

Transverse mode excitation and coupling in a waveguide free electron laser

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An analysis of excitation, gain-coupling and evolution of transverse electromagnetic modes in a waveguide free electron laser is presented. We use a formalism of coupled-mode analysis to find the gain-dispersion equation of the coupled waveguide modes and diagonalize the system to find the eigenmodes ("supermodes") of the coupled system and their gain.

We demonstrate the use of the method to find the mutual gain of the coupled TE_{21} TM_{21} modes which are degenerate. A noteworthy finding is that the gain of the supermode is higher than the gain of either of the separate waveguide modes when calculated in a model where coupling is neglected.

The analytically calculated results are compared to the results of a numerical 3-D FEL code.

1. Introduction

The extended one-dimensional linear model of the free electron laser (FEL) is a useful and quite satisfactory model to describe the interaction between the electromagnetic wave and the electron beam in this device [1–5]. This model assumes a single electromagnetic transverse mode and a finite cross section of the electron beam. The coupling between the radiation wave and the beam is proportional to the "power filling factor", defined as the ratio between the beam cross section area and the effective electromagnetic mode area.

The small signal gain analysis of a variety of FEL schemes leads in the framework of the extended 1-D model, to the same gain-dispersion relation [6,7]. This relation is valid for warm- and cold-electron-beam effects, and covers both the FEL low- and high-gain, single-particle (Compton) and collective (Raman) regimes.

In optical open resonators and overmoded waveguide resonators a more adequate 3-D model of the FEL interaction is necessary. Such a model for a FEL operating in the nonlinear regime was suggested in [8–10]. In this and other analysis [11–13], the transverse electromagnetic field was expanded in terms of the free-space Hermite–Gaussian modes.

A self-consistent theory which includes effects of finite cylindrical waveguide geometry was presented in ref. [14]. A simplified model was proposed in ref. [15]. In a linear 3-D model developed in refs. [16–18], the

transverse dependence of the electromagnetic field is represented by a Fourier series of plane waves leading to a gain-dispersion matrix.

In this paper we employ a coupled-mode analysis for solving the excitation and evolution of transverse modes in the FEL and calculating the eigenmodes of the coupled system. Here the electromagnetic wave is expanded by the transverse eigenmodes which are coupled through the e-beam current. Taking a parametric interaction approach, we derive a coupled set of gain-dispersion equations which describes the FEL interaction in the linear small-signal regime. The system is diagonalized to find the normal modes ("supermodes") of the FEL, and calculate their gain. This general analysis can be adopted for any waveguide or free-space FEL scheme.

The analysis given here relates to the case of coupled transverse modes that are degenerate in their longitudinal wavenumber k_z .

2. The excitation equation

The transverse electric and magnetic fields can be expressed in terms of forward (+q) and backward (−q) going waveguide eigenmodes [19–21]:

$$E_{\perp}(r, t) = \frac{1}{2} \sum_q C_{+q}(z) \tilde{E}_{\perp+q}(x, y) e^{+j(k_{zq}z - \omega_s t)} + C_{-q}(z) \tilde{E}_{\perp-q}(x, y) e^{-j(k_{zq}z - \omega_s t)} + \text{c.c.}, \quad (1)$$

$$\begin{aligned}
H_{\perp}(r, t) = & \frac{1}{2} \sum_q C_{+q}(z) \tilde{\mathcal{E}}_{\perp+q}(x, y) e^{+j(k_{zq}z - \omega_q t)} \\
& + C_{-q}(z) \tilde{\mathcal{E}}_{\perp-q}(x, y) e^{-j(k_{zq}z - \omega_q t)} \\
& + \text{c.c.}, \quad (2)
\end{aligned}$$

where $C_q(z)$ is the slowly varying amplitude of waveguide mode q , $\tilde{\mathcal{E}}_{\perp+q}(x, y)$ and $\tilde{\mathcal{E}}_{\perp-q}(x, y)$ are complex vectors representing the transverse profile and polarization of the electric and magnetic fields of the mode, ω_q and k_{zq} denote the angular frequency and wave-number of mode q , and satisfy the dispersion relation $k_{zq} = [(\omega_q/c)^2 - k_{\perp q}^2]^{1/2}$.

The total power transfer in the electromagnetic wave is given by:

$$P(z) = \sum_q |C_{+q}(z)|^2 \mathcal{P}_{+q} + |C_{-q}(z)|^2 \mathcal{P}_{-q}, \quad (3)$$

where \mathcal{P}_q is the normalized power of transverse mode q :

$$\begin{aligned}
\mathcal{P}_{+q} &= -\mathcal{P}_{-q} \\
&= \frac{1}{2} \text{Re} \int \int_{\text{c.s.}} [\tilde{\mathcal{E}}_{\perp+q}(x, y) \times \tilde{\mathcal{E}}_{\perp+q}^*(x, y)] \cdot \hat{z} \, dx \, dy. \quad (4)
\end{aligned}$$

The excitation equation for the slowly varying amplitude of waveguide mode q can be derived from the Maxwell equations by imposing the boundary conditions of the waveguide walls:

$$\frac{dC_q(z)}{dz} = -\frac{1}{4\mathcal{P}_q} e^{-jk_{zq}z} \int \int \tilde{J}(r) \cdot \tilde{\mathcal{E}}_q^*(x, y) \, dx \, dy. \quad (5)$$

The last equation is an exact formula describing the evolution of traveling waveguide modes excited by the current density $\tilde{J}(r)$ along the interaction region.

In a FEL, the transverse phasor component of the current density $\tilde{J}_{\perp}^s(r)$ is a product of the density modulation wave $\tilde{n}_i(r)$ and the wiggling transverse velocity $\tilde{\mathcal{V}}_{\perp}^w(r)$:

$$\tilde{J}_{\perp}^s(r) = -\frac{1}{2} e \tilde{n}_i(r) \tilde{\mathcal{V}}_{\perp}^w(x, y) e^{-jk_w z}. \quad (6)$$

Substituting the last equation into the excitation equation (eq. (5)) results in:

$$\begin{aligned}
\frac{dC_q(z)}{dz} &= \frac{e}{8\mathcal{P}_q} e^{-j(k_{zq} + k_w)z} \int \int \tilde{n}_i(r) \\
&\times [\tilde{\mathcal{V}}_{\perp}^w(x, y) \cdot \tilde{\mathcal{E}}_{\perp+q}^*(x, y)] \, dx \, dy. \quad (7)
\end{aligned}$$

The amplitude growth of a waveguide mode q , is observed to be associated with an overlap integral between its transverse profile $\tilde{\mathcal{E}}_{\perp+q}^*(x, y)$ and the spatial space charge density modulation $\tilde{n}_i(r)$. The density bunching will be calculated from the electron-beam fluid equations.

3. The electron-beam fluid equations

In a FEL, the velocity modulation and consequently the space charge density modulation are generated by the longitudinal ponderomotive force. The whole process is described by the linearized plasma moment equations. In a small-signal analysis, the ac perturbations are assumed to oscillate at a single angular frequency $\omega_i \equiv \omega_s - \omega_w$. This takes into account also an electromagnetic pump FEL scheme where ω_w is the angular frequency of the wiggler wave. For a magnetostatic wiggler $\omega_w = 0$. We also assume that there is no change in the electron-beam cross section profile during the interaction. Thus, in the following equations only the development along the z direction is considered. The longitudinal ac part of the current density is given by:

$$\tilde{J}_z^i(z) = -e [n_0 \tilde{v}_z^i(z) + V_{z0} \tilde{n}_i(z)]. \quad (8)$$

Here n_0 is the dc charge density, and $\tilde{v}_z^i(z)$ is the axial velocity modulation of the electron beam. From the continuity equation we get:

$$\frac{d\tilde{J}_z^i(z)}{dz} = -j\omega_i e \tilde{n}_i(z). \quad (9)$$

After substituting eq. (8) into the continuity equation (eq. (9)), a linear differential equation for the density bunching is derived:

$$\frac{d}{dz} \tilde{n}_i(z) - j \frac{\omega_i}{V_{z0}} \tilde{n}_i(z) = -\frac{n_0}{V_{z0}} \frac{d\tilde{v}_z^i(z)}{dz}. \quad (10)$$

The velocity modulation $\tilde{v}_z^i(z)$ which is required in the last equation can be found from the small-signal form of the axial force equation:

$$\begin{aligned}
\frac{d}{dz} \tilde{v}_z^i(z) - j \frac{\omega_i}{V_{z0}} \tilde{v}_z^i(z) \\
= -\frac{e}{\gamma_z^2 \gamma m V_{z0}} [\tilde{E}_{\text{pond}}(z) + \tilde{E}_{\text{sc}}(z)]. \quad (11)
\end{aligned}$$

Each waveguide mode independently interacts with the transverse components of the electron velocity to produce a combined longitudinal ponderomotive field:

$$\tilde{E}_{\text{pond}}(z) = \sum_q C_q(z) \tilde{\mathcal{E}}_q^{\text{pm}}(x, y) e^{+j(k_{zq} + k_w)z}, \quad (12)$$

where $\tilde{\mathcal{E}}_q^{\text{pm}} \equiv \frac{1}{2} [\tilde{\mathcal{V}}_{\perp+q}^s \times \tilde{\mathcal{B}}_{\perp+q}^{w*} + \tilde{\mathcal{V}}_{\perp+q}^{w*} \times \tilde{\mathcal{B}}_{\perp+q}^s] \cdot \hat{z}$.

The space-charge field $\tilde{E}_{\text{sc}}(z)$, which is caused by the density modulation, can be found by solving the Poisson equation:

$$\frac{d\tilde{E}_{\text{sc}}(z)}{dz} = -\frac{e}{\epsilon_0} \tilde{n}_i(z). \quad (13)$$

The FEL interaction is fully described by the linear set of eq. (7) and the cold-beam plasma fluid equations (eqs. (10)–(13)).

4. The coupled-mode dispersion equation

After some extensive algebraic steps, the FEL interaction equations can be reduced to a set of linear differential equations of the third order for the slowly varying amplitude of mode q :

$$\begin{aligned} \frac{d^3}{dz^3} C_q(z) - 2j\theta_q \frac{d^2}{dz^2} C_q(z) + (\theta_p^2 - \theta_q^2) \frac{d}{dz} C_q(z) \\ = \sum_{q'} Q_{qq'} C_{q'}(z) e^{-j\Delta k_{zqq'} z}, \end{aligned} \quad (14)$$

Here we use the following parameter notations.

(a) Detuning parameter:

$$\theta_q \equiv \frac{\omega_s}{V_{z0}} - (k_{zq} + k_w).$$

(b) Space charge parameter:

$$\theta_p^2 \equiv \frac{\omega_p^2}{V_{z0}^2} = \frac{n_0 e^2}{\gamma_z^2 \gamma m \epsilon_0 V_{z0}^2}.$$

(c) Gain parameter: $Q_{qq'} \equiv \kappa_{qq'} \theta_p^2$, where the coupling parameter is given by

$$\begin{aligned} \kappa_{qq'} = \frac{\epsilon_0}{8\mathcal{P}_q} (k_{zq} + k_w) \iint f(x, y) \tilde{\mathcal{E}}_q^{pm}(x, y) \\ \times [\tilde{\mathcal{E}}_1^w(x, y) \cdot \tilde{\mathcal{E}}_1^*(x, y)] dx dy, \end{aligned} \quad (15)$$

and $f(x, y)$ is the e-beam transverse profile.

(d) $\Delta k_{zqq'} \equiv k_{zq} - k_{zq'}$.

Assuming that there are no prebunching effects, i.e. $\tilde{n}_i(z=0) = \tilde{v}_z^i(z=0) = \tilde{E}_{sc}(z=0) = 0$, one can derive the gain-dispersion equation for the q th transverse mode from a Laplace transformation in the z variable of the FEL interaction equations:

$$\begin{aligned} \mathcal{E}_q(s) = G_{qq}(s) C_q(z=0) \\ + \sum_{q' \neq q} G_{qq'}(s) C_{q'}(s + j\Delta k_{zqq'}), \end{aligned} \quad (16)$$

where $G_{qq}(s)$ is the well known single-mode gain-dispersion relation developed previously in refs. [6,7]:

$$G_{qq}(s) \equiv \frac{(s - j\theta_q)^2 + \theta_p^2}{s[(s - j\theta_q)^2 + \theta_p^2] - Q_{qq}}. \quad (17)$$

It describes amplitude growth and phase development of the q th-order mode due to self-excitation. The mutual interaction between the q th-order mode and another $q' \neq q$ mode which is excited by it, is expressed by:

$$G_{qq'}(s) \equiv \frac{Q_{qq'}}{s[(s - j\theta_q)^2 + \theta_p^2] - Q_{qq}}. \quad (18)$$

The evolution of each transverse mode along the interaction is obtained from the solution of the set of

coupled mode equations (eq. (16)). However, in order to find the FEL gain at a certain frequency, it is not necessary to solve for all the waveguide modes. Only modes that are nearly phase matched to each other interact efficiently, and need to be taken into account in the coupled-mode gain calculation. These modes can be identified by inspection of their single mode gain curves and observation of overlap at some frequencies. In a case of a finite set of modes, the coupled-mode dispersion equations (eq. (16)) can be presented in a compact matrix form:

$$\mathcal{E}(s) = \Gamma(s) C(z=0). \quad (19)$$

The amplitude growth and phase development of the entire radiation field is thus expressed in terms of the initial values $C(z=0)$ of the transverse mode expansion at the entrance to the interaction region, and a gain-dispersion matrix $\Gamma(s)$.

Note that the free-space or waveguide modes are not in general the normal modes of the FEL system. Namely, if one starts with a certain transverse mode at $z=0$, it does not keep its transverse electromagnetic field profile and polarization along the interaction length. One can look for a new set of independent modes that are eigenmodes of the FEL system, and which are characterized by the feature that except for magnitude and phase, their field profiles and polarization do not change along the propagation coordinate z . These eigenmodes of the coupled system would be the steady state eigenmodes of the FEL oscillator if the resonator mirrors do not scatter the transverse modes to each other, or reflect them discriminatively.

5. The FEL "supermodes" – degenerate case

The derivation of the FEL normal modes consists of finding the eigensolutions of coupled differential equations (eq. (14)). Standard procedures for coupled-mode analysis can be utilized for this purpose [22]. However, since these standard methods were developed to solve a coupled first-order set of equations, it helps to transform the set of third-order differential equations, which describes the FEL modes excitation, into a new set of coupled first-order equations by definition of state variables [23].

For simplicity we limit our attention in the present paper to the special case of coupled waveguide modes which are degenerate in their longitudinal wave number k_{zq} . This can be of interest, for instance, in a FEL based on a rectangular waveguide where a number of excited waveguide modes may be degenerate, namely have the same wavenumber k_{zq} and consequently have the same detuning parameter θ . (There is always degeneracy between the TE and TM modes; if the wave-

guide cross section dimensions have integral ratios there can be many more degenerate modes).

In the case of degenerate modes we may skip the step of defining state variables and start from the third-order coupled differential equations set (eq. (14)). It can be written in a simple matrix form:

$$\frac{d^3}{dz^3} C(z) - 2j\theta \frac{d^2}{dz^2} C(z) + (\theta_p^2 - \theta^2) \frac{d}{dz} C(z) = \mathbf{Q} C(z). \quad (20)$$

The mode coupling is expressed by the matrix \mathbf{Q} , which consists of gain parameters $Q_{qq'}$, defined previously.

In every waveguide cross section, any FEL system normal mode can be written as a superposition of the uncoupled modes. The two representations can be related at each point through a linear matrix transformation:

$$C(z) = \mathbf{T} U(z). \quad (21)$$

This transformation together with eq. (20) is used to derive a new set of differential equations for the slowly varying amplitudes of the supermodes $U_i(z)$:

$$\frac{d^3}{dz^3} U(z) - 2j\theta \frac{d^2}{dz^2} U(z) + (\theta_p^2 - \theta^2) \frac{d}{dz} U(z) = \mathbf{T}^{-1} \mathbf{Q} \mathbf{T} U(z). \quad (22)$$

If the similarity transformation $\mathbf{T}^{-1} \mathbf{Q} \mathbf{T}$ produces a diagonal matrix on the left side of the above equation, eq. (22) is a complete set of uncoupled equations and $U_i(z)$ is said to be the slowly varying amplitude of the i th FEL supermode. The diagonal matrix elements are the eigenvalues λ_i of the gain parameter matrix \mathbf{Q} and fulfill the algebraic equation: $|\mathbf{Q} - \lambda \mathbf{I}| = 0$. The column vectors \mathbf{t}_i in the matrix \mathbf{T} are the eigenvectors which satisfy the relation $\mathbf{Q} \mathbf{t}_i = \lambda_i \mathbf{t}_i$.

The dispersion relation for the slowly varying amplitude of the normal mode i is found after a Laplace transformation of the last equation. Noting that if there are no prebunching effects, the initial conditions at the entrance of the FEL interaction region are

$$\frac{d^2}{dz^2} [U_i(z=0)] = \frac{d}{dz} [U_i(z=0)] = 0,$$

the gain-dispersion relation is directly found:

$$\Lambda_i(s) = \frac{\mathcal{Z}_i(s)}{U_i(z=0)} = \frac{(s - j\theta)^2 + \theta_p^2}{s[(s - j\theta)^2 + \theta_p^2] - \lambda_i}. \quad (23)$$

Thus for the case of degenerate coupled modes, the dispersion relation (eq. (23)) for the FEL normal modes resembles the single-mode gain-dispersion equation (eq. (17)) except for the gain parameters λ_i which are in this case the eigenvalues of matrix \mathbf{Q} .

6. Two-mode coupling

We demonstrate the coupled-mode formalism described in the previous section on a waveguide FEL in which only two modes are excited. In this case the set of eq. (20) consists of two equations which are coupled through a 2×2 gain parameter matrix \mathbf{Q} , and two supermodes need to be identified. First the eigenvalues of the matrix \mathbf{Q} are found from a quadratic determinantal equation:

$$\lambda_{1,2} = \frac{1}{2} \left[Q_{11} + Q_{22} \pm \sqrt{(Q_{11} - Q_{22})^2 + 4Q_{12}Q_{21}} \right]. \quad (24)$$

These are the "gain parameters" of the FEL normal modes needed in the expression for the gain-dispersion relation (eq. 23)) of the normal modes.

The relation between the slowly varying amplitudes of the uncoupled modes and the FEL normal modes is expressed by the transformation \mathbf{T} which contains the eigenvectors of \mathbf{Q} in its columns:

$$\begin{bmatrix} C_1(z) \\ C_2(z) \end{bmatrix} = \begin{bmatrix} Q_{12} & \lambda_2 - Q_{22} \\ \lambda_1 - Q_{11} & Q_{21} \end{bmatrix} \begin{bmatrix} U_1(z) \\ U_2(z) \end{bmatrix}. \quad (25)$$

Note that one element of each eigenvector is determined arbitrarily. This leaves some latitude of choice in the linear waveguide mode combination to make up the normal-mode solutions.

In certain cases (e.g. when the electron beam width is narrow relative to the profiles of the modes) the relation $|\mathbf{Q}| \equiv Q_{11}Q_{22} - Q_{12}Q_{21} = 0$ is satisfied (see eq. (15)). In this case $\lambda_1 = Q_{11} + Q_{22}$ and $\lambda_2 = 0$. In general we obtain the important result that one of the supermodes has gain that is higher than that of either separate waveguide modes when calculated in a single-mode gain analysis.

7. Numerical results

The numerical gain calculations presented here are of the millimeter wave free-electron maser (FEM) proposed by FOM for thermonuclear fusion [24,25]. This FEM is based on an electrostatic accelerator and designed to attain 1 MW cw from a 2 MeV, 10–20 A electron beam at a frequency band of 150–300 GHz.

The rf cavity is an overmoded rectangular waveguide in which several transverse modes may be excited. Since the FEM utilizes a magnetostatic planar wiggler ($\lambda_w = 4$ cm), only modes with an electric field component in the transverse wiggling transverse direction will be excited. The contribution of the mode to the interaction is also determined by its field intensity at the center of the waveguide where the electron beam passes. For the case of a narrow beam, modes

that are null at the center of the waveguide may not be excited effectively. The modes found to be within the frequency range of operation of the FOM-FEM and exhibit substantial gain are the TE_{01} , TE_{21} and TM_{21} modes.

In a rectangular waveguide with cross section dimensions $a \times b$, transverse modes TE_{mn} , TM_{mn} are degenerate in their longitudinal wave number k_{zmn} if they have the same $k_{\perp mn} = [(m\pi/a)^2 + (n\pi/b)^2]^{1/2}$. This points out that the TE_{21} and TM_{21} modes will operate at the same frequency and they can both be excited simultaneously. Moreover, these modes will be coupled to each other by the e-beam of finite cross section. It is thus necessary to use a coupled-mode theory for solving for the accurate gain in the frequency domain where they exhibit gain.

For the parameters of the FEL listed above, it is found that all other waveguide modes are far from being phase matched and their gain curves do not overlap. Non-synchronous modes do not contribute to the interaction with the e-beam and need not to be taken into account in the coupled-mode analysis.

Fig. 1 illustrates the gain curves of the TE_{01} , TE_{21} and TM_{21} waveguide modes as a function of the operating frequency. The results of the single mode gain calculations for modes TE_{21} and TM_{21} are given in dashed lines and their resultant supermode gain calculated from the coupled mode formalism is shown in a continuous line. Higher-order modes correspond to operating frequencies below the range of the present

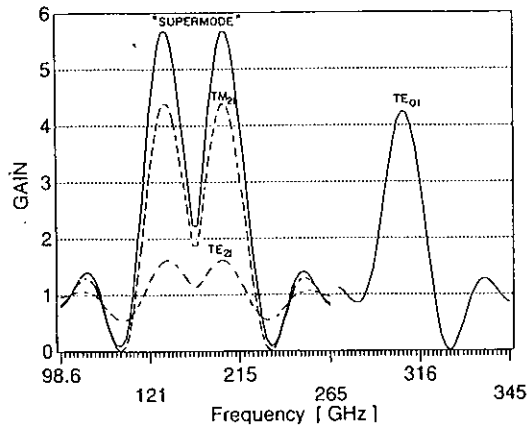


Fig. 1. Gain curves of transverse modes in FOMs free electron maser (FEM).

design. Evidently, substantially higher gain is attained in the more accurate coupled-mode model.

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