Wave energy converter control by wave prediction and dynamic programming

Guang Li a, *, George Weiss b, Markus Mueller c, Stuart Townley c, Mike R. Belmont a

a College of Engineering, Mathematics and Physical Sciences, University of Exeter, Harrison Building, North Park Road, Exeter EX4 4QF, UK
b Faculty of Engineering, Tel Aviv University, Ramat Aviv 69978, Israel
c College of Engineering, Mathematics and Physical Sciences, University of Exeter, Harrison Building, North Park Road, Exeter EX4 4QF, UK

ABSTRACT

We demonstrate that deterministic sea wave prediction (DSWP) combined with constrained optimal control can dramatically improve the efficiency of sea wave energy converters (WECs), while maintaining their safe operation. We focus on a point absorber WEC employing a hydraulic/electric power take-off system. Maximizing energy take-off while minimizing the risk of damage is formulated as an optimal control problem with a disturbance input (the sea elevation) and with both state and input constraints. This optimal control problem is non-convex, which prevents us from using quadratic programming algorithms for the optimal solution. We demonstrate that the optimum can be achieved by bang–bang control. This paves the way to adopt a dynamic programming (DP) algorithm to resolve the on-line optimization problem efficiently. Simulation results show that this approach is very effective, yielding at least a two-fold increase in energy output as compared with control schemes which do not exploit DSWP. This level of improvement is possible even using relatively low precision DSWP over short time horizons. A key finding is that only about 1 second of prediction horizon is required, however, the technical difficulties involved in obtaining good estimates necessitate a DSWP system capable of prediction over tens of seconds.

© 2012 Elsevier Ltd. All rights reserved.

1. Introduction

Ocean waves provide an enormous source of renewable energy [1,2]. Research into wave energy was initially stimulated by the oil crisis of the 1970s [3]. Since then many different types of sea wave energy converters (WECs) have been designed and tested [4,5], but this is still a relatively immature technology (compared to solar or wind energy) and is far from being commercially competitive with traditional fossil fuel or nuclear energy sources. Progress is hampered by two fundamental problems:

1. Inefficient energy extraction, often due to the fact that the WEC’s dynamic parameters are not optimally tuned and their control is not optimal for most wave profiles.
2. Risk of device damage. In order to prevent WECs from being damaged by large waves, they have to be shut down, especially during winter storms. Such periods of inactivity can last for days.

Extracting the maximum possible time average power from WECs, while reducing the risk of device damage involves a combination of good fundamental engineering design of the devices and effective control of their operation. The traditional approach to these issues exploits short term statistical properties of the sea [6] but it has been shown [7,8] that doing so severely limits the average power that can be extracted. We address the above two problems by considering schemes designed to achieve optimal control. It will be shown that (as [9–11] demonstrated in the 1970s) methods for achieving the maximum power output are inevitably non-causal and require prediction of the shape of the incident waves. The recent development of deterministic sea wave prediction (DSWP) as a scientific discipline [12–28], particularly real time DSWP [12–19,26,28] now makes such an approach realistic. For a variety of reasons high accuracy real time DSWP is very demanding. However it will be shown that the optimal control techniques described here provide considerable improvements over traditional WEC control methods, even with modestly accurate DSWP and relatively short prediction horizons.

The dimensions of point absorbers are small compared with the wave length of incoming waves and they are potentially very efficient if their frequency response function closely matches the spectrum of the incident waves (resonance). Passive control...
methods (such as impedance matching) have been explored to improve energy extraction by tuning the dynamical parameters of the devices [9,29–31]. Most of these approaches are linear control schemes. A non-linear control method that has received some attention is latching, [32–37]. This attempts to force the phase angle between the wave and the float at the WEC to be similar to conditions at resonance. The above control strategies do not use prediction of the forces acting on the WEC and thus inevitably lead to sub-optimal energy extraction. Since the early work [5,9,11] there have been a number of authors who have recognized the importance of DSWP in the control of a variety of floating body applications [7,8,17,37,38], but these have, as yet, not been incorporated into actual control schemes.

The point absorber model used is shown in Fig. 1 and roughly corresponds to the Power Buoy device PB150 developed by OPT Inc, see [39]. On the sea surface is a float, below which hydraulic cylinders are vertically installed. These cylinders are attached at the bottom to a large area anti-heave plate whose vertical motion is designed to be negligible compared with that of the float. The heave motion of the float drives the pistons inside the hydraulic cylinders to produce a liquid flow. The liquid drives hydraulic motors attached to a synchronous generator. From here, the power reaches the grid via back-to-back AC/DC/AC converters. The mechanical circuit corresponding to this simplified model is shown in Fig. 2. Here \( h_w \) is the water level, \( h_i \) is the height of the mid-point of the float and \( D \) is the hydrodynamic damping of the float including added damping due to the damping effect of the movement of the float [1]. \( K \) is the hydrostatic stiffness giving the buoyancy force, which can be calculated from the float geometry, while \( m \) is the mass of the float including “added mass” [1]. The friction force acting on the float is \( f_f = D_y h_i \). In order to simplify the model we neglect the frequency dependence of both \( D \) and \( m \) (see, e.g., [29]). We also neglect the static component of the friction force \( f_s \). For a more thorough investigation of the modeling issues of point absorbers, see [1,40,41].

The control input is the \( q \)-axis current in the generator-side power converter, to control the electric torque of the generator [42]. The generator torque is proportional to the force \( f \) acting on the pistons from the fluid in the cylinders. Since the motion of the float imposes a velocity \( v \) on the piston, the extracted power \( P(t) \) at time \( t \) is expressed as

\[
P(t) = f(t)v(t).
\]

This power is smoothed by the capacitors on the DC link of the converters. In our model the modest power losses in the hydraulic transmission, the generator and the converters will be neglected.

To avoid damage, and for overall performance reasons, two constraints have to be considered in any real WEC. One concerns the relative motion of the float to the sea surface (it should neither sink nor raise above the water and then slam), which can be expressed as

\[
|h_w - h_i| \leq \phi_{\text{max}}.
\]

The other constraint is on the control signal set by limitations on the allowable converter current. This constraint can be expressed as

\[
|f| \leq \gamma.
\]

The control objective is to maximize the extracted energy subject to the constraints (2) and (3). We remark that there is a further constraint on the motion of the float because of the limited excursion of the piston with respect to the cylinder (see Fig. 1). This constraint has the form \( |h_i| \leq \lambda \). However, we shall not consider this constraint, since we assume that \( \lambda \) is large enough compared with the expected excursion.

The constraints imposed on WECs significantly affect the power that can be extracted. It has been shown that by using control strategies that incorporate these constraints, considerable increases in the energy output can be obtained without increasing the risk of damage [7,8]. The ability to handle constraints combined with the development of real time wave prediction methods has recently led to an interest in the use of model predictive control (MPC) for wave energy devices [43–45]. The work published to date has used standard MPC techniques and they rely on the formulation of a convex quadratic programming (QP) problem. The underlying problem formulations and the cost function representations in [43–45] differ from our case. We leave the question open if the convexity assumption holds for a broad class of constrained optimal control problems for WECs. However, we find that this assumption does not hold for the problem formulated in this paper and many other similar optimal control problems [46,47].

In this paper, the constrained optimal control problem is solved using fundamental principles from optimal control theory [48–50], see Section 3, and real time deterministic sea wave prediction [12–19,26,28]. We demonstrate that a nearly optimal control is of bang–bang type, meaning that the control input \( f \) is always at one edge of the allowed range, see Subsection 3.2. As will be shown, for an arbitrary sea wave input known over an interval of time (not a sine wave), direct numerical computation of the optimal control scheme is not realistic. Consequently, we employ dynamic programming (DP) [46], which is well suited for constrained optimal control problems [51]. We have sacrificed some detail in the hydrodynamic modeling, leading to a model of manageable complexity for on-line DP.

The implementation of DP on a WEC control system is based on the assumption that the sea surface shape can be predicted for a short time period. This requirement has been a bottleneck for the development of suitable optimal control strategies for WECs. However, the developments in deterministic sea wave modeling techniques have made real time sea wave prediction for a short time period realizable. [12–19,26,28]. A key finding from this work is that the prediction horizons required are considerably smaller than those resulting from the previous studies [43–45].
For the sake of comparison, simulations have also been performed for various non-prediction-based WEC control strategies. The results demonstrate the following benefits of our approach. 1) Significant increase in energy output. We get up to a two-fold increase in energy output when compared with rival control algorithms which do not exploit sea state prediction. 2) Robustness to the prediction accuracy and prediction horizon of DSWP. Especially important is the possibility of reducing the prediction horizon, because the difficulties associated with real time DSWP increase significantly with the prediction horizon.

The structure of the paper is as follows. Section 2 formulates the constrained optimal control problem for the WEC. Section 3 provides a detailed analysis for this optimal control problem. The dynamic programming control strategy is developed in Section 4, and compared in simulation with several other control methods in Section 5.

2. The WEC model

In this section, we find it convenient to replace the mechanical circuit in Fig. 2 with its electrical equivalent, shown in Fig. 3. The relationship between the electrical and mechanical elements from Figs. 2 and 3 is:

\[ R = \frac{1}{D} \quad L = \frac{1}{R} \quad m = C \quad R_f = \frac{1}{D_f} \]

Moreover, force and velocity in Fig. 2 correspond to current and voltage in Fig. 3. Hence, \( w \) and \( v \) are voltages associated with the vertical velocities of the sea level and float, respectively; the current \( i_L \) corresponds to the friction force between the float and the lower part of the WEC that is rigidly connected to the heave plate. The control input \( u \) is the current associated with the control force \( f \) in Fig. 2 (numerically, they are the same).

From the circuit we obtain

\[
\begin{bmatrix}
\frac{di_L}{dt} \\
\frac{dv}{dt}
\end{bmatrix} = \begin{bmatrix}
0 & -\frac{1}{L} \\
\frac{1}{C} & \frac{1}{RC} & \frac{1}{RC}
\end{bmatrix} \begin{bmatrix}
i_L \\
v
\end{bmatrix} + \begin{bmatrix}
\frac{1}{L} \\
\frac{1}{RC}
\end{bmatrix} w + \begin{bmatrix}
0 \\
\frac{1}{C}
\end{bmatrix} u. \tag{4}
\]

The extracted energy in the time interval \([0,T]\) is \( E = \int_0^T v w \, dt \). The two constraints corresponding to (2) and (3) can be expressed as \( |i_L| \leq \delta \) and \( |u| \leq \gamma \), respectively, where \( \delta = \Phi_{\text{max}}/L \).

If the constraint \( |i_L| \leq \delta \) is violated, which means that the float moves outside some acceptable limit with respect to the water surface, then the incremental buoyancy force gets much smaller, due to the smaller cross-section of the buoy at the water level, and therefore a nonlinearity is introduced into the WEC model. We denote by \( \Phi \) the magnetic flux of the inductor, which corresponds in the mechanical circuit to the vertical displacement difference between the water level and the mid-point of the float \( \Phi(t) = \int_0^t (w - v) \, dt \). Then the non-linear behavior of the inductor can be approximated by

\[
i_L = \begin{cases}
\Phi_{\text{max}}/L + (\Phi - \Phi_{\text{max}})/kL, & \text{if } \Phi > \Phi_{\text{max}} \\
\Phi/L, & \text{if } |\Phi| \leq \Phi_{\text{max}} \\
-\Phi_{\text{max}}/L + (\Phi + \Phi_{\text{max}})/kL, & \text{if } \Phi < -\Phi_{\text{max}}
\end{cases} \tag{5}
\]

where \( k > 1 \) is a constant and \( \Phi_{\text{max}} = L \delta \).

3. The WEC optimal control problem

3.1. Problem formulation

Introducing the state vector \( x = [i_L, v]^T \) allows (4) to be rewritten as

\[
\dot{x} = A_x x + B_{cw} u + B_{cw} w, \tag{6a}
\]

\[
v = C_x x, \quad C_c = \begin{bmatrix} 0 & 1 \end{bmatrix}.
\tag{6b}
\]

The control objective is to maximize the energy

\[
E = \int_0^T v(t) u(t) \, dt
\]

deextracted over a time interval \([0,T]\), assuming that the disturbance input \( w \) is known. The following constraints must also be satisfied (for all \( t \in [0,T] \)):

\[
|x_1| \leq \delta, \quad |u| \leq \gamma.
\tag{7}
\]

Thus, the optimization problem is maximized \( E \) subject to \( (7) \).

We remark that if the constraints \( (7) \) were removed and if \( w \) were a sine wave then, for large \( T \), this would become an impedance matching problem, which has a simple analytic solution. If \( w \) is not a sine wave, and it is not known in advance, then this “solution” becomes non-causal \cite{1} and various causal approximations to it have been studied in \cite{52–55}.

3.2. Optimal control analysis

In the following analysis, the problem \( (8) \) is recast by introducing the cost functional

\[
J(u) = \int_0^T \left[ -v(t) u(t) + \frac{e}{\delta - |x_1(t)|^2} \right] \, dt \tag{9}
\]

where, for some (small) \( e > 0 \), the interior penalty term \( e/(\delta - |x_1(t)|^2) \) replaces the state constraint \cite{56}. The modified cost function does not allow the optimal state trajectory to approach the boundary of the permitted region, because \( e/(\delta - |x_1(t)|^2) \rightarrow \infty \) as \( |x_1(t)| \rightarrow \delta \), leading to a large cost. Put

\[
U = \{ u : [0,T] \rightarrow [-\gamma, \gamma] \mid u \text{ measurable} \}
\]

then the problem is to minimize \( J(u) \) with respect to \( u \in U \) subject to \( x, v \) being the solution of (6). It is known that for \( e \rightarrow 0 (e > 0) \) the optimal cost of the above unconstrained (in state) optimal control problem (i.e., the minimum of \( J(u) \)) converges to the minimum of the cost \( -E \) of the constrained optimal control problem \( (8) \), see \cite{56}. Thus, for \( e \) sufficiently small, we get a control input \( u \) that produces an amount of energy \( E \) as close as desired to the optimal energy \( E \) of the original constrained problem \( (8) \).

![Fig. 3. The equivalent electric circuit representation of the WEC.](image)
To solve the optimization problem for $J$, we use Pontryagin’s Minimum Principle (PMP) ([48], Ch. 1), [49], [50] or [57]). We briefly recall the PMP in our context. First we introduce the pre-Hamiltonian

$$\mathcal{H}(t, x, \lambda, u) = L(t, x, u) + \lambda^T (A_x x + B_{cu} u + B_{cw} w(t))$$

(10)

where $L(t, x, u) = -x_2 + e/(\delta - |x_1|)$ is the integrand of $J(u)$, see (9), and $\lambda = [\lambda_1, \lambda_2]^T \in \mathbb{R}^2$ denotes the costate. Define the Hamiltonian $H_0$ by

$$H_0(t, x, \lambda) = \min_{u \in [-\gamma, \gamma]} H(t, x, \lambda, u_0).$$

Suppose that there exists an optimal control function $u^*$ (defined on $[0, T]$) for the problem of minimizing $J$, with a given initial state $x(0) = [x_1(0), x_2(0)]^T$. Let $x^*$ be the state trajectory corresponding to the optimal control $u^*$, with the initial state $x(0)$. Let $\lambda^* = [\lambda_1^*, \lambda_2^*]^T$ be the corresponding costate trajectory defined as the solution of the differential equation

$$\frac{d\lambda^*_i}{dt} = -\frac{\partial H_0}{\partial x} (t, x^*(t), \lambda^*(t), u^*(t)), \quad i = 1, 2, \quad \lambda^*(T) = 0. \quad (11)$$

The zero end condition for $\lambda^*$ follows from the fact that there is no terminal cost in (9). The PMP states that $u^*$ minimizes $H$ at every moment:

$$H(t, x^*(t), \lambda^*(t), u^*(t)) \leq H(t, x^*(t), \lambda^*(t), u_0)$$

for all $t \in [0, T]$ and all $u_0 \in [-\gamma, \gamma]$. Moreover, the canonical equations hold:

$$\dot{x}^* = \frac{\partial H_0}{\partial x} \quad \text{and} \quad \dot{\lambda}^* = -\frac{\partial H_0}{\partial x}$$

Using the notation $R_p = \frac{RR_c}{R + R_f}$ and $v(t) = x_2^*(t)$, by a short computation

$$H(t, x^*(t), \lambda^*(t), u_0) = \frac{1}{2} \left( v(t) + \frac{\lambda_2^*(t)}{C} \right) u_0 + \frac{e}{\delta - |x_1^*(t)|}$$

$$+ \frac{x_1^*(t)}{L} (w(t) - v(t))$$

$$+ \frac{x_2^*(t)}{C} \left( x_1^*(t) - \frac{w(t) - v(t)}{R_p} \right).$$

Since $H$ is linear in $u_0$, its minimum with respect to $u_0$ is always achieved at $u_0 = \gamma$ or at $u_0 = -\gamma$. More precisely,

$$u^*(t) = \begin{cases} \gamma & \text{if} \quad v(t) + \frac{\lambda_2^*(t)}{C} > 0, \\ -\gamma & \text{if} \quad v(t) + \frac{\lambda_2^*(t)}{C} < 0. \end{cases} \quad (12)$$

If $v(t) + \frac{\lambda_2^*(t)}{C} = 0$ then the minimum principle does not give us $u^*(t)$, It seems that the times when this happens are negligible (they form a set of measure zero), but we do not have a rigorous proof for this.

This type of control, jumping between a finite number of extreme points, is called bang–bang control. To find the optimal state and costate trajectories, one approach is to solve the two-point boundary value problem corresponding to the canonical equations:

$$\dot{x}_1^1(t) = \frac{1}{L} x_2^1(t) + \frac{1}{L} w(t)$$

$$\dot{x}_2^1(t) = \frac{1}{C} v^1(t) - \frac{1}{R_p C} x_2^1(t) + \frac{1}{R} v^1(t) + \frac{1}{R C} w(t)$$

$$\dot{\lambda}_1^1(t) = \frac{\pm e}{(\delta - |x_1^1(t)|)^2} - \frac{\lambda_2^1(t)}{C}, \quad \left( + \text{sign if } x_1^1(t) < 0 \right)$$

$$\dot{\lambda}_2^1(t) = \frac{\pm \gamma + \lambda_2^1(t)}{L} + \frac{\lambda_2^1(t)}{R}, \quad i = 1, 2.$$
4. Dynamic programming for WEC control

4.1. A brief introduction to dynamic programming

Dynamic programming (DP), originally developed by [46], is a powerful approach for numerically solving discrete optimization problems [47]. It is a multi-stage decision process based on Bellman's principle of optimality [46]. Based on this principle, we can simplify the process of making a decision by breaking the process down into a sequence of decision steps.

DP has long been recognized as being naturally suited for resolving a wide variety of optimal control problems [47,59]. This is mainly because it allows diverse forms of the criterion functions and broad classes of systems to be controlled, which can be non-linear and time varying.

We briefly describe the DP optimization procedure. The first step is to discretize the continuous system model, $x = f(x, u, w, t)$ to its discrete time form $x(k+1) = g(x(k), u(k), w(k), k)$ and quantify each state variable $x_j(k)$ and each control variable $u_k(k)$. The second step is to define a sequence of criterion functions $J_k(y(u), k = 1, ..., N)$ representing the performance values (energy in the present work) from the time $k$ to the final time $N$ for all admissible inputs $u(i)$, outputs $y(i)$ and states $x(i)$, $i = k, ..., N$, that satisfy the system's dynamic equations. The third step is to find the optimal control sequence in a recursive manner. The conventional computational procedure of the DP algorithm starts with determining the optimal values of $J_0$ obtained at the final stage $N$ for all possible states and control inputs. Then the optimal values $J_k$, $k = 1, ..., N$, can be found backwards by a recurrence relationship. The crucial point of the algorithm is that since the optimal values of $J_k$, $k = 1$ for all required states of the system have already been calculated in the previous step, only one stage of optimization is required for obtaining the optimal value of $J_0$. This is very efficient. When this recursive calculation procedure finishes at the initial stage $k = 1$, the optimal value of $J_1$ for the whole process and the corresponding optimal control sequence at each quantized initial state can be determined. Because this DP algorithm performs the calculation recursively from the last stage backward to the initial stage, it is usually called backward dynamic programming (BDP).

4.2. Implementation of dynamic programming for optimal control

Optimal control based on DP can be implemented off-line or on-line, according to the characteristics of the control problem. Some optimal control problems can be solved by running the BDP algorithm off-line and storing in memory the optimal control values for all the quantized initial states. At each sampling instant, the control signal is generated by looking up the memory based on the measured or estimated state. Using this off-line method, the real time implementation speed can be very fast, because the main computational burden is removed to off-line. However, if there is a disturbance acting on the system, then the required memory capacity increases exponentially with the number of time steps for which this disturbance is predicted. For WEC control, such an off-line approach seems unrealistic.

In the on-line implementation of the DP algorithm for WECs, at each sampling instant, the DP algorithm determines the optimal control input to achieve maximal energy extraction for the WEC over the prediction horizon, while satisfying input and state constraints. Denote the sampling period by $T_s$. Then the $N$-stage receding horizon is $T_0 + N T_s$, where $H_p = N \times T_s$ is the prediction time and $T_0$ is the present time. The receding horizon slides forward by one sampling period after each execution of the algorithm. At $T_0$, the DP algorithm produces an optimal control input sequence for the interval $[T_0, T_0 + H_p]$; however, the control action which is applied to the system is only the first value (at time $t_0$) of this control input sequence. The next execution of the algorithm computes the optimal input for the system within the interval $[t_0 + T_s, t_0 + 2T_s + T_s]$, with updated sea surface/wave prediction, but only the control input value at $t_0 + T_s$ is applied to the system, and so forth. This on-line optimization control method is known as model predictive control (MPC), and also as moving horizon control [60,61].

For this on-line implementation of the DP algorithm in the WEC control, forward dynamic programming (FDP) is better suited than BDP. The FDP algorithm is also based on the principle of optimality, but it calculates the cost by sweeping from the initial stage to the last stage. There are two reasons for us adopting FDP as an on-line optimization algorithm. First, FDP can significantly reduce the on-line computational burden. In FDP, at each sampling instant only the optimal control signal corresponding to one initial state is calculated, i.e., the state measured at the present time. In BDP, the optimal control signal is calculated at each possible grid point. Hence the on-line computational speed and memory storage space for FDP are significantly reduced compared with the BDP. Second, in FDP there is no constraint imposed on the final state.

4.3. The FDP algorithm for WEC control

4.3.1. Problem setup

In view of the recursive nature of the FDP, it is first necessary to discretize the WEC's continuous time model (6). The discretized model is

$$x(k + 1) = A x(k) + B_d u(k) + B_d w(k), \tag{13a}$$

$$y(k) = C_d x(k). \tag{13b}$$

Corresponding to the input and state constraints (7), the control input and the state are constrained in the sets $U$ and $X$, respectively:

$$U := \{u \in \mathbb{R} | -\gamma \leq u \leq \gamma\}. \tag{14}$$

$$X := \{x \in \mathbb{R}^2 | -\delta \leq x_1 \leq \delta\}. \tag{15}$$

Given a predicted wave velocity sequence $w(N) = \{w(0), \ldots, w(N-1)\}$ and a measured initial state $x(0) \in \mathbb{R}^2$, DP aims to resolve the following constrained optimization problem:

$$\min_{k=0}^{N-1} \sum_{k=0}^{N-1} [-y(k)u(k)] \quad \text{subject to} \quad (13)$$

$$x(k) \in X \quad \text{for} \quad k = 0, \ldots, N-1.$$  \tag{16}

4.3.2. Quantization

An essential step in performing DP optimization within reasonable computational limits is to quantize the state space $X$ and the input space $U$. We assume that $u$ can only take one of the two boundary values of $U$, $-\gamma$ and $\gamma$. This is a reasonable assumption in view of the analysis in Section 3.

The quantization of the two-dimensional state space is shown in Appendix A. Note that in the algorithm, it is not necessary to perform the state quantization physically, since the only purpose of the state quantization is to divide the state space into equivalence classes over which the minimum costs are compared [47]; this will be explained in Subsubsection 4.3.4. For a system with a higher state dimension, more sophisticated quantization methods should be used, see [62].
4.3.3. DP recurrence relation

For every \( k \in \{1, 2, \ldots, N\} \) we denote \( u(k) = (u(0), u(1), \ldots, u(k-1)) \). To proceed the DP optimization, it is necessary to derive the recurrence relation. Define, for \( k \geq 1 \), the cost function \( J_k \) from the initial state \( x(0) \) to the present time \( k \) by

\[
J_k(u(k)) = \sum_{i=0}^{k-1} [-y(i)u(i)]
\]

which depends on \( u(k) \). Note that the sequences of states \( x(0), x(1), \ldots, x(k) \) and outputs \( y(0), y(1), \ldots, y(k) \) are uniquely determined by the initial state \( x(0) \) and the wave velocity sequence \( w(N) \). Define, for \( k = 12, \ldots, N \), the minimum cost \( I_k(x) \) from the initial state \( x(0) \) to any reachable present state \( x \) by

\[
I_k(x) = \min_{u(k)} \left\{ J_k(u(k)) \mid x(k) = x \right\}
\]

and set \( I_0 = 0 \). This function \( I_k(x) \) is defined only for those states \( x \) that can be reached from \( x(0) \) without violating the constraints (7) and the minimum is taken over those input sequences \( u(k) \) for which the constraints remain satisfied up to the time \( k \). According to (13), a reachable state \( x \) at time \( k \) can be reached from possibly several states \( z \) at time \( k-1 \) using the input \( u(k-1) \) at time \( k-1 \), if they satisfy

\[
x = A_xz + B_xu(k-1) + B_{xw}w(k-1).
\]

Note that, since \( A_x \) is invertible and \( w(k-1) \) is given, this determines \( z \) as a function of \( x \) and \( u(k-1) \). The function \( I_k \) satisfies the recurrence relation

\[
I_k(x) = \min_{u(k)} \left\{ -(C_xz)u(k-1) + I_{k-1}(z) \mid (19) \text{ holds} \right\}.
\]

The minimum is taken over all values \( u(k-1) \in [-\gamma, \gamma] \) for which \( z \) resulting from (19) is also a reachable state.

We prove (20), using (18):

\[
I_k(x) = \min_{u(k)} \left\{ -y(k-1)u(k-1) + \sum_{i=0}^{k-2} [-y(i)u(i)] \mid x(k) = x \right\}
\]

\[
= \min_{u(k)} \min_{u(k-1)} \left\{ -(C_xz)u(k-1) + \sum_{i=0}^{k-2} [-y(i)u(i)] \mid x(k-1) = z \right\}
\]

\[
\times \text{Min \( z \) such that \( (19) \) holds}
\]

\[
= \min_{u(k)} \left\{ -(C_xz)u(k-1) + I_{k-1}(z) \mid (19) \text{ holds} \right\}.
\]

4.4. The implementation of the FDP for WEC control

In summary, the implementation for WEC control based on the FDP can be represented by the framework shown in Fig. 4, where \( E = \{0, 1, \ldots, 0\} \) is used to extract the first control value \( u^*(0) \) of the control sequence \( u^*(N) \). At each sampling instant, the WEC control follows this procedure:

Step 1 Predict the sea wave magnitude and speed for the next \( N \) steps, i.e., the values of \( w(k) \) with \( k = 0, \ldots, N-1 \).

Step 2 Implement the DP algorithm using the measured state value \( x(0) \) and \( w(k) \) with \( k = 0, \ldots, N-1 \), which is estimated in Step 1; an optimal control sequence \( u^*(N) = (u^*(0), u^*(1), \ldots, u^*(N-1)) \) is determined.

Step 3 Implement only the first control of \( u^*(N) \), i.e., \( u^*(0) \), on the WEC for one sampling period \( T_s \).

Step 4 At the next sampling instant, repeat the procedure from the Step 1.

5. Numerical simulation

The numerical parameters used approximately reflect those of a moderate sized point absorber such as the PB150, but are not intended to be a precise description of any actual system. The float diameter is \( d = 9 \) m. The height of the float is \( \Delta h = 2.4 \) m, hence the range of the float heave motion is \([-1.2, 1.2]\) m. The sea water density is \( \rho = 1025 \) kg/m\(^3\). The gravity constant is 9.8 N/kg. The stiffness is \( K = \pi(d/2)^2 \rho g = \pi 	imes (9/2)^2 	imes 1025 	imes 9.8 = 6.39 \times 10^5 \) N/m. The mass of the float is \( m_f = 10 \times 10^3 \) kg. For a circular cylinder, the added mass is equivalent to the displaced mass of the sea water [63]. Here we estimate the added mass to be \( 7 \times 10^4 \) kg. Then the total mass is \( m = m_f + m_f = 8 \times 10^4 \) kg. The damping coefficient is taken as \( D = 2 \times 10^4 \) Nm/s. The damping ratio corresponding to the friction is \( D_f = 2 \times 10^4 \) Nms. In the simulations, \( \Phi_{\max} = 1.2 \) m. The coefficient \( k \) for the nonlinearity of the inductor is chosen as \( k = 4 \). The maximum control input is chosen to be \( \gamma = 3 \times 10^5 \) N. These parameters are summarized in Table 1.

![Fig. 4. Block diagram for WEC control using FDP (0 stands for the present time).](image-url)
The sea level and velocity for a period of 50 s used for simulations in this paper are shown in Fig. 5.

We have used MATLAB® and SIMULINK® to perform the simulations, but real time code would be written in a language such as C combined with dedicated hardware realized functions. The continuous time model of the WEC is built using SIMULINK® as C combined with dedicated hardware realized functions. The simulations, but real time code would be written in a language such as C combined with dedicated hardware realized functions. The simulations, but real time code would be written in a language such as C combined with dedicated hardware realized functions. The simulations, but real time code would be written in a language such as C combined with dedicated hardware realized functions.

In the dynamic programming algorithm, the ranges for the prediction time grows. The values taken are deliberately very conservative (i.e., in a real application we expect the prediction errors to be smaller) [12]. Note that $e_0$ is not zero; see Section 5.2 for an explanation.

To demonstrate the effectiveness of the DP control, its performance is compared with two alternative control strategies:

a) Linear control:

$$u(k) = \begin{cases} \gamma, & \text{if } Fy(k) > \gamma; \\ Fv(k), & \text{if } -\gamma \leq Fv(k) \leq \gamma; \\ -\gamma, & \text{if } Fy(k) < -\gamma. \end{cases}$$

where $F$ is a constant feedback gain and $v$ is the voltage associated with the vertical velocity of the buoy.

b) Bang–bang control:

$$u(k) = \begin{cases} \gamma, & \text{if } v(k) > 0; \\ -\gamma, & \text{if } v(k) < 0. \end{cases}$$

In order to consider the constraint on $\Phi$ when implementing these two elementary control strategies, two methods are employed:

**Method I** The magnitude of control signals are reduced to lessen the constraint violation on $\Phi$. For bang–bang control, the magnitude of the control input signal is reduced by choosing a smaller value of $\gamma$. For linear control, either a smaller $\gamma$ is chosen and/or the feedback gain $F$ is reduced so that the constraint on $\Phi$ is satisfied. However, the simulations show that accuracy. Fig. 5 shows the example of wave speed used in our simulations, over a time interval of 50 s. In the simulation, the predicted wave data for the DP algorithm is created by adding a sequence of prediction errors $e_k$ to the real wave speed data at each sampling instant from the present time to the end of the prediction time. This $e_k$ is generated by $e_k = \lambda e_{k-1} + w_k$, with $k = 1,2,\ldots,N$. Here $w_k$ is Gaussian white noise, $w_k \sim \mathcal{N}(0,0.1)$ and the initial value of $e_k$ is $e_0 \sim \mathcal{N}(0,0.8)$. We take $\lambda = 1.001$, making the filter unstable, to match with realistic prediction errors that grow as the prediction time grows. The values taken are deliberately very conservative (i.e., in a real application we expect the prediction errors to be smaller) [12]. Note that $e_0$ is not zero; see Section 5.2 for an explanation.

### 5.1. DP control and a comparison with other control strategies

In this subsection, we present two sets of simulations with the prediction horizon for DP control fixed as $H_P = 1$ s, i.e. $N = 25$ (we will show why this horizon is chosen in the next subsection). Firstly, it is assumed that the FDP algorithm is implemented when the future wave profiles can be perfectly predicted without any prediction error. Secondly, since the sea wave prediction error is unavoidable, it is assumed that, at each sampling instant, the standard deviation of the prediction error increases, due to the fact that the longer the prediction time, the worse the prediction

### Table 1

<table>
<thead>
<tr>
<th>Description</th>
<th>Notation</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density of sea water</td>
<td>$\rho$</td>
<td>1025 kg/m$^3$</td>
</tr>
<tr>
<td>Gravity</td>
<td>$g$</td>
<td>9.8 N/kg</td>
</tr>
<tr>
<td>Diameter</td>
<td>$d$</td>
<td>9 m</td>
</tr>
<tr>
<td>Height</td>
<td>$\Delta h$</td>
<td>2.4 m</td>
</tr>
<tr>
<td>Maximum heave motion</td>
<td>$\Phi_{\text{max}}$</td>
<td>1.2 m</td>
</tr>
<tr>
<td>Damping</td>
<td>$D$</td>
<td>$2 \times 10^3$ N/m$^2$</td>
</tr>
<tr>
<td>Damping (friction)</td>
<td>$D_f$</td>
<td>$2 \times 10^3$ N/m$^2$</td>
</tr>
<tr>
<td>Float mass</td>
<td>$m_f$</td>
<td>$10 \times 10^3$ kg</td>
</tr>
<tr>
<td>Added mass</td>
<td>$m_a$</td>
<td>$70 \times 10^3$ kg</td>
</tr>
<tr>
<td>Stiffness</td>
<td>$K$</td>
<td>$6.39 \times 10^4$ N/m</td>
</tr>
<tr>
<td>Maximum input magnitude</td>
<td>$\gamma$</td>
<td>$3 \times 10^4$ N</td>
</tr>
</tbody>
</table>
Fig. 6. The control input as a function of time. The inputs generated by all the candidate controllers satisfy the input constraint.
a better performance can usually be achieved by reducing the value of $F$ alone than by reducing $\gamma$ (or both $\gamma$ and $F$ together).

This is because reducing $\gamma$ can cause more severe input saturation, which can further degrade control performance, while reducing $F$ leads to a smaller control signal, which can lessen or even avoid the saturation, see Fig. 6. In the simulations, $\gamma = 9 \times 10^4$ is chosen for the bang–bang control and $F = 4.5 \times 10^5$ is chosen for the linear control.

Method 2 The control signals are modified to lessen the constraint violation on $\Phi$ during the simulation period as follows: 1) the signal $\Phi$ is measured at each sampling instant; 2) if this measured $\Phi$ exceeds a predefined limit, denoted by $[-\Phi_{\text{max}}, \Phi_{\text{max}}]$, then the control input is set to zero, i.e., $u = 0$.

Note that the value of $\Phi_{\text{max}}$ is usually not the same as the actual limit of $\Phi$. $\Phi_{\text{max}}$ is purely used as a tuning parameter, and its magnitude is influenced by the waves’ (statistical) speed profile. In the simulations, we tune $\Phi_{\text{max}} = 0.4 m$ for both the bang–bang control and the linear control.

In summary, the simulation results are compared in the following cases:

i) Dynamic programming without prediction error;
ii) Dynamic programming with prediction error;
iii) Bang–bang control method 1;
iv) Bang–bang control method 2;
v) Linear control method 1;
v) Linear control method 2;

The control input signals are plotted for all the control methods in Fig. 6, and it is clear that the inputs are all limited within $[-3 \times 10^5, 3 \times 10^5]$. Fig. 7 shows the buoy movement displacement $\Phi$ (in meters) when using all the control methods. It can be seen that the constraint on $\Phi$ is satisfied for the cases i) ii) and vi), but not for the cases iii), iv) and v), no matter what tuning parameters are tried. Although the control methods in cases iii), iv) and v) cannot be used in reality due to the constraint violations, we still include the simulations here for comparison.

Fig. 8 shows the power generated in each case. The power generated by DP control can be negative at some time instants. The energy generated in each of the cases is plotted in Fig. 9. It can be seen that the energy generated using DP without prediction error $(2.39 \times 10^7$ J) is 2 times of the amount generated using linear control with method 2, i.e. case vi) $(1.19 \times 10^7$ J). When the prediction error exists, the amount of generated energy by DP $(2.30 \times 10^7$ J) is $4\%$ less than the amount when perfect prediction is assumed. Fig. 9 demonstrates the robustness of the DP algorithm and also the importance of sea wave prediction in improving the performance of the DP control of WECs. Note that, to improve the numerical accuracy, the generated power and its integration (i.e., energy output) are both calculated by sampling the WEC continuous model at a frequency of 1000 Hz.

In practice it is possible to use the modified bang–bang control or the linear control for the small sea waves and a large limit $\Phi_{\text{max}}$; however, it is more difficult to choose the appropriate values of $\Phi_{\text{max}}, \gamma$ and $F$ in practice than in simulation, because these tuning parameters can vary dramatically due to the changing statistical properties of the waves. Choosing safe values would cause an extra amount of generated energy to be sacrificed.

5.2. The influence of the prediction horizon on DP control performance

In this discussion we assume that there is no prediction error, because the results with prediction error are very similar. Fig. 10 shows the average power generation using the DP control strategy at different prediction horizons from $H_p = 0.4 s$ (corresponding to $N = 10$) to $H_p = 2 s$ (corresponding to $N = 50$). The average power is $P_{\text{avg}} = E/T$, where $T = 50 s$ and $E$ is the energy generated during 50 s. It can be seen that the average power increases sharply with $H_p$ when $H_p \leq 1 s$ (corresponding to $N = 25$), and it saturates after $H_p = 1 s$. This means that a prediction horizon of only $H_p = 1 s$ can be used without degrading the WEC performance significantly, while reducing the computational burden.

The simulations show that the prediction horizon required to secure the majority of the benefits from optimal control is very short, typically 1 s. The natural conclusion is that a very short wave prediction time is required from the DSWP, so short in fact that it suggests that simple linear extrapolation from the present wave profile data should be effective. This at first appears surprising because the early work by Falnes suggested long prediction times would be required. The explanation is that Falnes’ approach was to obtain an estimate of the Fourier transform $W(j\omega)$ of the incoming wave profile and then conjugate match the frequency response of the WEC to this. Clearly this requires a long enough predictive time window to obtain a reasonable estimate of $W(j\omega)$.

The obvious conclusion would appear to be that the extra effort involved in providing DSWP is not justified in practice, suggesting that by knowing the WEC motion the wave profile can be determined. For totally linear WECs, obtaining the profile of a multidirectional sea in this manner would imply de-convoluting the WEC impulse response from the DSWP to determine the wave system in the $x, y, t$ domain. However the WEC motion is subject to constraints and other nonlinearities, so that it is not possible to estimate the wave profile from local data. It is necessary to use wave sensors placed sufficiently far from the WEC to minimize the effect of the wave system created by the WEC motion. This distance causes a propagation delay which once more necessitates substantial prediction times, of the order of 10 s. Thus the conclusion is that while in principle a long prediction horizon of tens of seconds is not required for optimal WEC control, in practice it is the only way that the predictive control can be realized.

![Fig. 7](image_url)

**Fig. 7.** The vertical displacement difference between the water level and the mid-point of the float. The state constraint is satisfied for the DP control with or without prediction error and the control methods i), ii) and vi); but a constraint violation occurs for the control methods iii), iv) and vi) at about 14 s.
Fig. 8. Extracted power as a function of time, over 50 s. Note that the power generated by DP control can be negative at some time instants.
In this paper, the constrained optimal control problem of WECs is addressed. After addressing modeling and optimal control issues, we have proposed an on-line control algorithm for a point absorber, based on DP, MPC and wave prediction. Two other elementary control strategies have also been explored for the sake of comparison. Numerical simulations demonstrate the potential of our control strategies have also been explored for the sake of comparison. Numerical simulations demonstrate the potential of our control strategies.

6. Conclusions

In this paper, the constrained optimal control problem of WECs is addressed. After addressing modeling and optimal control issues, we have proposed an on-line control algorithm for a point absorber, based on DP, MPC and wave prediction. Two other elementary control strategies have also been explored for the sake of comparison. Numerical simulations demonstrate the potential of our control strategies have also been explored for the sake of comparison. Numerical simulations demonstrate the potential of our control strategies.

3) Optimal control of a whole wave farm, taking coupling into account.
4) Experimental tests for the control algorithms.

Acknowledgments

The authors acknowledge the support of European Commission, Seventh Framework Programme, “Demonstration & Deployment of a Commercial Scale Wave Energy Converter with an Innovative Real Time Wave by Wave Tuning System (WAVEPORT)”. The second author thanks the Royal Academy of Engineering for a “distinguished visiting fellowship” that enabled him to visit Exeter in August–September 2010. The second author was partially supported by Israel Science Foundation grant no. 701/10.

Appendix A. Quantization of the state space

Suppose \( x_i \in \mathbb{R} \), with \( i = 1, 2 \), are evenly divided by \( N_\text{xi} \) points in the range of \([x_\text{min}, x_\text{max}]\), so that the length of each interval is

\[
\Delta x_i = \frac{x_\text{max} - x_\text{min}}{N_\text{xi} - 1}
\]

For the WEC control problem, we have \( x_1 \text{ min} = -\delta \), \( x_1 \text{ max} = \delta \). Since no constraint is imposed on \( x_2 \), the values of \( x_2 \text{ min} \) and \( x_2 \text{ max} \) should be chosen such that the interval \([x_2 \text{ min}, x_2 \text{ max}]\) is large enough to guarantee that the trajectory of \( x_2 \) is always within this range.

The grid points of \( \mathbb{X} \cap (\mathbb{R}^2 | x_2 \text{ min} \leq x_2 \leq x_2 \text{ max}) \) are

\[
x_{1t_i} = x_{1\text{ min}} + (t_i - 1) \Delta x_1 \quad \text{with} \quad t_i = 1, \ldots, N_{\text{xi}}, \quad i = 1, 2
\]

where the first subscript \( i \) represents the \( i \)th component of \( x \), while the second subscript \( t_i \) represents the \( t \)th quantized value of the corresponding \( i \)th state component. The set of quantized state vectors is given by

\[
\chi^Q = \left\{ x_i^Q \mid x_i^Q = \left[ x_{1t_1} \ x_{2t_2} \right], \quad t_1 = 1, \ldots, N_{x1}, \ t_2 = 1, \ldots, N_{x2} \right\}
\]

where the superscript \( Q \) represents the index of each vector in \( \chi^Q \).

Appendix B. Associating a quantized state

Given a state \( x(k + 1) = [x_1(k), x_2(k)] \), the index \( j \) of the nearest point \( x^Q \in \chi^Q \) can be determined by:

\[
 j_1 = \left[ \frac{\left| x_1(k) - x_{1\text{ min}} \right|}{\Delta x_1} + 0.5 \right]
 j_2 = \left[ \frac{\left| x_2(k) - x_{2\text{ min}} \right|}{\Delta x_2} + 0.5 \right]
 j = j_2N_{x1} + j_1 + 1.
\]

Here \( \left| c \right| \) denotes the largest integer that is \( \leq c \in \mathbb{R} \).

References


