Computer Arithmetic - Spring 1998

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Rounding Computation

Introduction:

In this lecture next concepts will be defined and discussed:

- *p-equivalent* numbers,
- *p-representatives*,
- *sticky-bit* and *sticky*,
- Rounding unit will be presented,
- Addition algorithm will be described.

Definition: Two real numbers x and y are *p-equivalent* (שקולים - p) if:

1. $x = y$ or
2. $x, y \in (q \cdot 2^{-p}, [q+1] \cdot 2^{-p})$ for any integer q ($q \in Z$).

This denoted as $x \overset{p}{=} y$.

In other words, binary representation of two p-equivalent numbers is the same at least for p positions to the right of the binary point.

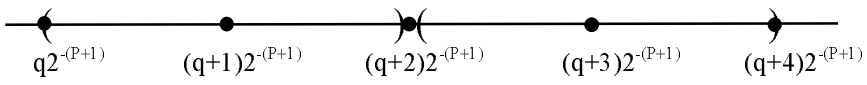
For every group of p-equivalent real numbers, let's define *p-representative* (נציג) as follows:

$$rep_p(x) = \begin{cases} q \cdot 2^{-p} & \text{if } x = q \cdot 2^{-p} \\ (q + 1/2) \cdot 2^{-p} & \text{if } x \in (q \cdot 2^{-p}, [q+1] \cdot 2^{-p}) \end{cases}$$

Note: all p-representative numbers are integral multipliers of $2^{-(p+1)}$.

The graphic representation of above definitions is presented below (q is even).

For example: all numbers into the ( ) brackets are p-equivalent, and their p-representative is in the interval middle (black circles).



Note:

- if binary representation of x and y are <u>not the same</u> for the first p places after the binary point, then x and y are not *p-equivalent*;
- if binary representation of x and y are <u>the same</u> for the first p places after the binary point, then the p-equivalency is not guaranteed.

Claim 9: Let x and y will be real numbers such, that $x \overset{\alpha}{=} y$ for an integer α, let α' will be integer such, that $\alpha' \le \alpha$, then:

1. $rep_\alpha(-x) = -rep_\alpha(x)$, and therefore $-x \overset{\alpha}{=} -y$;
2. if $x \overset{\alpha}{=} y$, then $x \overset{\alpha'}{=} y$;
3. for every integer i: $x \cdot 2^{-i} \overset{\alpha+i}{=} y \cdot 2^{-i}$;
4. for every integer q': $x + q' \cdot 2^{-\alpha'} \overset{\alpha}{=} y + q' \cdot 2^{-\alpha'}$.

Proof:

1. There are two cases:

if $x = 2^{-\alpha} \cdot q$, then: $\begin{cases} rep_\alpha(x) = x \\ -x = 2^{-\alpha} \cdot (-q) \end{cases} \Rightarrow rep_\alpha(-x) = -x = -rep_\alpha(x)$

if $x \in (q \cdot 2^{-\alpha}, [q+1] \cdot 2^{-\alpha})$, then:

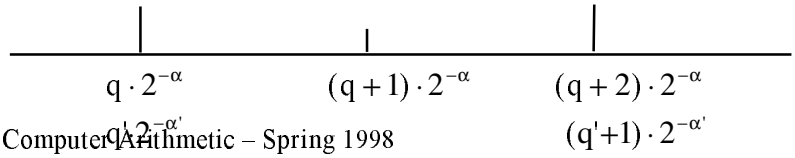
$$\begin{cases} rep_\alpha(x) = (q + \tfrac{1}{2}) \cdot 2^{-\alpha} \\ -x \in (-[q+1] \cdot 2^{-\alpha}, -q \cdot 2^{-\alpha}) \end{cases} \Rightarrow rep_\alpha(-x) = (-q - \tfrac{1}{2}) \cdot 2^{-\alpha} = -rep_\alpha(x)$$

$rep_\alpha(x) = rep_\alpha(y) \Rightarrow rep_\alpha(-x) = -rep_\alpha(x) = -rep_\alpha(y) = rep_\alpha(-y)$, and then $-x \overset{\alpha}{=} -y$.

From this claim follows that every rounding may be made independently of the number sign.

2. If $x \overset{\alpha}{=} y$, then $x, y \in (q \cdot 2^{-\alpha}, [q+1] \cdot 2^{-\alpha})$.

If we look at the interval $(q' \cdot 2^{-\alpha'}, [q'+1] \cdot 2^{-\alpha'})$, when $\alpha' \le \alpha$, it is obvious, that the interval is bigger then the initial interval and actually includes it into itself (look the figure below) and therefore $x \overset{\alpha'}{=} y$ ($\alpha' = \alpha - 1$ in the below figure).



3. If $x, y \in (q \cdot 2^{-\alpha}, [q+1] \cdot 2^{-\alpha})$, then $x \cdot 2^{-i}, y \cdot 2^{-i} \in (q \cdot 2^{-(\alpha+i)}, [q+1] \cdot 2^{-(\alpha+i)})$, and therefore $x \cdot 2^{-i} \overset{\alpha+i}{=} y \cdot 2^{-i}$.

Intuitively - if two α-equivalent binary numbers are shifted right by the *i* bits, then they become $(\alpha+i)$-equivalent.

4.

$$x + q' \cdot 2^{-\alpha'}, y + q' \cdot 2^{-\alpha'} \in (q \cdot 2^{-\alpha} + q' \cdot 2^{-\alpha'}, [q+1] \cdot 2^{-\alpha} + q' \cdot 2^{-\alpha'}) =$$
$$= ([q + q' \cdot 2^{\alpha-\alpha'}] \cdot 2^{-\alpha}, [q+1+q' \cdot 2^{\alpha-\alpha'}] \cdot 2^{-\alpha}).$$

The both expressions it the [ ] brackets are integers, because of q and q` are integers, and because of $\alpha` \leq \alpha$, $\alpha-\alpha`$ is positive integer (or zero) and $q' \cdot 2^{\alpha-\alpha'}$ is integer too. Therefore $x + q' \cdot 2^{-\alpha'} \overset{\alpha}{=} y + q' \cdot 2^{-\alpha'}$.

Intuitively, adding $q' \cdot 2^{-\alpha'}$ to *x* and *y* still keeps them in the same interval with endpoints $(q'' \cdot 2^{-\alpha}, [q''+1] \cdot 2^{-\alpha})$.

Question: what value of α we must choose in order two α-equivalent numbers will be rounded to the same value?

- For the *RNE* rounding it is enough $\alpha = p$.
- For the *Round to zero* and *Round to* $\pm\infty$ it is enough $\alpha = p-1$.

Claim 5: If $f' = \text{rep}_p(f)$, then:

1. sig_rnd (f) = sig_rnd (f');
2. sig_rnd $(f) = f \Leftrightarrow$ sig_rnd $(f') = f' \Leftrightarrow f = f' = q \cdot 2^{-(p-1)}$.

Proof:

1. Reminder: for the RNE rounding we have:

$$\text{sig_rnd}(f) = \begin{cases} q \cdot 2^{-(p-1)} & \text{if} \quad f \in (q \cdot 2^{-(p-1)}, [q + \tfrac{1}{2}] \cdot 2^{-(p-1)}) \\ (q+1) \cdot 2^{-(p-1)} & \text{if} \quad f \in ([q + \tfrac{1}{2}] \cdot 2^{-(p-1)}, [q+1] \cdot 2^{-(p-1)}) \\ q' \cdot 2^{-(p-1)} & \text{if} \quad f = (q + \tfrac{1}{2}) \cdot 2^{-(p-1)} \end{cases}$$

In fact, operation $\text{rep}_p(f)$ doesn't influence on the sig_rnd operation, because if, for example, f belongs to the first interval, then f' belongs to the same interval - actually it is now at the middle of the interval, the same happens, when f belongs to the second interval. In third case, $f = f'$.

2. $\text{sig_rnd}(f) = f \Rightarrow f = q \cdot 2^{-(p-1)} \Rightarrow f = f' \Rightarrow \text{sig_rnd}(f') = f'$.

As follows from the Claim, we can use $\text{rep}_p(f)$ value in $\text{sig_rnd}(f)$ computation, rather then the f value itself.

Claim 6: Suppose $\eta(x) = (s_x, e_x, f_x)$ and $\eta(y) = (s_y, e_y, f_y)$. If $x \overset{p-e_x}{=} y$, then:

1. $s_x = s_y, e_x = e_y, f_x \overset{p}{=} f_y$;

2. $r(x) = r(y)$.

Proof:

If $x = y$, the proof is obvious.

From now, let's discuss the case $x \neq y$.

If $x \overset{p-e_x}{=} y$, then $x, y \in I = (q \cdot 2^{e_x - p}, [q+1] \cdot 2^{e_x - p})$. Because the interval is open, and q and (q+1) have the same sign, zero is not in the interval, therefore or x,y>0, or x,y<0, and $s_x = s_y$. Let's suppose $s_x = s_y = 0$.

Support Claim: If x and y belongs to the interval I, then or $f_x, f_y \in [0,1)$, or $f_x, f_y \in [1,2)$.

Proof: Suppose, that the above claim is not true, i.e. $f_x \in [0,1)$ and $f_y \in [1,2)$, for example. Then $e_x = e_{min}$ and $e_y \geq e_{min}$ and follows, that $2^{e_{min}} \in I$ and it is impossible case.

- If $f_x, f_y \in [0,1)$, then $e_x = e_y = e_{min}$ and follows from Claim 9, part 3

$(x \cdot 2^{-i} \overset{\alpha+i}{=} y \cdot 2^{-i})$, that $f_x \overset{p}{=} f_y$.

- When $f_x, f_y \in [1,2)$: let's suppose, that $e_y < e_x$, for example. It is follows that:

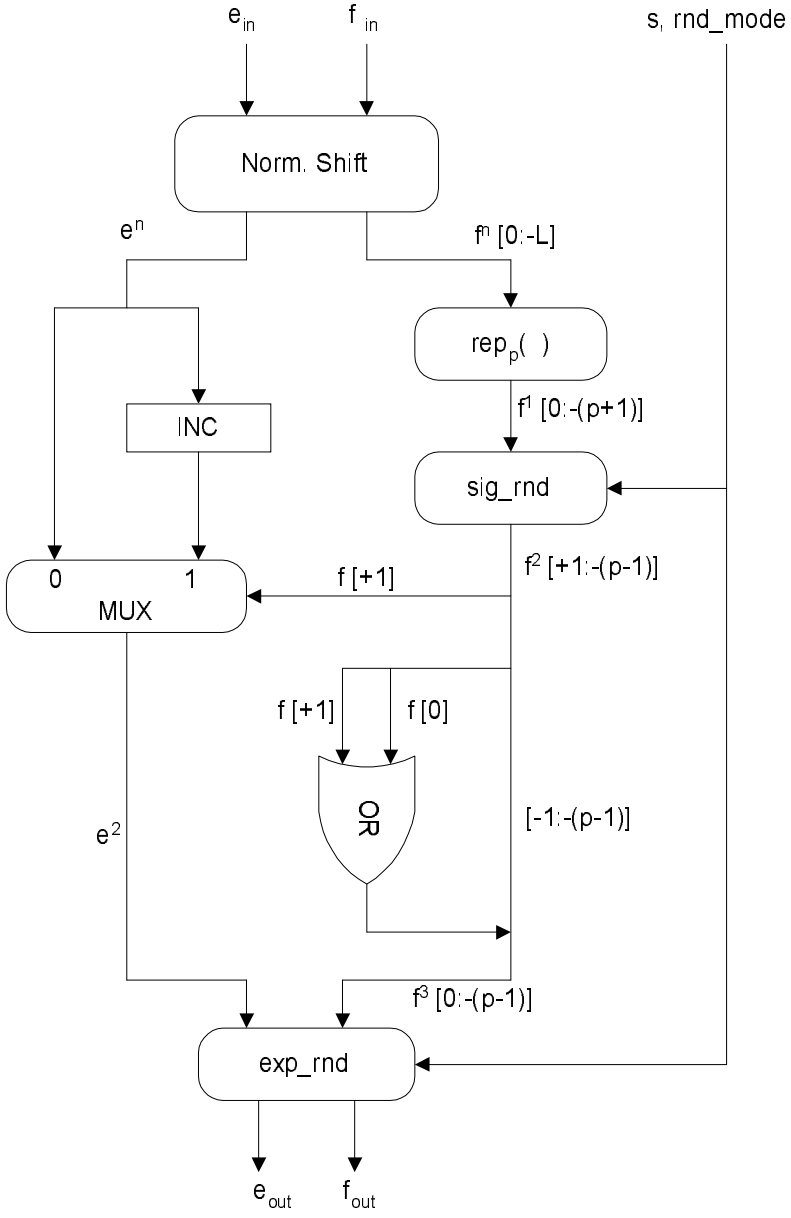
$$y = f_y \cdot 2^{e_y} < 2 \cdot 2^{e_y} \leq 2^{e_x} \leq f_x \cdot 2^{e_x} = x.$$

From above equation follows, that $2^{e_x} \in I$. Because 2^{e_x} is integral multiply of $2^{e_x - p}$, statement, that $2^{e_x} \in I$ is not true, and therefore follows, that $e_x = e_y$ and $f_x \overset{p}{=} f_y$.

Proof of the second part of the Claim - r(x) = r(y), follows from Claim 5.

Rounding Unit

(without exceptions handling)



Assumptions: the value we want to round is: $x = val(s, e_{in}, f_{in})^{p-e_x}$, i.e. the input to the unit is not the exact value of x, but $val(s, e_{in}, f_{in})$.

Functionality: 1. The *normalization shift* box computes the value of $\eta(val(s, e_{in}, f_{in})) = (s, e^n, f^n)$.

Reminder: Normalization shift maps every real number x to (s,e,f) so that x = val(s,e,f). If $asb(x) \geq 2^{e_{min}}$, then $1 \leq f < 2$, if $asb(x) < 2^{e_{min}}$, then $0 \leq f < 1$.

2. The $f^1 = rep_p(f^n)$ value is computed, i.e. final Sticky-bit is computed.
3. Significand rounding: $f^2 = sig_rnd(f^1)$. It is obvious, that the *sign* and the Rounding mode must be input to this block.
4. Now, if f^2 is rounded to 2 (f [1,0] = 10), then e^n must be incremented by one, this is done by INC and MUX, and f must be set to one - f [0] = 1, by OR, according to the algorithm.
5. Exponent rounding computed:

$$(s, e_{out}, f_{out}) = exp_rnd(s, e^2, f^2) = \begin{cases} (s, e_\infty, f_\infty) & \text{if } f \cdot 2^e \geq x_{max} \\ \eta(val(s, e_{out}, f_{out})) & \text{otherwise} \end{cases}$$

Of course, this unit uses in computations two additional inputs: sign and Rounding mode.

Correctness: Actually we need to proof that $r(x) = (s, e_{out}, f_{out})$.

Assume that $x` = (s, e_{in}, f_{in})$.

According to the Claim 5, $f^2 = sig_rnd(f^n)$.

It is easy to see from the above chart, that $(s, e^2, f^3) = pors_norm(s, e^n, f^2)$.

According to the rounding algorithm: $r(x`) = (s, e_{out}, f_{out})$.

In fact, the input to the unit – x` is (p-e)-equivalent to the original value x. Claim 6 implies, that for two (p-e)-equivalent numbers, computing of r(x) yields the same results, i.e. $r(x) = r(x`) = (s, e_{out}, f_{out})$.

Sticky-bit and Sticky

As known, if binary representation of the number *f* is as follows: $f_0.f_{-1}f_{-2}\ldots f_{-p}\ldots f_{-l}$, $(l \geq p+1)$, then $f = \sum_{i=0}^{l} 2^{-i} \cdot f_i$. Let's define *Sticky-bit* as OR of all bits after the f_{-p} bit:

$$\text{Sticky-bit } (f, p) = f'_{-(p+1)} = OR(f_{-(p+1)}, f_{-(p+2)}, \ldots) = \bigvee_{i=p+1}^{l} f_{-i}, \text{ and}$$

$$\text{Sticky } (f, p) = f_0.f_{-1}f_{-2}\ldots f_{-p}f_{-(p+1)}.$$

The Relationship between *Sticky-bit* and Representatives.

Claim 8: $rep_p(f) = \text{Sticky } (f, p)$.

Proof: If all the bits of *f*, starting at *p+1* position after the binary point are zero: $f = f_0.f_{-1}f_{-2}\ldots f_{-p}000\ldots$, then f is integral multiple of 2^{-p} and therefore $f = rep_p(f)$ and, according to the above definitions Sticky-bit(f, p) = 0, and

Sticky (f, p) = f = $rep_p(f)$.

If f is not integral multiple of 2^{-p}, then Sticky-bit (f, p) = 1, and therefore $rep_p(f) =$ Sticky (f, p).

Addition

Input: (s_1, e_1, f_1) and (s_2, e_2, f_2) - normalized factorings.

Let's choose: $x = val(s_1, e_1, f_1)$ and $y = val(s_2, e_2, f_2)$.

Output: (s', e', g'), when $x + y \overset{p-e}{=} val(s', e', g')$, $\eta(x+y) = (s, e, f)$.

The output is then rounded by Rounding unit, which is described above.

The algorithm is divided into three steps: Preprocessing, Adding and Rounding.

Preprocessing:

- Swap: First of all both exponents are compared, and operands selected such, that $e_1 \geq e_2$. From now, we assume that $e_1 \geq e_2$.
- Alignment Shift: Replace f_2 with $f'_2 = f_2 \cdot 2^{-\delta}$, when $\delta = \min\{e_1 - e_2, p+2\}$.

- Representative Computing: $f_2'' = \mathrm{rep}_{p+1}(f_2')$. Now $f_2'' = f_0 . f_{-1} f_{-2} \ldots f_{-(p+2)}$.

Adding:

$s' = s_1$

$e' = e_1$

$g' = (-1)^{s_1} \cdot f_1 + (-1)^{s_2} \cdot f_2$

Now $g' = g_1 g_0 . g_{-1} g_{-2} \cdots g_{-(p+2)}$.

We need to proof, that if (s',e',g') value will be rounded, the result will be $(s_{out}, e_{out}, f_{out})$, such, that $r(x+y) = (s_{out}, e_{out}, f_{out})$. From correctness proof of the Rounding unit follows, that enough to proof only that $x + y \overset{p-e}{=} \mathrm{val}(s',e',g')$.

1. If $\delta = \min\{e_1 - e_2, p+2\} \le 1$, then $f_2' = f_2 \cdot 2^{-\delta}$ and $f_2'' = \mathrm{rep}_{p+1}(f_2')$ and follows that $x+y = \mathrm{val}(s',e',g')$.

Note: the above equalities is true because if $f_2[0 : -(p-1)]$ then $f_2' = f_2 >> 1 = [0 : -p]$.

2. If $\delta = \min\{e_1 - e_2, p+2\} \ge 2$:

Support Claims:

- $|e - e_1| \le 1$,
- even if $e_2 - e_1 > p+2$, $f_2'' = \mathrm{rep}_{p+1}(f_2 \cdot 2^{-(e_1 - e_2)})$.

From the second claim follows, that $y \overset{p+1-e_1}{=} y''$ $[= (-1)^{s_2} \cdot 2^{e_1} \cdot f_2'']$.

Because x is integer multiplier of $2^{e_1-(p-1)}$ follows (part 4 of the Claim 9) $x + y \overset{p+1-e_1}{=} x + y''$. From the first support claim: $e \ge e_1 - 1$, and therefore $x + y \overset{p-e}{=} x + y'' = \mathrm{val}(s',e',g')$.