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Gain analysis of a strong-pump FEL operating at the fundamental and high harmonics

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An analytical study of a planar wiggler FEL in the cold electron-beam strong-pump regime is presented. Such an FEL can lase at high odd harmonics due to the periodic variation in the longitudinal electron velocity, and can also emit superradiant emission at harmonic frequencies. Based on a parametric interaction approach, we develop an extended gain dispersion equation for a laser operating at the fundamental or high harmonics taking collective effects (Raman regime) into account. The analysis applies also to the case $a_w/(\beta\gamma) \sim 1$, where the customary use of the Taylor expansion of V_z does not apply. This case is of interest mostly for mildly relativistic FELs, and allows excitation of very high harmonics.

1. Introduction

In a basic free electron laser (FEL), the fundamental operation frequency is mainly determined by the axial velocity V_z of the electrons, which is often taken to be constant over the interaction region. When the pump strength is increased, the axial velocity becomes z-dependent, and periodic variation of half the wiggler's period emerges. Since the electron's axial velocity is modulated, its spontaneous emission radiation observed in the laboratory frame is also at higher odd harmonics besides the fundamental frequency.

Similar higher-harmonic operation can take place in stimulated emission and can be utilized in FEL amplifiers or oscillators [1]. This can help attaining short-wavelength FEL devices with moderate energy accelerators.

Previous analysis of the FEL operation at fundamental or high harmonics [2,3] was valid under the condition $a_w/(\beta\gamma) \ll 1$ which allows using a first-order Taylor approximation of $1/V_z(z)$, where $V_z(z)$ is the longitudinal z-dependent electron velocity in the wiggler. This approximation becomes inaccurate when the electron beam is not relativistic, as in strong-pumped microwave devices. Moreover, it may give a wrong estimation of the FEL operation frequencies and gain also at the fundamental harmonic.

The fact that the axial electron velocity in the wiggler is periodic in the z-direction enables expanding $1/V_z(z)$ as a Fourier series. This expansion applies for any $a_w/(\beta\gamma) \leq 1$. In this article we present a cold-beam FEL gain analysis using the Fourier expansion. An extended and corrected gain dispersion equation for the fundamental and harmonic frequencies is developed by using the parametric interaction approach described in ref. [4], and by assuming a fluid model for the electron beam.

2. The kinetics of an electron beam in a strong wiggler

We first solve for the electron velocity in the wiggler in the absence of the electromagnetic signal field. The signal strength is assumed to be much smaller relating to the wiggler. Hence its effect on the electron's axial velocity is negligible. We assume a magnetostatic planar wiggler with a periodic transverse magnetic induction field of period λ_w approximated by

$$\boldsymbol{B}_w(z) = \mathscr{B}_\perp^w \cos(k_w z). \tag{1}$$

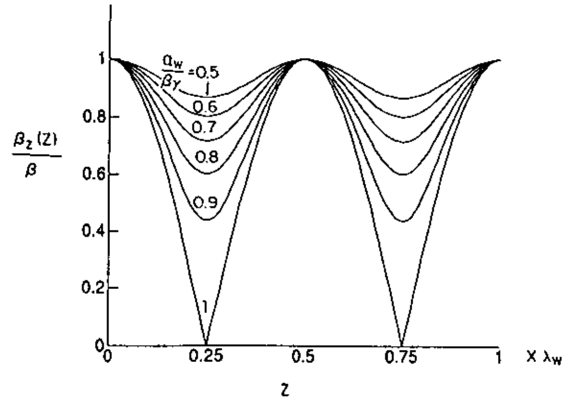

Fig. 1. The z-dependence of the axial velocity over a single wiggling period.

The transverse momentum of the electron produced by the Lorentz force $\boldsymbol{F}_\perp^w(z) = -eV_z(z) \times \boldsymbol{B}_\perp^w(z)$ is proportional to the magnetic potential vector $A_w(z)$ of the wiggler:

$$P_\perp^w(z) = -e\int^z B_w(z')\,dz' = -mca_w \sin(k_w z), \tag{2}$$

where m is the electron's rest mass, c is the velocity of light and $a_w \equiv e\mathscr{B}_\perp^w/(mck_w)$ is the wiggler parameter. From the last equation, we derive the electron transverse velocity:

$$V_\perp^w(z) = \mathscr{V}_\perp^w \ \sin(k_w z), \tag{3}$$

where $\mathscr{V}_\perp^w = -a_w c/\gamma$. The axial velocity is derived from the longitudinal momentum:

$$V_z(z) = \frac{1}{\gamma m}\sqrt{P^2 - P_\perp^2} = \beta c\left[1 - \left(\frac{a_w}{\beta\gamma}\right)^2 \sin^2(k_w z)\right]^{1/2}. \tag{4}$$

The z-dependence of the axial velocity over a single wiggling period is shown in fig. 1. Note that the axial velocity is modulated with a period that is half the period of the wiggler. This effect leads to the excitation of odd harmonics of the fundamental frequency. In order to calculate the gain at the harmonic frequencies, usually the inverse axial velocity

$$\frac{\beta}{\beta_z(z)} = \left[1 - \left(\frac{a_w}{\beta\gamma}\right)^2 \sin^2(k_w z)\right]^{-1/2} \tag{5}$$

is approximated by a Taylor expansion:

$$\frac{\beta}{\beta_z(z)} \approx \left[1 - \frac{1}{2}\left(\frac{a_w}{\beta\gamma}\right)^2\right]^{-1/2}\left[1 - \frac{1}{2}\,\frac{\frac{1}{2}(a_w/(\beta\gamma))^2}{1 - \frac{1}{2}(a_w/(\beta\gamma))^2}\cos(2k_w z)\right]. \tag{6}$$

This approximation is acceptable only if $a_w/(\beta\gamma) \ll 1$. However, this approximation is not necessary since what is required for the FEL gain calculation is the Fourier expansion of the inverse axial velocity,

$$\frac{\beta}{\beta_z(z)} = A_0 + 2\sum_{n=1}^{\infty} A_n \cos(2nk_w z), \tag{7}$$

and the expansion coefficients

$$A_n \equiv \frac{2}{\lambda_w}\int_0^{\lambda_w/2} \frac{\beta}{\beta_z(z)} e^{-j2nk_w z}\,dz \tag{8}$$

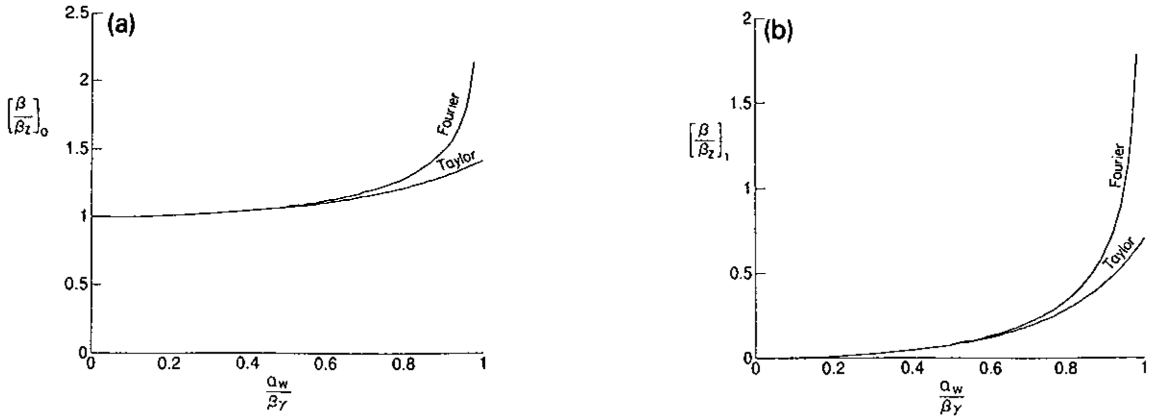

Fig. 2. Taylor and Fourier zeroth-order (a) and first-order (b) coefficient expansion of $\beta/\beta_z(z)$.

can be readily calculated numerically using an FFT algorithm. This expansion is accurate for any $a_w/(\beta\gamma) \leq 1$.

In the present work we use this more general model in order to apply the theory also to cases of nonrelativistic or moderately relativistic electrons.

Fig. 2 illustrates the difference between the Taylor and Fourier expansions of the zeroth $[\beta/\beta_z]_0$ and first $[\beta/\beta_z]_1$ order. The disparity between these two expansions becomes more severe as $a_w/(\beta\gamma)$ increases. The Taylor expansion is no longer a valid approximation of the Fourier expansion for those values.

3. The excitation equation

The signal electromagnetic wave is taken to be:

$$E_s(r, t) = \tfrac{1}{2}C_s(z)\tilde{\mathscr{E}}^s_\perp(x, y)\, e^{j(\omega_s t - k_s z)} + \text{c.c}, \tag{9}$$

$$B_s(r, t) = \tfrac{1}{2}C_s(z)\tilde{\mathscr{B}}^s_\perp(x, y)\, e^{j(\omega_s t - k_s z)} + \text{c.c}. \tag{10}$$

$C_s(z)$ is the slowly varying field amplitude of the signal wave. $\tilde{\mathscr{E}}^s_\perp(x,y)$ and $\tilde{\mathscr{B}}^s_\perp(x,y)$ are complex vectors representing the mode profile and polarization of the transverse electric and magnetic field, respectively. ω_s and k_s denote the angular frequency and the wave number of the signal wave, and satisfy the medium's dispersion relation.

From Maxwell's equations, assuming a small amplitude growth and neglecting the transverse variation of the mode profile of the electromagnetic wave over the electron-beam cross section, we get the excitation equation for the slowly varying signal amplitude:

$$\frac{dC_s(z)}{dz} = -\frac{1}{4\mathscr{P}_s} e^{jk_s z} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{J}^s_\perp(r)\cdot\tilde{\mathscr{E}}^{s*}_\perp(x, y)\, dx\, dy \cong -\frac{A_e}{4\mathscr{P}_s} e^{jk_s z}\tilde{J}^s_\perp(z)\cdot\tilde{\mathscr{E}}^{s*}_\perp(x_e, y_e), \tag{11}$$

where (x_e, y_e) are the transverse electron beam coordinates, and $\mathscr{P}_s$ is the normalized power of the mode given by

$$\mathscr{P}_s \equiv \tfrac{1}{2}\mathscr{R}\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\tilde{\mathscr{E}}^s_\perp(x, y)\times\tilde{\mathscr{H}}^s_\perp(x, y)\right]\cdot\hat{z}\, dx\, dy\right\} \cong \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\, A_{em}\,|\tilde{\mathscr{E}}^s_\perp(x_e, y_e)|^2. \tag{12}$$

The transverse phasor components of the current density $\tilde{J}^s_\perp(z)$ that would excite in eq. (11) the signal electromagnetic field radiation, are the result of the density modulation wave $\tilde{n}_i(z)$, generated by the "beating" of the signal and wiggler fields (the ponderomotive force), and a transverse velocity, eq. (3), caused by the wiggler:

$$\tilde{J}_{\perp}^{s}(z) = -e\tilde{n}_{i}(z)V_{\perp}^{w}(z) = -e\tilde{\mathscr{V}}_{\perp}^{w}\tilde{n}_{i}(z)\sin(k_{w}z). \tag{13}$$

Substitution of the last equation into the excitation equation (11) results in

$$\frac{dC_{s}(z)}{dz} \cong -j\frac{eA_{e}}{8\mathscr{P}_{s}}\tilde{\mathscr{V}}_{\perp}^{w}\cdot\tilde{\mathscr{E}}_{\perp}^{s*}(x_{e}, y_{e})\tilde{n}_{i}(z)[e^{j(k_{s}+k_{w})z} + e^{j(k_{s}-k_{w})z}]. \tag{14}$$

This equation describes the growth of the signal amplitude $C_s(z)$, which is observed to be associated with the spatial bunching of the electron beam $\tilde{n}_i(z)$. Gain is achieved when the phase velocity of this density wave, V_z, is approximately equal to the phase velocity of the "ponderomotive wave" $\omega_s/(k_s + k_w)$. This well known phase match condition determines the frequency at which the FEL amplifies.

The spatial periodic variation of the axial velocity causes space harmonics in the electron bunching wave that have different phase velocities. Therefore, phase match can be produced at several signal frequencies that are the odd harmonics of the fundamental.

4. The electron beam density bunching

Now we solve for the electron density bunching $\tilde{n}_i(z)$ generated by the ponderomotive axial force. Here the cold plasma moment equations for the electron beam are used. The longitudinal ac part of the current density is

$$\tilde{J}_{z}^{i}(z) = -e\left[n_{0}\tilde{v}_{z}^{i}(z) + V_{z}(z)\tilde{n}_{i}(z)\right], \tag{15}$$

where n_0 is the dc charge density, and $\tilde{v}_z^i(z)$ is the ac axial velocity modulation of the electron beam. After substituting eq. (15) into the continuity equation,

$$\frac{d\tilde{J}_{z}^{i}(z)}{dz} = j\omega_{s}e\tilde{n}_{i}(z), \tag{16}$$

we get a linear differential equation of the first order for the density bunching $\tilde{n}_i(z)$:

$$\frac{d}{dz}\tilde{n}_{i}(z) + \left\{\frac{j\omega_{s}}{V_{z}(z)} + \frac{d}{dz}[\ln V_{z}(z)]\right\}\tilde{n}_{i}(z) = -\frac{n_{0}}{V_{z}(z)}\frac{d\tilde{v}_{z}^{i}(z)}{dz}. \tag{17}$$

This equation can be solved analytically, resulting in

$$\tilde{n}_{i}(z) = \frac{n_{0}}{V_{z}(z)}\prod_{p=1}^{\infty}\sum_{m=-\infty}^{+\infty}J_{m}\left(\frac{\alpha_{p}}{p}\right)e^{j(2mpk_{w}-(\omega_{s}A_{0}/(\beta c)))z}$$
$$\times\int_{0}^{z}\left[\frac{d\tilde{v}_{z}^{i}(x)}{dx}\prod_{q=1}^{\infty}\sum_{n=-\infty}^{+\infty}(-1)^{n}J_{n}\left(\frac{\alpha_{q}}{q}\right)e^{j(2nqk_{w}+(\omega_{s}A_{0}/(\beta c)))x}\right]dx, \tag{18}$$

where $\alpha_p \equiv \omega_s A_p/(\beta k_w c)$ and A_p are the Fourier coefficients given in eq. (8). The Bessel functions are defined by

$$J_{m}\left(\frac{\alpha_{p}}{p}\right) \equiv \frac{2p}{\lambda_{w}}\int_{0}^{\frac{\lambda_{w}}{2p}} e^{-j(\alpha_{p}/p)\sin(2pk_{w}z)}\,e^{-j2mpk_{w}z}\,dz. \tag{19}$$

Explicit solution of the density bunching $\tilde{n}_i(z)$ requires substitution of the axial velocity $\tilde{v}_z^i(z)$ in eq. (18). This can be found from the longitudinal force equation,

$$\left[\frac{d}{dz}+\frac{j\omega_s}{V_z(z)}\right]\left[\gamma m\tilde{v}_z^i(z)\right]=\frac{\tilde{F}_z^i(z)}{V_z(z)}. \tag{20}$$

The longitudinal force $\tilde{F}_z^i(z)$ is a sum of two forcing terms,

$$\tilde{F}_z^i(z)=-e\left[\tilde{E}_{pond}(z)+\tilde{E}_{sc}(z)\right], \tag{21}$$

The ponderomotive field,

$$\tilde{E}_{pond}(z)=\tilde{\mathscr{E}}_{pm}C_s(z)\left[e^{-j(k_s+k_w)z}+e^{-j(k_s-k_w)z}\right], \tag{22}$$

where $\tilde{\mathscr{E}}_{pm}=\frac{1}{2}[\tilde{\mathscr{V}}_\perp^s\times\tilde{\mathscr{B}}_\perp^w+\tilde{\mathscr{V}}_\perp^w\times\tilde{\mathscr{B}}_\perp^s]$, and the longitudinal space charge field which is caused by the density modulation and can be found by solving the Poisson equation:

$$\frac{d\tilde{E}_{sc}(z)}{dz}=-\frac{e}{\epsilon_0}\tilde{n}_i(z). \tag{23}$$

Eq. (20) is a first-order differential equation that is solved much like eq. (17):

$$\tilde{v}_z^i(z)\cong\frac{1}{\gamma_z^2\gamma m}\prod_{p=1}^{\infty}\sum_{m=-\infty}^{+\infty}J_m\left(\frac{\alpha_p}{p}\right)e^{j(2mpk_w-(\omega_sA_0/(\beta c)))^2}$$
$$\times\int_0^z\left[\frac{\tilde{F}_z^i(x)}{V_z(x)}\prod_{q=1}^{\infty}\sum_{n=-\infty}^{+\infty}(-1)^nJ_n\left(\frac{\alpha_q}{q}\right)e^{j(2nqk_w+(\omega_sA_0/(\beta c)))x}\right]dx, \tag{24}$$

where $\gamma_z\equiv[1-\beta_z^2]^{-1/2}\approx[1-(\beta/A_0)^2]^{-1/2}$.

The complete set of equations which govern the interaction, consists of eqs. (14), (18) and (21)–(24). In the next section we derive the gain dispersion relation after a linearization procedure that enables using the Laplace transformation.

5. Gain dispersion relation

Eqs. (18) and (24) can be simplified by assuming:

a) $J_0(\alpha_p/p)\simeq 1 \qquad \forall p>1;$

b) $J_m(\alpha_p/p)\simeq 0 \qquad \forall m>0, \quad p>1;$

c) $1/V_z(z)\approx A_0/(\beta c)\equiv 1/V_{z0}.$

Using these approximations and taking $n=p=0$ the Laplace transform of the density bunching is singled out:

$$\tilde{n}_i(s)=J_0^2(\alpha_1)\theta_p^2\frac{\epsilon_0\tilde{\mathscr{E}}_{pm}}{e}\times\left[\left\{\sum_{m=-\infty}^{+\infty}\sum_{q=-\infty}^{+\infty}J_m(\alpha_1)J_q(\alpha_1)(-1)^q\frac{s-j2mk_w}{\left[s-j(2mk_w-(\omega_sA_0/(\beta c)))\right]^2}\right\}\right.$$
$$\left.\Bigg/\left\{1+J_0^2(\alpha_1)\theta_p^2\sum_{n=-\infty}^{+\infty}\frac{J_n^2(\alpha_1)(s-j2nk_w)}{s\left[s-j(2nk_w-(\omega_sA_0/(\beta c)))\right]^2}\right\}\right]$$
$$\times\left\{C_s\left[s+j(k_s+k_w-2(m+q)k_w)\right]+C_s\left[s+j(k_s-k_w-2(m+q)k_w)\right]\right\}, \tag{25}$$

where θ_p is the relativistic longitudinal space charge parameter defined by: $\theta_p^2 \equiv \omega_p^2/V_{z0}^2 = n_0 e^2/(\gamma_z^2 \gamma m \epsilon_0 V_{z0}^2)$.

Since we are interested to evaluate the FEL gain at one of the harmonic frequencies, and because it is usually possible to obtain a near-resonance operation with a single harmonic, it can be assumed that the appropriate detuning parameter $|\theta_{2m+1}| \equiv |k_s + (2m+1)k_w - (\omega_s A_0/(\beta c))| \ll |\theta_{2n+1}|$ $\forall m \neq n$. Substituting eq. (26) into the Laplace transform of the electromagnetic excitation equation (14), the gain dispersion relation for the $(2m+1)$ harmonic frequency is derived:

$$\left[\frac{C_s(s)}{C_s(z=0)}\right]_{2m+1} = \left\{ s - \mathrm{j}\frac{\kappa}{k_s + k_w} J_0^2(\alpha_1)\theta_p^2 [J_m(\alpha_1) - J_{m+1}(\alpha_1)][s - \mathrm{j}(k_s + (2m+1)k_w)] \right.$$
$$\times \left[\frac{J_m(\alpha_1)[s - \mathrm{j}(k_s + k_w)]}{(s - \mathrm{j}(k_s + k_w))(s - \mathrm{j}\theta_{2m+1})^2 + J_0^2(\alpha_1)\theta_p^2[s - \mathrm{j}(k_s + (2m+1)k_w)]} \right.$$
$$\left.\left. - \frac{J_{m+1}(\alpha_1)[s - \mathrm{j}(k_s - k_w)]}{(s - \mathrm{j}(k_s - k_w))(s - \mathrm{j}\theta_{2m+1})^2 + J_0^2(\alpha_1)\theta_p^2[s - \mathrm{j}(k_s + (2m+1)k_w)]} \right] \right\}^{-1}. \tag{26}$$

When $k_s \gg (2m+1)k_w$, the gain dispersion relation can be reduced to

$$\left[\frac{C_s(s)}{C_s(z=0)}\right]_{2m+1} = \left\{ s - \mathrm{j}\kappa J_0^2(\alpha_1)\theta_p^2 [J_m(\alpha_1) - J_{m+1}(\alpha_1)]^2 \right.$$
$$\left. \times \frac{(s - \mathrm{j}\theta_{2m+1})^2 + J_m(\alpha_1)J_{m+1}(\alpha_1)J_0^2(\alpha_1)\theta_p^2}{\left[(s - \mathrm{j}\theta_{2m+1})^2 + J_m^2(\alpha_1)J_0^2(\alpha_1)\theta_p^2\right]\left[(s - \mathrm{j}\theta_{2m+1})^2 + J_{m+1}^2(\alpha_1)J_0^2(\alpha_1)\theta_p^2\right]} \right\}^{-1}. \tag{27}$$

The last result agrees with the cold beam limit of the kinetic linear model solution for a relativistic wiggle velocity given in ref. [3].

References

[1] J.M.J. Madey and R.C. Taber, Physics of Quantum Electronics, vol. 7, eds. S. Jacobs et al. (Addison-Wesley, Reading, 1980) chap. 30.
[2] W.B. Colson, IEEE J. Quantum Electron. QE-17 (1981) 1417.
[3] E. Jerbi and A. Gover, Nucl. Instr. and Meth. A250 (1982) 192.
[4] A. Gover, Opt. Commun. 45 (4) (1983) 281.