Computer Arithmetic - Spring 1998

Dr. Guy Even

Dept. of Electrical Engineering - Systems

Lecture of June 4, 1998

Written by Roman Tsikel,

e-mail: romant@nexus.co.il

Rounding Computation

Introduction:

In this lecture next concepts will be defined and discussed:

- *p-equivalent* numbers,
- *p-representatives*,
- *sticky-bit* and *sticky*,
- Rounding unit will be presented,
- Addition algorithm will be discussed.

Definition: Two real numbers x and y are *p-equivalent* (שקולים - p) if:

1. $x = y$ or
2. $x, y \in (q \cdot 2^{-p}, [q+1] \cdot 2^{-p})$ for any integer q ($q \in Z$).

This denoted as $x \stackrel{p}{=} y$.

In other words, binary representation of two p-equivalent numbers is the same at least for p positions to the right of the binary point.

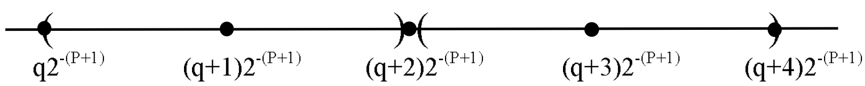
For every group of p-equivalent real numbers, let's define *p-representative* (נציג) as follows:

$$\mathrm{rep}_p(x) = \begin{cases} q \cdot 2^{-p} & \text{if } x = q \cdot 2^{-p} \\ (q + \tfrac{1}{2}) \cdot 2^{-p} & \text{if } x \in (q \cdot 2^{-p}, [q+1] \cdot 2^{-p}) \end{cases}$$

Note: all p-representative numbers are integral multipliers of $2^{-(p+1)}$.

The graphic representation of above definitions is presented below (q is even).

For example: all numbers into the ( ) brackets are p-equivalent, and their p-representative is in the interval middle.



Note:

- if binary representation of x and y are <u>not the same</u> for the first p places after the binary point, then x and y are not *p-equivalent*;
- if binary representation of x and y are <u>the same</u> for the first p places after the binary point, then the p-equivalency is not guaranteed.

Claim 9: Let x and y will be real numbers such, that $x \overset{\alpha}{=} y$ for an integer α, let $\alpha`$ will be integer such, that $\alpha` \le \alpha$, then:

1. $rep_\alpha(-x) = -rep_\alpha(x)$, and therefore $-x \overset{\alpha}{=} -y$;
2. if $x \overset{\alpha}{=} y$, then $x \overset{\alpha'}{=} y$;
3. for every integer i: $x \cdot 2^{-i} \overset{\alpha+i}{=} y \cdot 2^{-i}$;
4. for every integer q': $x + q' \cdot 2^{-\alpha'} \overset{\alpha}{=} y + q' \cdot 2^{-\alpha'}$.

Proof:

1. There are two cases:

if $x = 2^{-\alpha} \cdot q$, then: $\begin{cases} rep_\alpha(x) = x \\ -x = 2^{-\alpha} \cdot (-q) \Rightarrow rep_\alpha(-x) = -x = -rep_\alpha(x) \end{cases}$

if $x \in (q \cdot 2^{-\alpha}, [q+1] \cdot 2^{-\alpha})$, then:

$$\begin{cases} rep_\alpha(x) = (q + ½) \cdot 2^{-\alpha} \\ -x \in (-[q+1] \cdot 2^{-\alpha}, -q \cdot 2^{-\alpha}) \Rightarrow rep_\alpha(-x) = (-q - ½) \cdot 2^{-\alpha} = -rep_\alpha(x) \end{cases}$$

$rep_\alpha(x) = rep_\alpha(y) \Rightarrow rep_\alpha(-x) = -rep_\alpha(x) = -rep_\alpha(y) = rep_\alpha(-y)$, and then $-x \overset{\alpha}{=} -y$.

From this claim follows that every rounding may be made independently of the number sign.

2. If $x \overset{\alpha}{=} y$, then $x, y \in (q \cdot 2^{-\alpha}, [q+1] \cdot 2^{-\alpha})$.

If we look at the interval $(q' \cdot 2^{-\alpha'}, [q'+1] \cdot 2^{-\alpha'})$, when $\alpha' \le \alpha$, it is obvious, that the interval is bigger then the initial interval and actually includes it into itself (look the figure below) and therefore $x \overset{\alpha'}{=} y$ ($\alpha` = \alpha - 1$ in the below figure).

$q \cdot 2^{-\alpha}$ $(q+1) \cdot 2^{-\alpha}$ $(q+2) \cdot 2^{-\alpha}$

$q' \cdot 2^{-\alpha'}$ $(q'+1) \cdot 2^{-\alpha'}$

3. If $x, y \in (q \cdot 2^{-\alpha}, [q+1] \cdot 2^{-\alpha})$, then $x \cdot 2^{-i}, y \cdot 2^{-i} \in (q \cdot 2^{-(\alpha+i)}, [q+1] \cdot 2^{-(\alpha+i)})$, and therefore $x \cdot 2^{-i} \overset{\alpha+i}{=} y \cdot 2^{-i}$.

Intuitively - if two α-equivalent binary numbers are shifted right by the *i* bits, then they become (α+*i*)-equivalent.

4.

$$x + q' \cdot 2^{-\alpha'}, y + q' \cdot 2^{-\alpha'} \in (q \cdot 2^{-\alpha} + q' \cdot 2^{-\alpha'}, [q+1] \cdot 2^{-\alpha} + q' \cdot 2^{-\alpha'}) =$$
$$= ([q + q' \cdot 2^{\alpha-\alpha'}] \cdot 2^{-\alpha}, [q + 1 + q' \cdot 2^{\alpha-\alpha'}] \cdot 2^{-\alpha}).$$

The both expressions it the [ ] brackets are integers, because of q and q' are integers, and because of α'≤ α, α-α' is positive integer (or zero) and $q' \cdot 2^{\alpha-\alpha'}$ is integer too. Therefore $x + q' \cdot 2^{-\alpha'} \overset{\alpha}{=} y + q' \cdot 2^{-\alpha'}$.

Intuitively, adding $q' \cdot 2^{-\alpha'}$ to *x* and *y* still keeps them in the same interval with endpoints $(q'' \cdot 2^{-\alpha}, [q''+1] \cdot 2^{-\alpha})$.

Question: what value of α we must choose in order two α-equivalent numbers will be rounded to the same value?

For the *RNE* rounding it is enough α = *p*.

For the *Round to zero* and *Round to* ±∞ it is enough α = *p-1*.

Claim 5: If *f'* = $rep_p(f)$, then:

1. sig_rnd (*f*) = sig_rnd (*f'*);
2. sig_rnd (*f*) = *f* ⇔ sig_rnd (*f'*) = *f'* ⇔ *f* = *f'* = $q \cdot 2^{-(p-1)}$.

Proof:

1. Reminder: for the RNE rounding we have:

$$\mathrm{sig\_rnd}(f) = \begin{cases} q \cdot 2^{-(p-1)} & \text{if} \quad f \in (q \cdot 2^{-(p-1)}, [q + \frac{1}{2}] \cdot 2^{-(p-1)}) \\ (q+1) \cdot 2^{-(p-1)} & \text{if} \quad f \in ([q + \frac{1}{2}] \cdot 2^{-(p-1)}, [q+1] \cdot 2^{-(p-1)}) \\ q' \cdot 2^{-(p-1)} & \text{if} \quad f = (q + \frac{1}{2}) \cdot 2^{-(p-1)} \end{cases}$$

In fact, operation $\mathrm{rep}_p(f)$ doesn't influence on the sig_rnd operation, because if, for example, f belongs to the first interval, then f' belongs to the same interval - actually it is now at the middle of the interval, the same happens, when f belongs to the second interval. In third case, $f = f'$.

Note: the above Proof doesn't presented in the Class.

2. $\mathrm{sig\_rnd}(f) = f \Rightarrow f = q \cdot 2^{-(p-1)} \Rightarrow f = f' \Rightarrow \mathrm{sig\_rnd}(f') = f'$.

As follows from the Claim, we can use $\mathrm{rep}_p(f)$ value in $\mathrm{sig\_rnd}(f)$ computation, rather then the f value itself.

Claim 6: Suppose $\eta(x) = (s_x, e_x, f_x)$ and $\eta(y) = (s_y, e_y, f_y)$. If $x \overset{p-e_x}{=} y$, then:

1. $s_x = s_y, e_x = e_y, f_x \overset{p}{=} f_y$;

2. $r(x) = r(y)$.

Proof:

If $x = y$, the proof is obvious.

From now, let's discuss the case $x \neq y$.

If $x \overset{p-e_x}{=} y$, then $x, y \in I = (q \cdot 2^{e_x - p}, [q+1] \cdot 2^{e_x - p})$. Because the interval is open, and q and (q+1) have the same sign, zero is not in the interval, therefore or x,y>0, or x,y<0, and $s_x = s_y$. Let's suppose $s_x = s_y = 0$.

Support Claim: If x and y belongs to the interval I, then or $f_x, f_y \in [0,1)$, or $f_x, f_y \in [1,2)$.

Proof: Suppose, that the above claim is not true, i.e. $f_x \in [0,1)$ and $f_y \in [1,2)$, for example. Then $e_x = e_{min}$ and $e_y \geq e_{min}$ and follows, that $2^{e_{min}} \in I$ and in is impossible case.

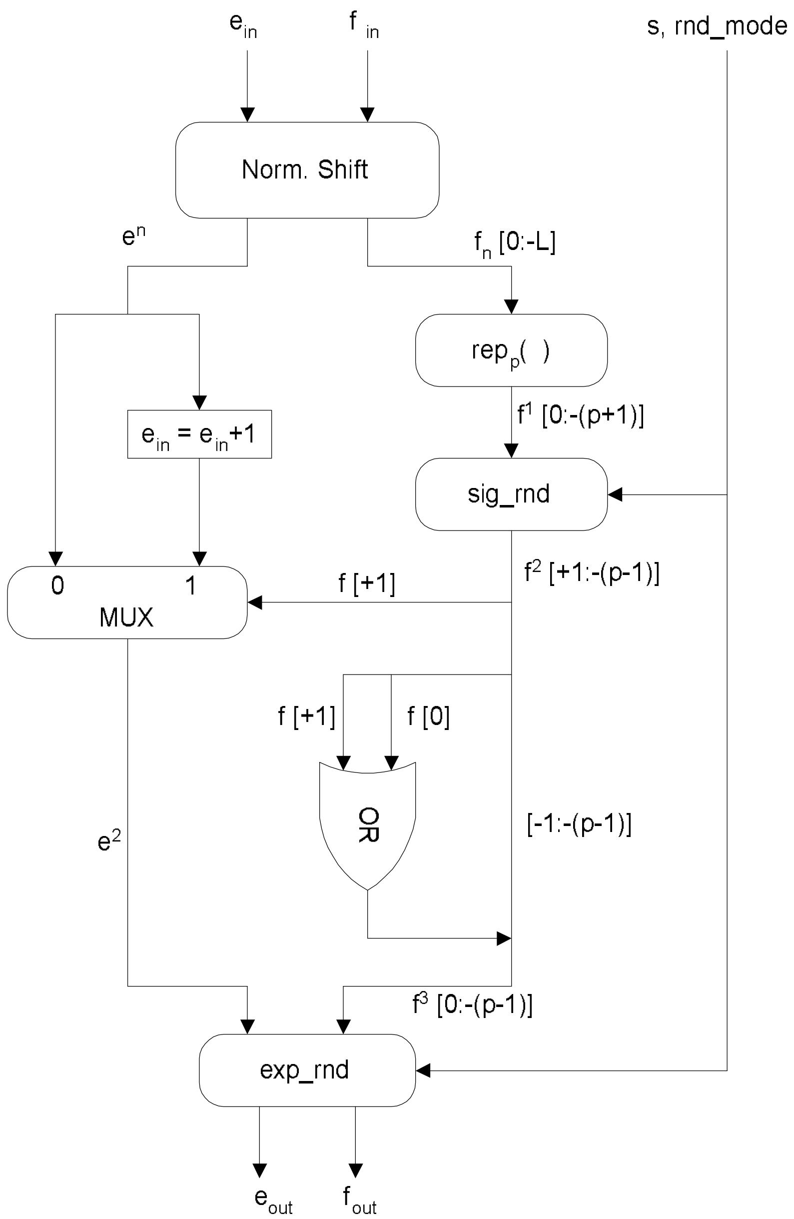
- If $f_x, f_y \in [0,1)$, then $e_x = e_y = e_{min}$ and follows from Claim 9, part 3 ($x \cdot 2^{-i} \overset{\alpha+i}{=\!=} y \cdot 2^{-i}$), that $f_x \overset{p}{=} f_y$.
- When $f_x, f_y \in [1,2)$: let's suppose, that $e_y < e_x$, for example. It is follows that:

$$y = f_y \cdot 2^{e_y} < 2 \cdot 2^{e_y} \le 2^{e_x} \le f_x \cdot 2^{e_x} = x.$$

From above equation follows, that $2^{e_x} \in I$. Because $2^{e_x} = i \cdot 2^{e_x - p}$ (i – integer) , statement, that $2^{e_x} \in I$ is not true, and therefore follows, that $e_x = e_y$ and $f_x \overset{p}{=} f_y$.

Proof of the second part of the Claim - r(x) = r(y), follows from Claim 5.

Rounding Unit
(without exceptions handling)



<u>Assumptions</u>: the value we want to round is: $x \overset{p-e_x}{=} val(s, e_{in}, f_{in})$, i.e. the input to the unit is not the exact value of x, but $val(s, e_{in}, f_{in})$.

<u>Functionality</u>: 1. The *normalization shift* box computes value of $\eta(val(s, e_{in}, f_{in})) = (s, e^n, f^n)$.

Reminder: Normalization shift maps every real number x to (s,e,f) so that x = val(s,e,f). If $asb(x) \geq 2^{e_{min}}$, then $1 \leq f < 2$, if $asb(x) < 2^{e_{min}}$, then $0 \leq f < 1$.

2. The $f^1 = rep_p(f^n)$ value is computed, i.e. final Sticky-bit is computed.
3. Significand rounding: $f^2 = sig\_rnd(f^1)$. It is obvious, that the *sign* and the Rounding mode must be input to this block.
4. Now, if f^2 is rounded to 2 (f [1,0] = 10), then e^n must be incremented by one, this is done by MUX, and f must be set to one - f [0] = 1, by OR, according to the algorithm.
5. Exponent rounding computed:

$$(s, e_{out}, f_{out}) = exp\_rnd(s, e^2, f^2) = \begin{cases} (s, e_\infty, f_\infty) & \text{if } f \cdot 2^e \geq x_{max} \\ \eta(val(s, e_{out}, f_{out})) & \text{otherwise} \end{cases}$$

Of course, this unit uses in computations two additional inputs: sign and Rounding mode.

Correctness:

Sticky-bit and Sticky

As known, if binary representation of the number *f* is as follows: $f_0.f_{-1}f_{-2}...f_{-p}...f_{-l}$, $(l \geq p+1)$, then $f = \sum_{i=0}^{l} 2^{-i} \cdot f_i$. Let's define *Sticky-bit* as OR of all bits after the f_{-p} bit:

$$\text{Sticky-bit (f, p)} = f'_{-(p+1)} = OR(f_{-(p+1)}, f_{-(p+2)}, \ldots) = \bigvee_{i=p+1}^{l} f_{-i}, \text{ and}$$

$$\text{Sticky (f, p)} = f_0.f_{-1}f_{-2}...f_{-p}f_{-(p+1)}.$$

The Relationship between *Sticky-bit* and Representatives.

Claim 8: $rep_p(f) = \text{Sticky (f, p)}$.

Proof: If all the bits of *f*, starting at *p+1* position after the binary point are zero: $f = f_0.f_{-1}f_{-2}...f_{-p}000...$, then f is integral multiple of 2^{-p} and therefore $f = rep_p(f)$ and, according to the above definitions Sticky-bit(f, p) = 0, and
Sticky (f, p) = f = $rep_p(f)$.

If f is not integral multiple of 2^{-p}, then Sticky-bit (f, p) = 1, and therefore $rep_p(f)$ = Sticky (f, p).

Addition

Input: (s_1, e_1, f_1) and (s_2, e_2, f_2) - normalized factorings.

Let's choose: $x = val(s_1, e_1, f_1)$ and $y = val(s_2, e_2, f_2)$.

Output: $(s`, e`, f`)$, when $x + y \overset{p-e}{=} val(s`, e`, g`)$, η(x+y) = (s, e, f).

The algorithm is divided into three steps: Preprocessing, Adding and Rounding.

Preprocessing:

- Swap: First of all both exponents are compared, and operands selected such, that $e_1 \geq e_2$. From now, we assume that $e_1 \geq e_2$.
- Alignment Shift: Replace f_2 with $f_2^` = f_2 \cdot 2^{-\delta}$, when $\delta = \min\{e_1 - e_2, p+2\}$.
- Representative Computing: $f_2^{``} = rep_{p+1}(f_2^`)$. Now $f_2^{``} = f_0.f_{-1}f_{-2}...f_{-(p+2)}$.

Adding:

$$s` = s_1$$

$$e` = e_1$$

$$g` = (-1)^{s_1} \cdot f_1 + (-1)^{s_2} \cdot f_2$$