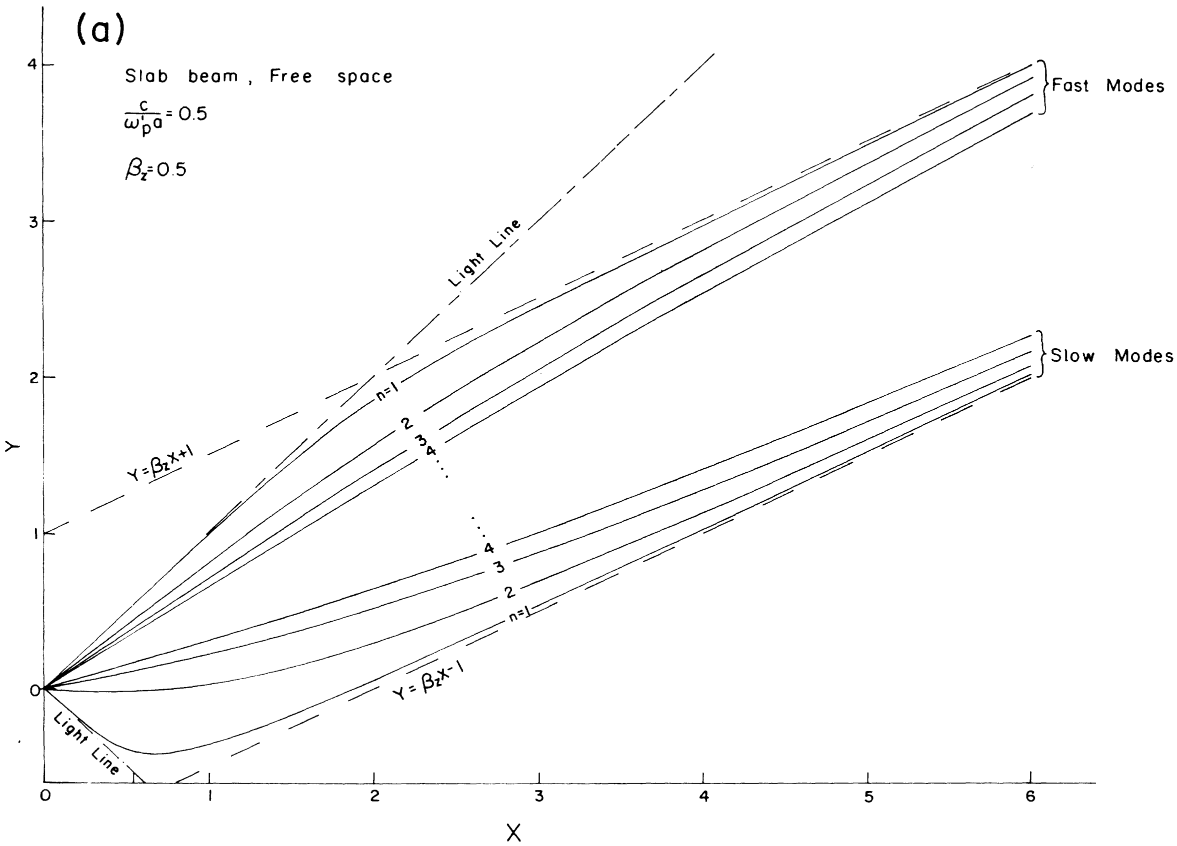


FIG. 3. The dispersion curves of the first four fundamental slow and fast beam modes of a confined uniform slab beam in free space, for various operating parameters. Note the asymptotic behavior of the fundamental dispersion waves as $x \to \infty$ for the cases of very wide systems (a) and (b). The light lines denote the dispersion curves $Y = \pm X$.

2. Beam modes including effect of rf boundary ripple

Analysis of beams without the magnetized beam approximation requires more careful consideration of the boundary conditions. The form of the boundary conditions (48a) and (48c) is invalid since the beam oscillates with the beat frequency ω_i and its boundary ripples. However, this effect may be well accounted for by the microwave-tube-theory model of surface charges and currents[21,22] with a relativistic generalization.

We replace the rippled boundary beam by surface currents and charges, carried on the surface of a beam with a smooth boundary and uniform profile. The boundary condition (48c) should be then modified accordingly while (48a) and (48b) stay the same. Equation (48c) is replaced by

$$\nabla_T \hat{\Phi}^{(2)}(\mathbf{r}_T)\,|_a \cdot \hat{\mathbf{e}}_n - \nabla_T \hat{\Phi}^{(1)}(\mathbf{r}_T)\,|_a \cdot \hat{\mathbf{e}}_n = \frac{\overline{\sigma}_s}{\epsilon_0} \tag{53}$$

where σ_s is the surface charge carried by the beam.

Neglecting azymuthal modes, the continuity equation of σ_s is

$$\frac{\partial}{\partial t}\sigma_s - V_{0z}\frac{\partial}{\partial z}\sigma_s = J_r^{\omega_i} , \tag{54}$$

where $J_r^{\omega_i}$ is the first-order radial current with the beat frequency ω_i:

$$J_r^{\omega_i} = -en_0 V_r^{\omega_i} \tag{55}$$

and $V_r^{\omega_i}$ is the radial electron velocity due to the self-fields of the electron beam, and it is derived from the force equation:

$$\left[\frac{\partial}{\partial t} + V_{0z}\frac{\partial}{\partial z}\right] V_r^{\omega_i} = -\frac{e}{\gamma_0 m}(-\nabla_T \hat{\Phi}^{(1)} + V_{0z}\hat{\mathbf{i}}_z \times \nabla \times \mathbf{A}^{\omega_i})\cdot \hat{\mathbf{e}}_n . \tag{56}$$

Using the Lorentz gauge condition $\overline{A}_z^{\omega_i} \simeq (i\omega_i / sc^2)\overline{\Phi}^{\omega_i}$, and Laplace-transforming Eq. (56) we find

$$\overline{V}_r^{\omega_i} = \frac{e/\gamma_0 m}{V_{0z}s - i\omega_i}\left[1 - \frac{i\omega_i V_{0z}}{sc^2}\right] \nabla_T \hat{\Phi}^{(1)}\cdot \hat{\mathbf{e}}_n . \tag{57}$$

After inserting Eqs. (57) and (55) in (54), we find

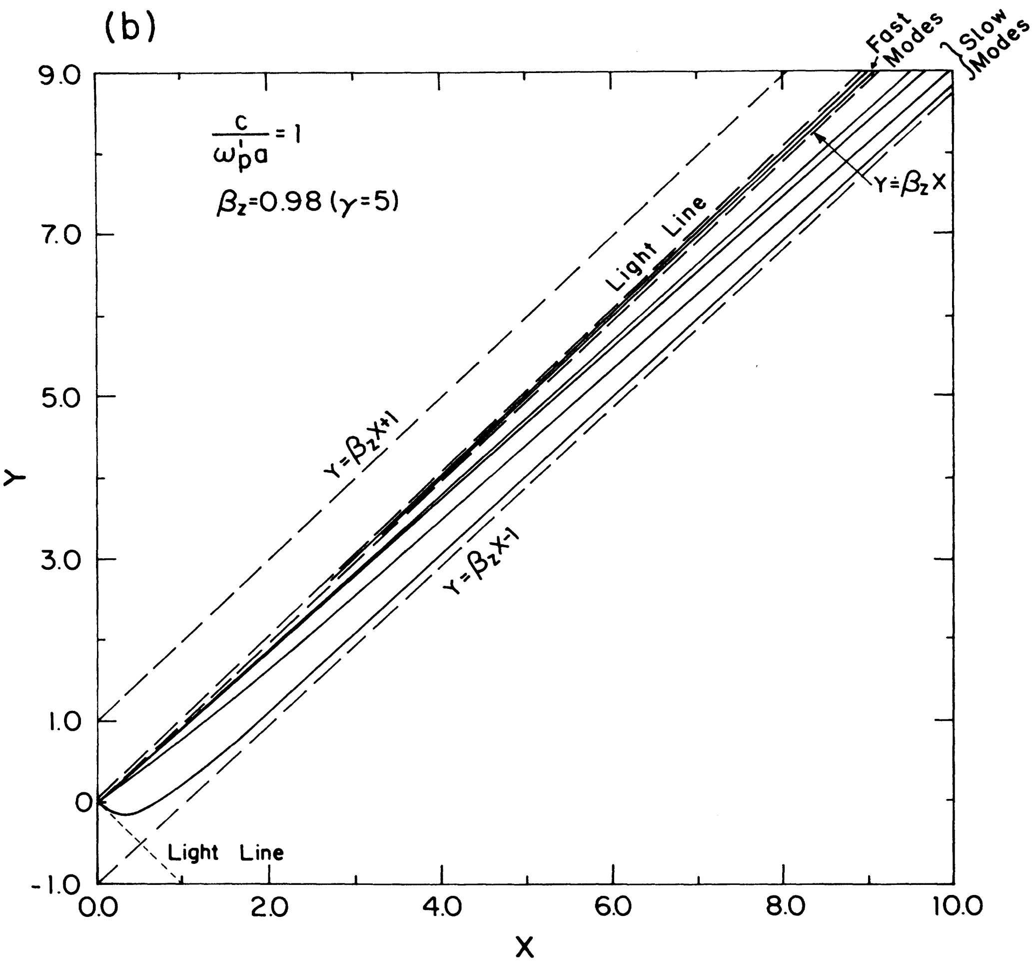


FIG. 3. (*Continued*).

$$\frac{\bar{\sigma}_s}{\epsilon_0}=-\frac{\omega_p^2}{\gamma_0}\frac{1-\dfrac{i\omega_i V_{0z}}{sc^2}}{(V_{0z}s-i\omega_i)^2}\nabla_T\hat{\Phi}^{(1)}(\mathbf{r}_T)\Bigg|_a\cdot\hat{\mathbf{e}}_n\ , \tag{58}$$

where $\omega_p^2=e^2n_0/\epsilon_0 m$. Using now (58), the boundary condition (53) becomes

$$\nabla_T\hat{\Phi}^{(2)}(\mathbf{r}_T)\,|_a\cdot\hat{\mathbf{e}}_n=\left[1-\frac{\omega_p^2}{\gamma_0}\frac{1-\dfrac{i\omega_i V_{0z}}{sc^2}}{(V_{0z}s-i\omega_i)^2}\right]\nabla_T\hat{\Phi}^{(1)}(\mathbf{r}_T)\Bigg|_a\cdot\hat{\mathbf{e}}_n\ . \tag{59}$$

The derivation of $\hat{\Phi}^{(1),(2)}$ and the transcendental dispersion equations of the beam modes can now be carried out with boundary condition (59) instead of (48c). The results are summarized in Table III for the example of a slab and a cylindrical waveguide geometry.

3. The gain-dispersion equation

In Secs. IV B 1 and IV B 2 we derived the beam modes in systems that consist of a uniform beam which partially fills the waveguide. We carried out the analysis with and without the magnetized beam approximation, and for various transverse configurations of the waveguide and the beam. For each of these cases we developed a transcendental dispersion equation, the solution at which, for a given frequency ω_i (or Y), determines an infinite set of eigenvalues $k_n=is_n$ (or X_n) and thus a set of Q_n and H_n [see Eqs. (44) and (45)]. In all these cases the eigenmodes are solutions of the homogeneous beam equation:

$$\nabla_T^2\hat{\Phi}_n(\mathbf{r}_T)+Q_n^2\left[\frac{g(\mathbf{r}_T)}{H_n^2}-1\right]\hat{\Phi}_n(\mathbf{r}_T)=0\ . \tag{60}$$

They are normalized according to

$$\int\int\hat{\Phi}_n^2(\mathbf{r}_T)dx\,dy=1 \tag{61}$$

and are orthogonal according to the rule (45). Following the procedure presented in Sec. IV A, we express the general solution of the beam equation $\bar{\Phi}^{\omega_i}(\mathbf{r}_T,s)$ as an infinite sum of the beam modes:

$$\bar{\Phi}^{\omega_i}(\mathbf{r}_T,s)=\sum_n\bar{A}_n(s)\hat{\Phi}_n(\mathbf{r}_T)\ . \tag{62}$$

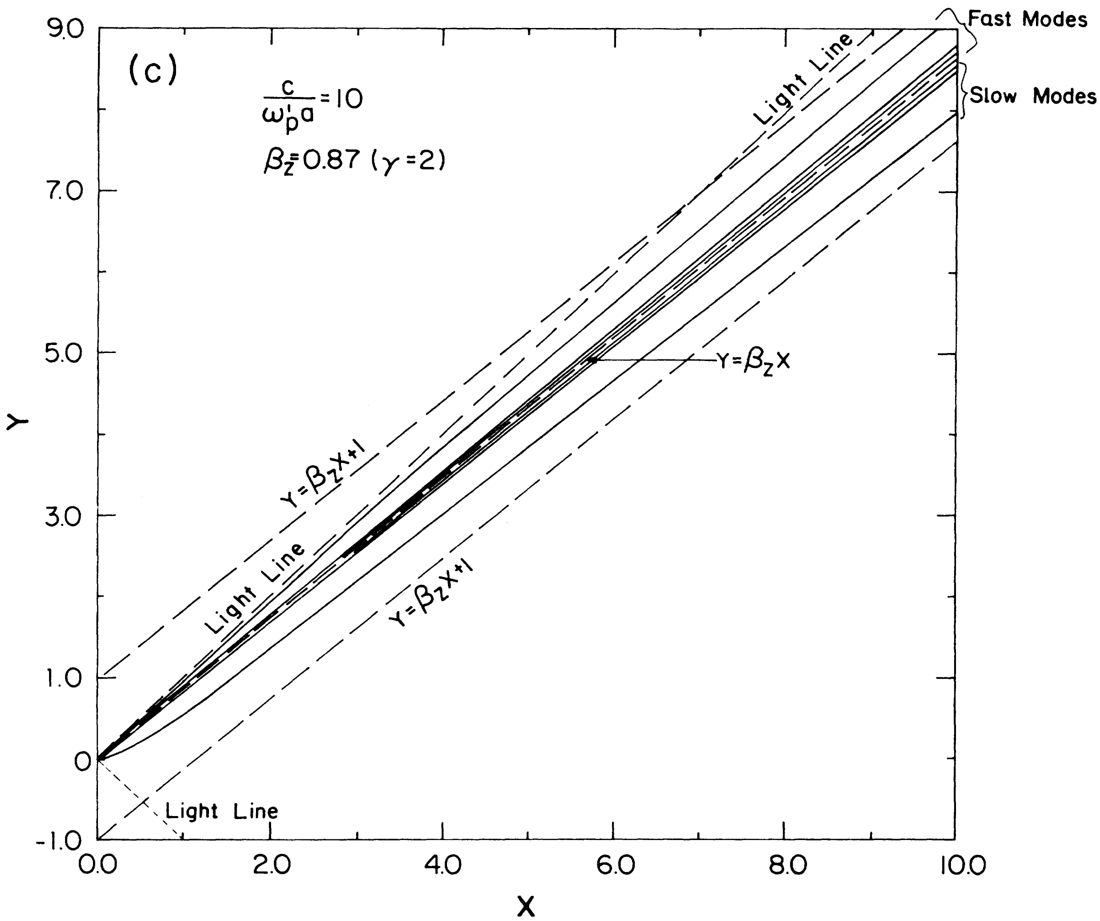


FIG. 3. (*Continued*).

By inserting Eq. (62) into the beam inhomogeneous Eq. (43) and using (60) we get

$$\sum_n \overline{A}_n(s)\left[Q^2\left[\frac{g(\mathbf{r}_T)}{H^2}-1\right]-Q_n^2\left[\frac{g(\mathbf{r}_T)}{H_n^2}-1\right]\right]\hat{\mathbf{\Phi}}_n(\mathbf{r}_T)$$
$$=-\frac{g(\mathbf{r}_T)s}{H^2}\overline{E}^{\,\omega_i}_{\text{pond}}(\mathbf{r}_T,s)\ . \quad (63)$$

Multiplying both sides by $\hat{\mathbf{\Phi}}_m(\mathbf{r}_T)$, and integrating over the cross-sectional area of the waveguide, one finds an infinite set of coupled algebraic equations for $\overline{A}_n(s)$:

$$\sum_n \overline{A}_n(s)C_{nm}(s)=d_m(s)\ , \quad (64a)$$

$$C_{nm}(s)=\int\int\left[Q^2\left[\frac{g(\mathbf{r}_T)}{H^2}-1\right]-Q_n^2\left[\frac{g(\mathbf{r}_T)}{H_n^2}-1\right]\right]$$
$$\times\hat{\mathbf{\Phi}}_n(\mathbf{r}_T)\hat{\mathbf{\Phi}}_m(\mathbf{r}_T)dx\,dy\ , \quad (64b)$$

$$d_m(s)=-\frac{s}{H^2}\int\int g(\mathbf{r}_T)\overline{E}^{\,\omega_i}_{\text{pond}}(\mathbf{r}_T,s)\hat{\mathbf{\Phi}}_m(\mathbf{r}_T)dx\,dy\ . \quad (64c)$$

Use of Helmholtz equation (22b) yields together with equations (60) and (62):

$$\overline{n}_1(\mathbf{r}_T,s)=\frac{\epsilon_0}{e}\sum_n \overline{A}_n(s)\left[s^2+\left[\frac{\omega_i}{c}\right]^2-Q_n^2\left[\frac{g(\mathbf{r}_T)}{H_n^2}-1\right]\right]\hat{\mathbf{\Phi}}_n(\mathbf{r}_T)\ . \quad (65)$$

In order to find the gain-dispersion equation, one must insert the formal solution of Eq. (64) into (65) and then substitute the resulting expression of $\overline{n}_1(\mathbf{r}_T,s)$ into the field equation (8). This procedure is extremely simplified if we can assume weak coupling between the modes. In such cases one can get an approximate solution for $\overline{A}_n(s)$:

$$\overline{A}_n(s)\simeq\frac{d_n(s)}{C_{nn}(s)}\ . \quad (66)$$

[This approximation becomes particularly good when it can be argued that the electromagnetic mode couples strongly to one of the beam modes n^*, in which case $s \approx s_{n^*}$, and by virtue of the orthogonality relation (45) $c_{nn^*}=0$ for $n \neq n^*$. This makes the reduction of (64a) into (66) well justified at least for $n=n^*$. Furthermore, since $C_{n^*n^*}(s) \approx 0$ near synchronization, the amplitude $\bar{A}_{n^*}(s)$ is resonantly large and it is valid to substitute (66) in (65) even for $n \neq n^*$, bearing in mind that these non-resonant terms will have negligible contribution to the sum.]

By substituting Eqs. (65) and (66) into the field equation (8) we find

$$\frac{\bar{C}_s(s)}{C_s(0)}=\left[s-\frac{|C_w|^2\epsilon_0\theta_p^2}{8|\mathcal{P}_s|}\sum_n \frac{(s+iK)G_n'\eta_n'}{\left[(s+iK)^2+\left[\frac{\omega_i}{c}\right]^2\right]\left[[(i\theta-s)^2+\theta_p^2\hat{\alpha}_n]-Q_n^2\left[\frac{\hat{\alpha}_n}{H_n^2}-1\right](i\theta-s)\right]}\right]^{-1}, \tag{67}$$

where

$$\begin{aligned}
G_n' &= \int\int g(\mathbf{r}_T)\tilde{\mathcal{E}}^{\,\omega_i}_{\rm pond}(\mathbf{r}_T)\hat{\Phi}_n(\mathbf{r}_T)dx\,dy\ ,\\
\eta_n' &= \int\int \left[(s+iK)^2+\left[\frac{\omega_i}{c}\right]^2-Q_n^2\left[\frac{g(\mathbf{r}_T)}{H_n^2}-1\right]\right]\hat{\Phi}_n(\mathbf{r}_T)\tilde{\mathbf{V}}_{Tw}(\mathbf{r}_T)\cdot\tilde{\mathcal{E}}^{\,*}_{Ts}(\mathbf{r}_T)dx\,dy\ ,\\
\hat{\alpha}_n &= \int\int g(\mathbf{r}_T)\hat{\Phi}_n^2(\mathbf{r}_T)dx\,dy\ ,\\
K &= k_s+k_w\ ,\\
\theta &= \frac{\omega_i}{V_{0z}}-K\ ,\\
\theta_p &= \frac{\omega_p'}{V_{0z}}\ .
\end{aligned} \tag{68}$$

As a check of consistency, when substituting in again $g(\mathbf{r}_T)=1$ (a uniform beam which fills the waveguide completely), one finds $\hat{\alpha}_n=1$, $Q_n^2(1/H_n^2-1)=k_{Tn}$ and Eqs. (67) and (68) are reduced to the simpler equations of Sec. IV A [(38) and (39)].

In cases where the interaction takes place essentially with the fundamental mode, and the higher interacting beam modes can be neglected ($G_n'\eta_n' \to 0$ for $n \neq 1$), again one can get a simpler expression:

$$\frac{\bar{C}_s(s)}{C_s(0)}=\frac{(i\theta-s)^2+\theta_{pr}^2}{s[(i\theta-s)^2+\theta_{pr}^2]-i\theta_p^2\kappa_r}\ , \tag{69}$$

where

$$\theta_{pr}^2=\theta_p^2\hat{\alpha}_1\left[1+\frac{Q_1^2\left[\frac{\hat{\alpha}_1}{H_1^2}-1\right]}{K^2-\left[\frac{\omega_i}{c}\right]^2}\right]^{-1}=\theta_p^2R'^2\ , \tag{70a}$$

$$\kappa_r=\frac{|C_w|^2\epsilon_0}{8|\mathcal{P}_s|}KG_1'\int\int\frac{\left[\frac{\omega_i}{c}\right]^2-K^2-Q_1^2\left[\frac{g(\mathbf{r}_T)}{H_1^2}-1\right]}{\left[\frac{\omega_i}{c}\right]^2-K^2-Q_1^2\left[\frac{\hat{\alpha}_1}{H_1^2}-1\right]}\hat{\Phi}_1(\mathbf{r}_T)\tilde{\mathbf{V}}_{Tw}(\mathbf{r}_T)\cdot\tilde{\mathcal{E}}^{\,*}_{Ts}(\mathbf{r}_T)dx\,dy\ , \tag{70b}$$

where again, as in Sec. IV A, Eqs. (69) and (70) differ from the well-known gain-dispersion relation of the transversely infinite FEL device mainly by the introduction of the plasma reduction factor

$$\hat{\alpha}_1^{1/2}\left\{1+Q_1^2\left[\frac{\hat{\alpha}_1}{H_1^2}-1\right]\Big/\left[K^2-\left[\frac{\omega_i}{c}\right]^2\right]\right\}^{-1/2}$$

which multiplies the plasma parameter θ_p in (70a), and by the inclusion of the beam mode profile $\hat{\Phi}_1(\mathbf{r}_T)$ in the overlap integral associated with the FEL coupling coefficient κ_r (70b).

It is of some practical interest at this point to examine whether the expression for the plasma reduction factor can be simplified, and to study its behavior as a function of some of the FEL operating parameters. Appreciable simplification is obtained by assuming coupling to a single beam mode (say, $n=1$). Hence we assume the phase matching condition $s=iK\simeq ik_1$. Under this condition the reduced plasma parameter given in Eq. (70a) becomes $\theta_{pr}^2=\theta_p^2 H_1^2$ and the simplified reduction factor is

$$R'=H_1=\frac{V_{0z}k_1-\omega_i}{\omega_p'}=\beta_z X_1 - Y \ , \qquad (71)$$

where Y and X are defined as ω_i/ω_p' and $k_1 c/\omega_p'$, respectively. Expression (71) enables us to assign an intuitive geometrical interpretation to the plasma mode reduction factor R: It is simply the normalized vertical distance of the plasma dispersion curve from the tenuous beam dispersion line $Y=\beta_z X$ as illustrated in Fig. 4. It is seen that for a given frequency Y, two choices of H_1 are possible, $H_1^{\rm sl}$ and H_1^f, corresponding to interaction with the slow and fast space-charge waves, respectively. When the problem of relevance is attaining positive gain in the conventional up conversion ($\omega_s>\omega_w$) FEL system, then $H_1^{\rm sl}$ is the relevant parameter; however, when considering special cases like attenuation conditions in the conventional FEL scheme (beam acceleration), or gain in the down conversion ($\omega_s<\omega_w$) absolute instability case,[7] H_1^f is the right reduction factor to choose.

Some of the qualitative behavior of H_1 can be understood from Fig. 4 without any numerical computations: For very wide systems ($c/\omega_p' a<1$) the space-charge-wave dispersion curves tend to the lines $Y=\beta_z X\pm 1$ and $Y=\pm X$ and we find for H_1:

$$H_1^{\rm sl}\to 1 \ ,$$
$$H_1^f \to \begin{cases} -1, & Y\geq(1-\beta_z)^{-1} \\ (\beta_z-1)Y, & Y<(1-\beta_z)^{-1} \ . \end{cases} \qquad (72)$$

Thus even for very wide systems (or very high frequencies) the reduction factor of the fast space-charge wave can approach zero if $\gamma_z \gg 1$. Note, however, that when one increases γ_z while keeping the width parameter $c/\omega_p' a$ fixed and finite, $H_1^{\rm sl}$ decreases also because of a relativistic effect.

$H_1^{\rm sl}, H_1^f$ were calculated numerically for the case of a magnetized slab beam with width $2a$ propagating in free space ($b \gg a$). A plot of $H_1^{{\rm sl},f}$ versus γ for an example of $Y=5$ and $c/\omega_p' a=10$ is shown in Fig. 5. It is seen that the reduction factors of both slow and fast waves fall down drastically for relativistic velocities of the e beam and this fall down is more pronounced for the fast wave.

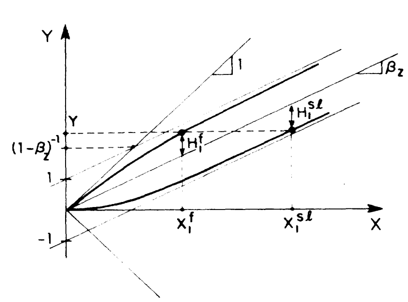


FIG. 4. The geometrical interpretation of the simplified reduction factor H_1.

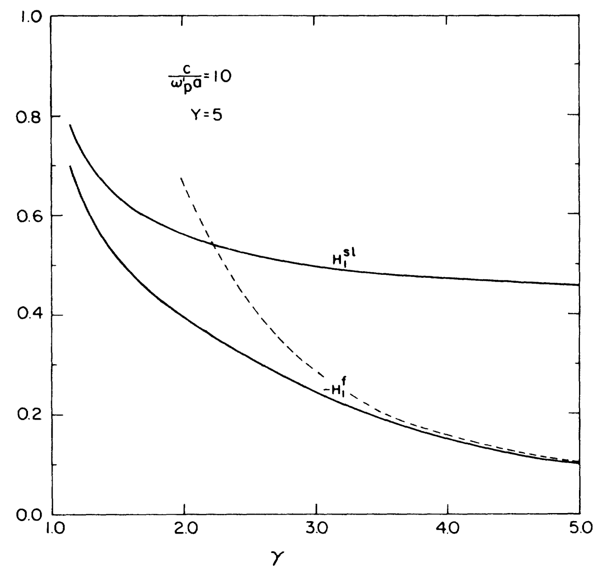


FIG. 5. The simplified reduction factors $H_1^{\rm sl}, H_1^f$ vs γ for a typical value of the width parameter. Note that the fast plasma-wave radiation factor is diminishing much faster than the factor of the slow plasma wave. The dashed line depicts the asymptotic expression of H_1^f [see Eq. (72)].

As a final remark for this section, we note that in the general case of multi (plasma) modes operation with a nonuniform beam in the limit of the tenuous beam region ($\theta_p \ll \pi$), the space-charge plasma waves should not play a role in the FEL interaction, and the gain-dispersion equation should reduce to Eq. (42) with $\theta_p \to 0$. We prove this assertion in Appendix B.

V. THREE-DIMENSIONAL PLASMA EFFECTS ON THE FEL OPERATING PARAMETERS

The application of the present theory to a practical FEL problem requires in general two steps: (a) solution of the beam eigenmodes and eigenvalues (beam mode dispersion relations) problem, and (b) solution of the FEL gain-dispersion relation. The second step may be carried out with various degrees of complexity. Considering the general case of a partially filled waveguide (or a finite beam in free space) we may carry out the second step in one of the three following ways:

(1) Exact solution of the set of Eqs. (64), (65), and (8). This approach considers all the beam modes and requires solution of the algebraic set (64a), substitution in (65) and (8), and consequently carrying out the inverse Laplace transformation.

(2) Solution of the approximate multimode gain-dispersion relation (67) by inverse Laplace transformation. Though all the beam modes are included, the validity of

the result depends on the validity of the simplifying approximation (66).

(3) Solution of the approximate single mode gain-dispersion relation (69). This only requires inverse Laplace transformation of the equation, which can be done numerically, and in many cases analytically, using the same computer codes and analytical approximations used to solve the 1D gain-dispersion equation.[11,19] As explained in the Introduction, this approximation will often be hard to satisfy because of lack of control over the beam transverse modes. Only if the wave numbers of the modes are sufficiently spaced to assure phase matching of the radiation mode to only one electrostatic mode will this approximation be valid. However, because of the simplicity of the final expression (69) and its similarity to the 1D expression (42), we use in the present chapter this form of the 3D plasma effect FEL gain-dispersion relation in a discussion of the 3D collective effects on the FEL operating parameter.

Since in the tenuous beam limit $\theta_p L \ll \pi$ the full 3D gain-dispersion equation reduces to the 1D expression (42), we need to consider the modification of the FEL gain parameters by plasma-3D effects only in the collective regimes. Starting with the low-gain collective regime, we Laplace-transform (69) by calculating the poles and residues of (69) in a first-order Taylor expansion in terms of κ_r.[11,19] Calculating the power input from $P(L)=[C(L)]^2$, one finds

$$\frac{\Delta P}{P(0)}=\kappa_r \theta_p^2 L^3 F(\bar{\theta},\bar{\theta}_{pr}) , \tag{73a}$$

$$F(\bar{\theta},\bar{\theta}_{pr})=\frac{1}{2\bar{\theta}_{pr}}\{\text{sinc}^2[(\bar{\theta}+\bar{\theta}_{pr})/2] - \text{sinc}^2[(\bar{\theta}-\bar{\theta}_{pr})/2]\} , \tag{73b}$$

where $\text{sinc}x=\sin x/x$, $\bar{\theta}=\theta L$, and $\bar{\theta}_{pr}=\theta_{pr}L$.

The maximum gain is attained at a reduced detuning parameter value $\bar{\theta}\simeq -\bar{\theta}_{pr}$ and is

$$\frac{\Delta P}{P(0)}=\frac{\kappa_r \theta_p^2 L^2}{2\theta_{pr}} . \tag{74}$$

If we may assume that the 3D effects on κ_r are small—$\kappa_r \approx \kappa$ and the major effect is on θ_{pr} (which is always reduced relative to θ_p)—we may conclude that the 3D effect is increasing the gain relative to the 1D model prediction.

In the high-gain regime the customary assumption allowing analytic solution of the gain-dispersion relation is synchronism with the slow beam plasma wave $\theta\simeq-\theta_{pr}$. Assuming further $|s| \ll 2\theta_{pr}$, the equation for the poles of expression (69) reduces to a quadratic equation. Keeping only the exponentially growing root, one finds[11,19]

$$\frac{P(L)}{P(0)}=\tfrac{1}{4}\exp[(2\kappa_r\theta_p^2/\theta_{pr})^{1/2}L] . \tag{75}$$

Again, under the assumption $\kappa_r \simeq \kappa$, the 3D effect will tend to increase the predicted gain.

Finally, we consider the 3D effect on the threshold condition for absolute instability backward-wave oscillation in a FEL structure, an effect which may set disrupting parasitic oscillations in the microwave regime in a FEL structure designed to operate at short (optical) wavelengths. As discussed in Ref. 15, the gain-dispersion equation of the absolute instability is given by the same dispersion equation (69) with the transformation $\kappa_r \to -\kappa_r$, $\theta \to -\theta$ and some minor changes of parameter definitions. Again the dispersion equation can be solved with the aid of the assumptions $\theta \simeq \theta_{pr}$, $|s| \ll 2\theta_{pr}$ resulting in a quadratic dispersion equation for the poles and yielding the following expression for the backward-going radiation field amplitude:

$$\frac{C_s(L)}{C_s(0)}=\cos[(\tfrac{1}{2}\kappa_r\theta_p^2/\theta_{pr})^{1/2}L] . \tag{76}$$

The threshold condition for oscillation in the lowest-order longitudinal mode is obtained from the requirement

$$(\tfrac{1}{2}\kappa_r\theta_p^2/\theta_{pr})^{1/2}L=\frac{\pi}{2} ,$$

which permits finite $C_s(0)$ for $C_s(L)=0$ (no input backward wave). This can be written as a condition for a threshold length

$$L_{\text{th}}=\pi\left[\frac{\theta_{pr}}{2\kappa_r\theta_p^2}\right]^{1/2} . \tag{77}$$

Again, with the assumption of $\kappa_r \simeq \kappa$, we may conclude that the 3D effect is to reduce the threshold for absolute instability oscillation eruption.

VI. CONCLUSIONS

In this work we developed a general 3D theory of the FEL in the collective regime which describes the FEL operation as a process of interaction between two kinds of modes: the waveguide radiation (electromagnetic) modes and the beam (plasma) modes. We developed the beam equation which describes the excitation of the beam modes by an electromagnetic mode of the waveguide, and we have shown that in the general case an infinite set of beam modes are excited during the interaction process.

We developed an approximate gain-dispersion equation of the FEL system which in the specific case of coupling to a single beam mode assumes the form of the known gain-dispersion relation of transversely infinite FEL's. It differs from the 1D dispersion relation by the introduction of the plasma reduction factor which modifies the plasma parameter θ_p, taking into account the finiteness of the beam. This reduction factor is similar to the one investigated nonrelativistically in the literature of microwave tubes.[20] It also differs in the expression for the coupling coefficient κ_r, which includes now the beam mode profile function in the coupled mode overlap integral.

Our analysis can be easily applied to three-dimensional generalization of many other scattering processes in the collective regime, like the forward Raman-scattering amplifiers and oscillators and the backward Raman-scattering oscillators.[15] It can be further extended and used to model the *collective* 3D aspects of the interaction in a comprehensive 3D analysis, which may include multiple radiation modes, optical guiding by the beam, and other 3D effects.

APPENDIX A

We derive here an orthogonality theorem for the beam modes, proving that they are orthogonal in the sense of Eq. (45).

Our starting point is the homogeneous beam equation:

$$\nabla_T^2\Phi+Q^2\left[\frac{g(\mathbf{r}_T)}{H^2}-1\right]\Phi=0 \; . \tag{A1}$$

Suppose we have found a set of solutions $\{\hat{\Phi}_n\}$ and a corresponding set of eigenvalues s_n (or ik_n). Let us write Eq. (A1) for the nth and mth solutions:

$$\nabla_T^2\hat{\Phi}_n+Q_n^2\left[\frac{g(\mathbf{r}_T)}{H_n^2}-1\right]\hat{\Phi}_n=0 \; , \tag{A2}$$

$$\nabla_T^2\hat{\Phi}_m+Q_m^2\left[\frac{g(\mathbf{r}_T)}{H_m^2}-1\right]\hat{\Phi}_m=0 \; . \tag{A3}$$

Multiplying Eq. (A2) by $\hat{\Phi}_m$ and Eq. (A3) by $\hat{\Phi}_n$, subtracting the second equation from the first, and integrating over the cross-sectional area of the waveguide, we find

$$\int\int(\hat{\Phi}_m\nabla_T^2\hat{\Phi}_n-\hat{\Phi}_n\nabla_T^2\hat{\Phi}_m)dx\,dy + \int\int\left[Q_n^2\left[\frac{g(\mathbf{r}_T)}{H_n^2}-1\right]-Q_m^2\left[\frac{g(\mathbf{r}_T)}{H_m^2}-1\right]\right] \times\hat{\Phi}_n\hat{\Phi}_m dx\,dy=0 \; . \tag{A4}$$

The potentials $\hat{\Phi}_n,\hat{\Phi}_m$ obey homogeneous boundary conditions on the waveguide walls:

$$\hat{\Phi}_n(\mathbf{r}_T)\,|_b=0 \; , \tag{A5}$$

where b represents the boundaries of the system. Thus by using Green's theorem (or integrating by parts) for the first integral of Eq. (A4), we find

$$\int\int\left[Q_n^2\left[\frac{g(\mathbf{r}_T)}{H_n^2}-1\right]-Q_m^2\left[\frac{g(\mathbf{r}_T)}{H_m^2}-1\right]\right] \times\hat{\Phi}_n\hat{\Phi}_m dx\,dy=0 \; , \tag{A6}$$

which is the orthogonality theorem given in Eq. (45).

In cases where a uniform beam completely fills the waveguide, we set $g(\mathbf{r}_T)=1$, then we define $k_{Tn}^2=Q_n^2(1/H_n^2-1)$ and we get from Eq. (A6):

$$(k_{Tn}^2-k_{Tm}^2)\int\int\hat{\Phi}_n\hat{\Phi}_m dx\,dy=0 \; , \tag{A7}$$

which is the orthogonality theorem of Eq. (32).

APPENDIX B

We prove here that in the limit of the tenuous beam region ($\theta_p \ll \pi$), Eq. (67), which describes the general case of multi (plasma) mode operation, reduces to the simpler Eq. (42) with $\theta_p\to 0$, while $\theta_p^2\kappa_r$ stays finite. We start by rewriting the orthogonality relation of Eq. (45) in the following way:

$$\left[\left[\frac{Q_n}{H_n}\right]^2-\left[\frac{Q_m}{H_m}\right]^2\right]\int\int g(\mathbf{r}_T)\hat{\Phi}_n(\mathbf{r}_T)\hat{\Phi}_m(\mathbf{r}_T)dx\,dy -(Q_n^2-Q_m^2)\int\int\hat{\Phi}_n(\mathbf{r}_T)\hat{\Phi}_m(\mathbf{r}_T)dx\,dy=0 \; . \tag{B1}$$

Assuming that $k_n \simeq K$ for all relevant n, then $Q_n\approx Q_m$, and one can approximate Eq. (B1) to the following reduced orthogonality relation:

$$\left[\left[\frac{Q_n}{H_n}\right]^2-\left[\frac{Q_m}{H_m}\right]^2\right]\int\int g(\mathbf{r}_T)\hat{\Phi}_n(\mathbf{r}_T)\hat{\Phi}_m(\mathbf{r}_T)dx\,dy =0 \; . \tag{B2}$$

This approximation can also be justified by noting that H_n^2 is always smaller than one and is close to zero for large n. We turn now to rewrite Eq. (67):

$$\frac{\bar{C}_s(s)}{C_s(0)}=\left[s-\frac{|C_w|^2\epsilon_0\theta_p^2}{8|\mathcal{P}_s|}M\right]^{-1} \; , \tag{B3}$$

where M is the sum written on the right-hand side of Eq. (67). Assuming that $\theta_p^2\hat{\alpha}_n \ll |i\theta-s|^2$, $|s|\ll K$, and that $k_n\simeq K$ for all relevant n, we find after some manipulations:

$$M=\frac{iK}{(i\theta-s)^2}\sum_n\frac{\left[\int\int g(\mathbf{r}_T)\hat{\Phi}_n(\mathbf{r}_T)\tilde{\mathbf{V}}_{Tw}(\mathbf{r}_T)\cdot\tilde{\mathcal{E}}_{Ts}^{*}(\mathbf{r}_T)dx\,dy\right]\left[\int\int g(\mathbf{r}_T)\tilde{\mathcal{E}}_{\rm pond}^{\omega_i}(\mathbf{r}_T)\hat{\Phi}_n(\mathbf{r}_T)dx\,dy\right]}{\hat{\alpha}_n}$$

or

$$M=\frac{iK}{(i\theta-s)^2}\int\int g(\mathbf{r}_T)\tilde{\mathbf{V}}_{Tw}(\mathbf{r}_T)\cdot\tilde{\mathcal{E}}_{Ts}^{*}(\mathbf{r}_T)\left[\sum_n\frac{\hat{\Phi}_n(\mathbf{r}_T)}{\sqrt{\hat{\alpha}_n}}\int\int\frac{\hat{\Phi}_n(\mathbf{r}_T)}{\sqrt{\hat{\alpha}_n}}g(\mathbf{r}_T)\tilde{\mathcal{E}}_{\rm pond}^{\omega_i}(\mathbf{r}_T)dx\,dy\right]dx\,dy \; . \tag{B4}$$

The sum in the parentheses is easily identified as a generalized Fourier expansion of $\tilde{\mathcal{E}}_{\rm pond}^{\omega_i}(\mathbf{r}_T)$ in terms of the orthogonal set $\hat{\Phi}_n(\mathbf{r}_T)/\sqrt{\hat{\alpha}_n}$ and the weighting function $g(\mathbf{r}_T)$; this can be proved by writing down the equality $\tilde{\mathcal{E}}_{\rm pond}^{\omega_i}=\sum_n a_n\hat{\Phi}_n(\mathbf{r}_T)/\sqrt{\hat{\alpha}_n}$, multiplying both sides by $g(\mathbf{r}_T)\hat{\Phi}_m(\mathbf{r}_T)/\sqrt{\hat{\alpha}_n}$, integrating over the cross-sectional area of the waveguide, and using the reduced orthogonality relation (B1). Thus (B4) reduces to

$$M=\frac{iK}{(i\theta-s)^2}\int\int g(\mathbf{r}_T)\tilde{\mathcal{E}}^{\omega_i}_{\text{pond}}(\mathbf{r}_T)\mathbf{\mathcal{V}}_{Tw}(\mathbf{r}_T)\cdot\tilde{\mathcal{E}}^{*}_{Ts}(\mathbf{r}_T)dx\,dy\ . \tag{B5}$$

Inserting (B5) into (B3) we find

$$\frac{\bar{C}_s(s)}{C_s(0)}=\frac{(i\theta-s)^2}{s(i\theta-s)^2-i\theta_p^2\kappa}\ ,$$
$$\kappa=\frac{|C_w|^2\epsilon_0}{8|\mathcal{P}_s|}K\int\int g(\mathbf{r}_T)\tilde{\mathcal{E}}^{\omega_i}_{\text{pond}}(\mathbf{r}_T)\mathbf{\mathcal{V}}_{Tw}(\mathbf{r}_T)\cdot\tilde{\mathcal{E}}^{*}_{Ts}(\mathbf{r}_T)dx\,dy\ . \tag{B6}$$

Equation (B6) is the well-known gain-dispersion equation of the tenuous beam FEL, and was derived here from Eq. (67). However, by using the asymptotic orthogonality relation (B2) (asymptotic in the sense that one should assume the limit $k_n \simeq K$ for every n) and following exactly the same analysis that was introduced above, one can reduce also the more complicated formulation of Eqs. (64) and (65) into the limiting case given in Eq. (B6).

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