Controllability and observability of a well-posed system coupled with a finite-dimensional system

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***Abstract*—We consider coupled systems consisting of a well-posed and strictly proper (hence regular) subsystem and a finite-dimensional subsystem connected in feedback. The external world interacts with the coupled system via the finite-dimensional part, which receives the external input and sends out the output. Under several assumptions, we derive well-posedness, regularity, exact (or approximate) controllability and exact (or approximate) observability results for such coupled systems.**

***Index Terms*—coupled system, well-posed system, exact controllability, approximate controllability, exact observability, approximate observability, simultaneous observability.**

I. INTRODUCTION

THIS paper is about coupled systems in which a well-posed and strictly proper linear system Σ_d is connected to a finite-dimensional linear system Σ_f with an invertible first component in its feedthrough matrix. We consider two kinds of structures: the special structure shown in Figure 1 and the general structure shown in Figure 3. We show that these coupled systems are well-posed and actually regular (this is easy). Then we address the question of exact (or approximate) controllability of the coupled system. For this we need that the two subsystems should be exactly (or approximately) controllable and we need also additional assumptions of an algebraic nature. We derive analogous results for exact (or approximate) observability.

Coupled infinite-dimensional systems have attracted much interest in recent years. For example, the book of Dáger and Zuazua [2] is devoted mainly to the study of flexible strings connected to form a planar graph. The theses of Villegas [14] and Pasumarthy [8] study the power-preserving interconnection of several port-Hamiltonian systems, possibly infinite-dimensional, using the formalism of Dirac structures developed by Arjan van der Schaft. The book of Lasiecka [5] is devoted mainly to the structural acoustic model, where a plate and a wave equation are coupled to create a model of an aircraft cockpit. These works contain a lot of further references on the topic of coupled systems.

Recently we have developed in [19] a theory for the regularity and controllability of coupled systems consisting of an infinite-dimensional subsystem Σ_d and a finite-dimensional subsystem Σ_f connected in feedback. In [19] we assume that the feedthrough matrix of Σ_f is zero and that Σ_d is such that it becomes well-posed and strictly proper when connected in cascade with an integrator (we call such systems SPI systems). We have shown that such coupled systems are well-posed and regular, when we use as state space a certain subspace of the product of the state spaces of the subsystems. Moreover, under certain assumptions, we have derived exact and approximate controllability results for this class of coupled systems. There is a certain analogy between the main controllability results in this paper and those in our paper [19]. In this paper the well-posedness and controllability results are simpler and neater since the assumptions allow us to work with the natural product state space. The well-posedness and controllability results here cannot be derived from those in [19] (or the other way round). In this paper we give also observability results (unlike in [19], where the asymmetric assumptions on Σ_d made this difficult).

We have applied the results in this paper to prove the well-posedness, exact (or approximate) controllability and the exponential (or strong) stabilization of wind turbine tower models consisting of a SCOLE beam system coupled to a two-mass drive-train model. We have considered various kinds of inputs and various state spaces. These results will be published in separate papers, see for example [20].

First we consider a coupled system with the special structure in Figure 1, denoted by Σ_{cs}. We assume that the external world interacts with Σ_{cs} via the finite-dimensional subsystem Σ_f, which receives the input $v = u_e - y$, where u_e is the input of Σ_{cs} and the signal y comes from Σ_d. The system Σ_f sends out the output u, which is also the output of the coupled system Σ_{cs}. The equations of Σ_f are

$$\begin{cases} \dot{q}(t) = aq(t) + bu_e(t) - by(t), & (1.1) \\ u(t) = cq(t) + du_e(t) - dy(t), & (1.2) \end{cases}$$

where $a \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^{n\times m}$, $c \in \mathbb{C}^{m\times n}$, $d \in \mathbb{C}^{m\times m}$ and $q(t) \in \mathbb{C}^n$ is the state of Σ_f.

Let p be a function defined on some domain in $\mathbb{C}$ that contains a right half-plane, with values in a normed space. We say that p is *strictly proper* if

$$\lim_{\mathrm{Re}\, s \to \infty} \|p(s)\| = 0, \qquad \text{uniformly with respect to } \mathrm{Im}\, s.$$

A linear system is called *strictly proper* if its transfer function is strictly proper.

The well-posed linear system Σ_d, with input function u, input space $\mathbb{C}^m$, state trajectory z, output function y and output space $\mathbb{C}^m$ is assumed to be strictly proper (hence regular). It

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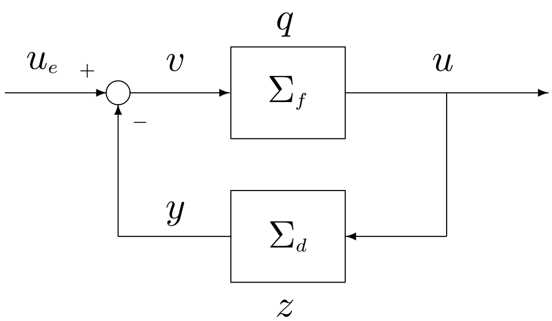

Fig. 1. A coupled system Σ_{cs} consisting of a well-posed and strictly proper system Σ_d and a finite-dimensional system $\Sigma_f = (a, b, c, d)$, connected in feedback.

is determined by its generating triple (A, B, C) via

$$\dot{z}(t) = Az(t) + Bu, \qquad y(t) = C_\Lambda z. \tag{1.3}$$

Here A is the semigroup generator of Σ_d, which generates a strongly continuous semigroup $\mathbb{T}$ on the state space X (a Hilbert space), $B \in \mathcal{L}(\mathbb{C}^m, X_{-1})$ is the control operator of Σ_d and $C \in \mathcal{L}(X_1, \mathbb{C}^m)$ is its observation operator and C_Λ is the Λ-extension of C. As Σ_d is strictly proper, its feedthrough operator is zero. The transfer function $\mathbf{G}$ of Σ_d is given by

$$\mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B, \qquad \forall\, s \in \rho(A).$$

For the terminology on regular systems that has been used here we refer to the background in Section II.

Our approach to proving controllability properties of Σ_{cs} is to consider it as a cascaded system Σ_{casc} (the open loop system in Figure 2) with a feedback. The input of Σ_{casc} is v (see Figure 1), and its outputs are u and y. We obtain Σ_{cs} via the feedback $v = u_e - y$. The cascaded system is easier to analyze than the coupled system and its controllability properties are invariant under feedback. Our approach to proving observability properties of Σ_{cs} is similar, but we use a different cascaded system (with the order reversed), as shown in Figure 5. The main idea for proving the controllability properties of Σ_{casc} (from Figure 2) is to perform a flow-inversion on Σ_f, so that u becomes the common input of the two subsystems. Now we can use the simultaneous controllability results presented in Tucsnak and Weiss [13]. The generator of the flow-inverted Σ_f is $a^\times = a - bd^{-1}c$, and this matrix plays a role in all our main results.

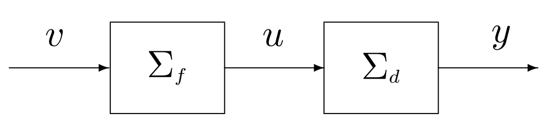

Fig. 2. A cascaded system Σ_{casc} consisting of a well-posed and strictly proper system Σ_d and a finite-dimensional system $\Sigma_f = (a, b, c, d)$.

For the well-posedness, controllability and observability properties of the coupled system Σ_{cs} we have the following:

Theorem I.1. *Let Σ_d be a well-posed and strictly proper (hence regular) system with input space $\mathbb{C}^m$, state space X (a Hilbert space), output space $\mathbb{C}^m$, semigroup $\mathbb{T}$ and generating triple (A, B, C). Let a, b, c, d be matrices as in* (1.1)–(1.2). *Then the coupled system Σ_{cs} from Figure* 1 *described by* (1.1), (1.2) *and* (1.3), *with input u_e, state $\left[\begin{smallmatrix} z \\ q \end{smallmatrix}\right]$ and output $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$, is well-posed and regular with the state space $X \times \mathbb{C}^n$.*

Now assume additionally the following:
(i) *(A, B) is exactly controllable in time T_0;*
(ii) *(a, b) is controllable;*
(iii) *$d \in \mathbb{C}^{m\times m}$ is invertible;*
(iv) *Denote $a^\times = a - bd^{-1}c$. Then A^* and $a^{\times *}$ have no common eigenvalue.*

Then Σ_{cs} is exactly controllable in any time $T > T_0$.

Theorem I.2. *We use the assumptions and the notation from the first part of Theorem* I.1. *We also assume the following:*
(i) *(A, C) is exactly observable in time T_0;*
(ii) *(a, c) is observable;*
(iii) *$d \in \mathbb{C}^{m\times m}$ is invertible;*
(iv) *A and $a^\times$ have no common eigenvalue.*

Then Σ_{cs}, with output u only, is exactly observable in any time $T > T_0$.

For approximate controllability and approximate observability we have weaker results, in which we cannot tell the approximate controllability (or observability) time of the coupled system. We denote by $\rho_\infty(A)$ the connected component of $\rho(A)$ containing some right half-plane.

Proposition I.3. *We use the assumptions and the notation from the first part of Theorem* I.1. *We also assume the following:*
(i) *(A, B) is approximately controllable in some time;*
(ii) *(a, b) is controllable;*
(iii) *$d \in \mathbb{C}^{m\times m}$ is invertible;*
(iv) *Denote $a^\times = a - bd^{-1}c$. We have $\sigma(a^\times) \subset \rho_\infty(A)$.*

Then Σ_{cs} is approximately controllable in some time.

Proposition I.4. *We use the assumptions and the notation from the first part of Theorem* I.1. *We also assume the following:*
(i) *(A, C) is approximately observable in some time;*
(ii) *(a, c) is observable;*
(iii) *$d \in \mathbb{C}^{m\times m}$ is invertible;*
(iv) *Denote $a^\times = a - bd^{-1}c$. We have $\sigma(a^\times) \subset \rho_\infty(A)$.*

Then Σ_{cs}, with output u only, is approximately observable in some time.

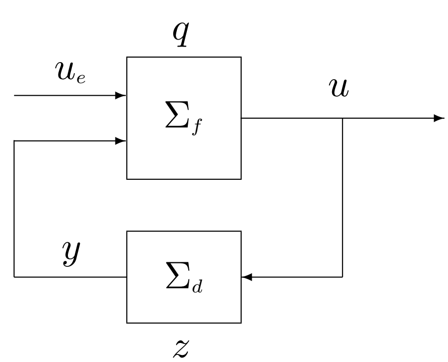

Fig. 3. A more general coupled system Σ_c consisting of a well-posed and strictly proper system Σ_d and a finite-dimensional system $\Sigma_f = (a, b, b_f, c, d, d_f)$.

Now we consider coupled systems with the general structure as shown in Figure 3, denoted by Σ_c and described by the equations

$$\begin{cases} \dot{q}(t) = aq(t) + bu_e(t) - b_f y(t), & (1.4) \\ u(t) = cq(t) + du_e(t) - d_f y(t), & (1.5) \\ \dot{z}(t) = Az(t) + Bu(t), & (1.6) \\ y(t) = C_\Lambda z(t). & (1.7) \end{cases}$$

This general structure allows the external input u_e and the feedback signal y to be two separate inputs of Σ_f described by (1.4)-(1.5). Here $a \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^{n\times m}$, $c \in \mathbb{C}^{m\times n}$, $d \in \mathbb{C}^{m\times m}$, $b_f \in \mathbb{C}^{n\times p}$ and $d_f \in \mathbb{C}^{m\times p}$. The system Σ_{cs} from Figure 1 is a particular case of this system, corresponding to $b_f = b$ and $d_f = d$. The well-posed subsystem Σ_d described by (1.6)–(1.7) is again assumed to be strictly proper, but now its input and output dimensions may be different (m and p).

For the well-posedness and exact controllability of the coupled system Σ_c, we have the following theorem, which guarantees the exact controllability of Σ_c for a very large (open and dense) set of the pair $\left(b_f \in \mathbb{C}^{n\times p}, d_f \in \mathbb{C}^{m\times p}\right)$, but not for all. Thus, exact controllability is a generic property with respect to b_f and d_f.

Theorem I.5. *Let Σ_d be a well-posed and strictly proper (hence regular) system with input space $\mathbb{C}^m$, state space X (a Hilbert space), output space $\mathbb{C}^p$, semigroup $\mathbb{T}$ and generating triple (A, B, C). Let a, b, b_f, c, d, d_f be matrices as in* (1.4)–(1.5)*. Then the coupled system Σ_c from Figure* 3 *described by* (1.4)–(1.7)*, with input u_e, state $\left[\begin{smallmatrix} z \\ q \end{smallmatrix}\right]$ and output $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$, is well-posed and regular with the state space $X \times \mathbb{C}^n$.*

Now assume additionally that conditions (i)–(iv) *from Theorem* I.1 *hold. Then for every $T > T_0$ there is an open dense set $\mathcal{O}_T \subset \left(\mathbb{C}^{n\times p} \times \mathbb{C}^{m\times p}\right)$ (that may depend on A, B, C, a, b, c, d and T) such that for every pair $(b_f, d_f) \in \mathcal{O}_T$, the coupled system Σ_c is exactly controllable in time T. If we set $d_f = 0$, then the exact controllability holds for b_f in an open and dense subset of $\mathbb{C}^{n\times p}$. For every $b_1 \in \mathbb{C}^{n\times p}$ and $d_1 \in \mathbb{C}^{m\times p}$ there exists a set $F_T \in \mathbb{C}$ with at most n elements such that the coupled system Σ_c with $b_f = \lambda b_1$ and $d_f = \lambda d_1$, with $\lambda \in \mathbb{C} \setminus F_T$, is exactly controllable in time T.*

We mention the obvious facts that $\mathcal{O}_T$ is non-decreasing as a function of T, while (for every fixed pair (b_f, d_f)) F_T is non-increasing. We give below a simple finite-dimensional example that shows that the set $\mathcal{O}_T$ in Theorem I.5 is not necessarily equal to $\mathbb{C}^{n\times p} \times \mathbb{C}^{m\times p}$.

Example I.6. Let $U = X = Y = \mathbb{C}$ and take $A = 0$, $B = 1$, $C = 1$,

$$a = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b_f = \begin{bmatrix} 0 \\ \lambda \end{bmatrix},$$

$$c = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad d = 1, \quad d_f = 0.$$

We show that all the assumptions in Theorem I.5 are satisfied. It is easy to see that both (A, B) and (a, b) are controllable and that d is invertible, so that assumptions (i) (ii) and (iii) are true. By computation we have

$$a^\times = a - bd^{-1}c = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix},$$

which has eigenvalues 1 and -1, so that assumption (iv) is satisfied. By Proposition IV.2, we get the following matrices A^c and B^c for the coupled system Σ_c:

$$A^c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -\lambda & 0 & 1 \end{bmatrix}, \quad B^c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

By computation we have

$$\begin{bmatrix} B^c & A^c B^c & A^{c2} B^c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -\lambda + 1 & -2\lambda + 1 \end{bmatrix}.$$

The corresponding determinant is

$$\det \begin{bmatrix} B^c & A^c B^c & A^{c2} B^c \end{bmatrix} = 2\lambda - 1.$$

Thus, Σ_c is not controllable for $\lambda = \frac{1}{2}$. This means that for this λ, the pair (b_f, d_f) is not in the set $\mathcal{O}_T$ from Theorem I.5, for every $T > 0$.

Remark I.7. There is no need to construct an observability counterpart to Theorem I.5. Indeed, for the system in Figure 3, when we discuss observability we take $u_e = 0$, so that it reduces to the system in Figure 1 (with $u_e = 0$).

The structure of the paper is as follows: Section II is dedicated to the background. Here we give the necessary preliminaries about admissible control and observation operators, well-posed linear systems, and regular linear systems. We recall the concept of the closed-loop system associated to a well-posed linear system with an admissible feedback operator. We also discuss controllability and observability, in particular simultaneous controllability (or observability).

In Section III we analyze the cascaded system Σ_{casc} from Figure 2. We prove its well-posedness, regularity and its exact (or approximate) controllability (depending on the assumptions), with the state space $X \times \mathbb{C}^n$. We derive its generating operators and transfer function. We also consider a slightly different cascaded system, needed for the study of the observability properties of Σ_{cs}.

In Section IV we consider coupled systems as in Figures 1 and 3 and we prove our main results. The idea of the proof for Theorem I.1 and Proposition I.3 is to consider the coupled system from Figure 1 as being obtained from Σ_{casc} via an admissible output feedback. Then, the controllability properties of the closed-loop system are inherited from the open-loop system. Theorem I.2 and Proposition I.4 can be obtained by similar arguments, using a different cascaded system. The idea of the proof for Theorem I.5 is to consider the input maps of Σ_c as finite-rank perturbations of the input maps of Σ_{cs}, while regarding u as the input signal.

In Section V we present an example to illustrate Theorem I.1 and Proposition I.4. The physical system being modeled is a flexible shaft with one end connected to a rigid body that is attached to a beam. The other end of the beam is clamped. The control signal is the angular velocity at which the free end of the flexible shaft is being turned. We choose the angular velocity of the rigid body as well as the torque acting on this rigid body from the shaft as output signals. It is not easy to analyze the well-posedness, controllability and observability of this system directly, but we derive them using Theorem I.1 and Proposition I.4.

II. Some background on infinite-dimensional systems

In this section we introduce some concepts and results on infinite-dimensional linear time invariant systems, without proof. For the details we refer to the literature.

A. Admissible control and observation operators

The material of this section can be found (in much greater detail and with many references) in Tucsnak and Weiss [13].

Let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on a Hilbert space X. Then $\mathbb{T}$ and X determine two additional Hilbert spaces: X_1 is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, and X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|$, where $\beta \in \rho(A)$ is fixed. These spaces are independent of the choice of β, since different values of β lead to equivalent norms on X_1 and X. The norm $\|z\|_1$ is equivalent to the graph norm of A. We have $X_1 \subset X \subset X_{-1}$ densely and with continuous embeddings. We can extend A to a bounded operator from X to X_{-1}, still denoted by A. The semigroup generated by this extended A is the extension of $\mathbb{T}$ to X_{-1}, which is still denoted by $\mathbb{T}$.

In this section, U, X and Y are Hilbert spaces, $\mathbb{T}$ is a strongly continuous semigroup on X, with generator A, $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$.

We define the operators Φ_τ (for $\tau > 0$) by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau - t} B u(t) \mathrm{d}t,$$

where $u \in L^2_{loc}([0,\infty); U)$. Clearly Φ_τ is bounded from $L^2([0,\infty); U)$ to X_{-1}. These operators are called the *input maps* of (A, B).

Definition II.1. B is said to be an *admissible control operator* for the semigroup $\mathbb{T}$ if $\mathrm{Ran}\ \Phi_\tau \subset X$ for some $\tau > 0$.

The admissibility of B implies that the solutions $z(\cdot)$ of

$$\dot{z}(t) = Az(t) + Bu(t), \tag{2.1}$$

with initial state $z(0) = z_0 \in X$ and with $u \in L^2_{loc}([0,\infty); U)$ remain in X. Moreover it follows that $z(\cdot)$ is a continuous X-valued function of t and

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u. \tag{2.2}$$

The Laplace transform of $z(\cdot)$ is

$$\hat{z}(s) = (sI - A)^{-1} z_0 + (sI - A)^{-1} B \hat{u}(s).$$

The operator B is said to be *bounded* if $B \in \mathcal{L}(U, X)$ and *unbounded* otherwise.

In the sequel, we denote by ω_0 the growth bound of $\mathbb{T}$. We also use the notation $\mathbb{C}_\alpha$ for the right half-plane determined by the real number α:

$$\mathbb{C}_\alpha = \{ s \in \mathbb{C} \mid \mathrm{Re}\, s > \alpha \}.$$

Proposition II.2. *If B is admissible then for every $\alpha > \omega_0$ there exists a constant $K_\alpha \geq 0$ such that*

$$\|(sI - A)^{-1} B\|_{\mathcal{L}(U,X)} \leq \frac{K_\alpha}{\sqrt{\mathrm{Re}\, s - \alpha}} \qquad \forall\, s \in \mathbb{C}_\alpha.$$

We define the operators Ψ_τ (for $\tau > 0$) by

$$(\Psi_\tau z_0)(t) = \begin{cases} C\mathbb{T}_t z_0 & \text{for } \ t \in [0, \tau], \\ 0 & \text{for } \ t > \tau. \end{cases}$$

Clearly Ψ_τ is bounded from X_1 to $L^2([0,\infty); Y)$. These operators are called the *output maps* of (A, C). C is said to be *bounded* if it can be extended such that $C \in \mathcal{L}(X, Y)$ and *unbounded* otherwise.

Definition II.3. C is said to be an *admissible observation operator* for the semigroup $\mathbb{T}$ if Ψ_τ has a continuous extension to X for some $\tau > 0$.

The admissibility of C is equivalent to the fact that for some (hence, for every) $\tau > 0$ there is a $K_\tau \geq 0$ such that

$$\int_0^\tau \|C\mathbb{T}_t z_0\|^2 \mathrm{d}t \leq K_\tau^2 \|z_0\|^2 \qquad \forall\, z_0 \in \mathcal{D}(A).$$

We regard $L^2_{loc}([0,\infty); Y)$ as a Fréchet space with the seminorms being the L^2 norms on the intervals $[0, n],\ n \in \mathbb{N}$. Then the admissibility of C means that there exists a continuous operator $\Psi : X \to L^2_{loc}([0,\infty); Y)$ such that

$$(\Psi z_0)(t) = C\mathbb{T}_t z_0 \qquad \forall\, z_0 \in \mathcal{D}(A). \tag{2.3}$$

The operator Ψ is completely determined by (2.3), because $\mathcal{D}(A)$ is dense in X.

We introduce the Λ-extension of C, denoted C_Λ, by

$$C_\Lambda z_0 = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} z_0, \tag{2.4}$$

whose domain $\mathcal{D}(C_\Lambda)$ consists of all $z_0 \in X$ for which the limit exists. If we replace C by C_Λ, formula (2.3) becomes true for all $z_0 \in X$ and for almost every $t \geq 0$. If $y = \Psi z_0$, then its Laplace transform is $\hat{y}(s) = C(sI - A)^{-1} z_0$.

B. Well-posed linear systems

A *well-posed linear system* with *input space U, state space* X and *output space* Y is a family of bounded linear operators (parametrized by $\tau \geq 0$) that associates to every initial state $z_0 \in X$ and every input signal $u \in L^2([0,\tau]; U)$ a final state $z(\tau)$ and an output signal $y \in L^2([0,\tau]; Y)$. These operators have to satisfy certain natural functional equations, for the formal definition we refer to Weiss [15].

By continuous extension, for any well-posed linear system, we can define state trajectories and output signals for any initial state in the state space X and for any input signal in $L^2_{loc}([0,\infty); U)$; the output signal is then in $L^2_{loc}([0,\infty); Y)$. For more detailed background about well-posed systems we refer to Salamon [9], Staffans [10], Staffans and Weiss [11], Weiss, Staffans and Tucsnak [18].

We recall some facts about well-posed linear systems from [15], [16]. Let Σ be a well-posed system with input space U, state space X and output space Y. Then Σ is completely determined by its *generating triple* (A, B, C) and its transfer function $\mathbf{G}$. Here, A is the *semigroup generator* of Σ, which generates a strongly continuous semigroup $\mathbb{T}$ on X, $B \in \mathcal{L}(U, X_{-1})$ is the *control operator* of Σ and $C \in \mathcal{L}(X_1, Y)$ is its *observation operator*. The transfer function $\mathbf{G}$ satisfies

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C[(sI - A)^{-1} - (\beta I - A)^{-1}]B \tag{2.5}$$

for all $s, \beta \in \rho(A)$. The state trajectories of Σ satisfy (2.1), hence also (2.2). If $u \in L^2_{loc}([0,\infty);U)$ is the input function of Σ, $z_0 \in X$ is its initial state and $y \in L^2_{loc}([0,\infty);Y)$ is the corresponding output function, then

$$y = \Psi z_0 + \mathbb{F} u. \tag{2.6}$$

Here, Ψ is the operator from (2.3). The operator $\mathbb{F}$ appearing above is continuous from $L^2_{loc}([0,\infty);U)$ to $L^2_{loc}([0,\infty);Y)$ (which we regard as Fréchet spaces, see the comments around (2.3)). It is easiest to represent $\mathbb{F}$ using Laplace transforms, as follows: if $u \in L^2([0,\infty);U)$ and $y = \mathbb{F}u$, then y has a Laplace transform $\hat{y}$ and

$$\hat{y}(s) = \mathbf{G}(s)\hat{u}(s) \tag{2.7}$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s$ sufficiently large. This determines $\mathbb{F}$, since $L^2([0,\infty);U)$ is dense in $L^2_{loc}([0,\infty);U)$. $\mathbf{G}$ is *proper* which means that its domain contains a right half-plane $\mathbb{C}_\alpha$ such that $\mathbf{G}$ is uniformly bounded on $\mathbb{C}_\alpha$.

Conversely, if $\mathbf{G}$ is an analytic and proper $\mathcal{L}(U,Y)$-valued function, then $\mathbf{G}$ determines a continuous operator $\mathbb{F}$ from $L^2_{loc}([0,\infty);U)$ to $L^2_{loc}([0,\infty);Y)$ via (2.7) (see for example [15, Theorem 3.6]). (In (2.7) we only take $u \in L^2([0,\infty);U)$, but this determines $\mathbb{F}$, as explained earlier.) We define the *input-output maps* of $\mathbf{G}$, denoted by $\mathbb{F}_\tau$ ($\tau \geq 0$), by truncating the output to $[0,\tau]$:

$$\mathbb{F}_\tau u = (\mathbb{F}u)|_{[0,\tau]}.$$

The operator $\mathbb{F}$ (defined above via $\mathbf{G}$) is *causal*, which means that $\mathbb{F}_\tau u$ depends only on the truncation $u|_{[0,\tau]}$. It follows that we may regard $\mathbb{F}_\tau$ as a bounded operator from $L^2([0,\tau];U)$ to $L^2([0,\tau];Y)$.

Definition II.4. Let U, X and Y be complex Hilbert spaces. A triple of operators (A,B,C) is called *well-posed* on (U,X,Y) if there exists a well-posed linear system Σ on (U,X,Y) such that (A,B,C) is the generating triple of Σ.

This definition and the Proposition below are taken from Curtain and Weiss [1].

Proposition II.5. *A triple of operators (A,B,C) is well-posed on (U,X,Y) if and only if:*
(1) *A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X, $B \in \mathcal{L}(U,X_{-1})$ and $C \in \mathcal{L}(X_1,Y)$,*
(2) *B is an admissible control operator for $\mathbb{T}$,*
(3) *C is an admissible observation operator for $\mathbb{T}$,*
(4) *some (hence every) transfer function $\mathbf{G}$ associated with (A,B,C) (i.e., satisfying (2.5)) is proper.*

Let Σ be a well-posed linear system on (U,X,Y) with generating triple (A,B,C) and transfer function $\mathbf{G}$. An operator $K \in \mathcal{L}(Y,U)$ is called an *admissible feedback operator* for Σ (or for $\mathbf{G}$) if $I - \mathbf{G}K$ has a proper inverse (equivalently, if $I - K\mathbf{G}$ has a proper inverse). If this is the case, then the system with output feedback shown in Figure 4 is well-posed on (U,X,Y) (its input is v, its state and output are the same as for Σ). This new system is called the *closed-loop system* corresponding to Σ and K, and it is denoted by Σ^K. Its transfer function is

$$\mathbf{G}^K = \mathbf{G}(I - K\mathbf{G})^{-1} = (I - \mathbf{G}K)^{-1}\mathbf{G}. \tag{2.8}$$

We have that $-K$ is an admissible feedback operator for Σ^K and the corresponding closed-loop system is Σ. Let us denote by (A^K, B^K, C^K) the generating triple of Σ^K. Then for every $x_0 \in \mathcal{D}(A^K)$ and for every $z_0 \in \mathcal{D}(A)$,

$$A^K x_0 = \left(A + BKC^K\right) x_0, \quad A z_0 = \left(A^K - B^K K C\right) z_0.$$

For more details on closed-loop systems we refer to [16].

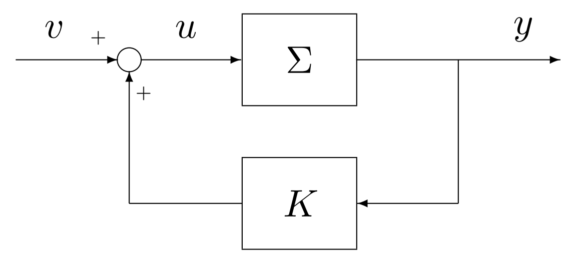

Fig. 4. A well-posed linear system Σ with output feedback via K. If K is admissible, then this is a new well-posed linear system Σ^K, called the closed-loop system.

Definition II.6. The well-posed linear system Σ is called *regular* if the limit

$$\lim_{s \to +\infty} \mathbf{G}(s)\mathrm{v} = D\mathrm{v}$$

exists for every $\mathrm{v} \in U$, where s is real. Then the operator $D \in \mathcal{L}(U,Y)$ is called the *feedthrough operator* of Σ.

We mention a few facts about regular systems, following [15]. Regularity is equivalent to the fact that the product $C_\Lambda(sI - A)^{-1}B$ makes sense, for some (hence for every) $s \in \rho(A)$. Here C_Λ is the Λ-extension of C defined in (2.4). If Σ is regular then for every initial state $z_0 \in X$ and every $u \in L^2_{loc}([0,\infty);U)$, the solution of $\dot{z} = Az + Bu$ with $z(0) = z_0$ satisfies $z(t) \in \mathcal{D}(C_\Lambda)$ for almost every $t \geq 0$ and the corresponding output from (2.6) is given by

$$y(t) = C_\Lambda z(t) + Du(t) \qquad \text{for almost every} \quad t \geq 0. \tag{2.9}$$

The transfer function of the regular system Σ is given by

$$\mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B + D \qquad \forall\, s \in \rho(A). \tag{2.10}$$

The operators A, B, C, D are called the *generating operators* of Σ. This is because they determine Σ via (2.1) and (2.9).

The following proposition follows from the results in [16, Sections 4, 7]. We need the space Z introduced as follows:

$$Z = X_1 + (\beta I - A)^{-1}BU.$$

Proposition II.7. *Suppose that Σ is a regular linear system on (U,X,Y) with generating operators A, B, C and D. We assume that U is finite-dimensional. Let K be an admissible feedback operator for Σ and let Σ^K be the corresponding closed-loop system. Then the following holds:*

(1) *$I - DK$ (and hence also $I - KD$) is invertible.*

(2) *Σ^K is regular.*

(3) *Let A^K, B^K, C^K and D^K be the generating operators of Σ^K. Then*

$$A^K = A + BK(I - DK)^{-1}C_\Lambda,$$

$$\mathcal{D}(A^K) = \{z \in Z \mid Az + BK(I-DK)^{-1}C_\Lambda z \in X\},$$
$$B^K = B(I-KD)^{-1}, \qquad C^K = (I-DK)^{-1}C_\Lambda,$$
$$D^K = D(I-KD)^{-1} = (I-DK)^{-1}D.$$

C. Controllability and observability

Let U, X, Y, $\mathbb{T}$, A, B, C, Φ_τ and Ψ_τ be as in Subsection II-A. We assume that B and C are admissible for $\mathbb{T}$.

Definition II.8. The pair (A, B) is said to be *exactly controllable in time* $\tau > 0$ if Ran $\Phi_\tau = X$; (A, B) is said to be *approximately controllable in time* $\tau > 0$ if Ran Φ_τ is dense in X.

Definition II.9. The pair (A, C) is said to be *exactly observable in time* $T > 0$ if Ψ_T is bounded from below, i.e., there exists $\kappa_T > 0$ such that

$$\int_0^T \|C\mathbb{T}_t z_0\|_Y^2 \mathrm{d}t \geq \kappa_T^2 \|z_0\|^2. \tag{2.11}$$

(A, C) is said to be *approximately observable in time* $T > 0$ if Ker $\Psi_T = \{0\}$.

We often need the controllability concepts without specifying a time τ. Therefore the following definition is introduced.

Definition II.10. The pair (A, B) is said to be *exactly controllable* if it is exactly controllable in some finite time $\tau > 0$. (A, B) is said to be *approximately controllable* if it is approximately controllable in some finite time.

Observability concepts without a specified time are introduced in a similar way.

Proposition II.11. *The pair (A, C) is exactly observable in time $\tau > 0$ if and only if (A^*, C^*) is exactly controllable in time τ. (A, C) is approximately observable in time $\tau > 0$ if and only if (A^*, C^*) is approximately controllable in time τ.*

For much more details on the above concepts we refer to [13]. The following invariance result is taken from Section 6 of Weiss [16].

Proposition II.12. *Let Σ be a well-posed linear system, let K be an admissible feedback operator for Σ and let Σ^K be the corresponding closed-loop system. Let (A, B, C) and (A^K, B^K, C^K) be the generating triples of Σ and Σ^K, respectively.*

Then (A, B) is exactly (approximately) controllable in time T, if and only if (A^K, B^K) has the same property.

Moreover, (A, C) is exactly (approximately) observable in time T, if and only if (A^K, C^K) has the same property.

We quote the following definition and results on simultaneous controllability and simultaneous observability from [13, Chapters 6, 11].

Definition II.13. For $i \in \{1, 2\}$ let A^i be the generators of strongly continuous semigroups $\mathbb{T}^i$ on the Hilbert spaces X^i. Let U, Y be Hilbert spaces. Assume that $B^i \in \mathcal{L}(U, X_{-1}^i)$ are admissible control operators for $\mathbb{T}^i$ and that $C^i \in \mathcal{L}(\mathcal{D}(A^i), Y)$ are admissible observation operators for $\mathbb{T}^i$.

The pairs (A^i, B^i) are said to be *simultaneously exactly controllable* in time $T > 0$, if for every $x_1^i \in X^i$ there exists a function $u \in L^2([0, T]; U)$ such that

$$\int_0^T \mathbb{T}_{T-\sigma}^i B^i u(\sigma) d\sigma = x_1^i, \quad i \in \{1, 2\}.$$

The pairs (A^i, B^i) are said to be *simultaneously approximately controllable* in time $T > 0$, if the equality above holds for (x_1^1, x_1^2) in a dense subspace of $X^1 \times X^2$.

The pairs (A^i, C^i) are said to be *simultaneously exactly observable* in time $T > 0$, if there exists $k_T > 0$ such that for all $(z_0^1, z_0^2) \in \mathcal{D}(A_1) \times \mathcal{D}(A_2)$ the following inequality holds:

$$\int_0^T \|C_1 \mathbb{T}_t^1 z_0^1 + C_2 \mathbb{T}_t^2 z_0^2\|_Y^2 \mathrm{d}t \geq k_T^2 \left(\|z_0^1\|_{X^1}^2 + \|z_0^2\|_{X^2}^2\right). \tag{2.12}$$

The pairs (A^i, C^i) are said to be *simultaneously approximately observable* in time $T > 0$, if the fact that $(z_0^1, z_0^2) \in X^1 \times X^2$ satisfies

$$C_{1\Lambda} \mathbb{T}_t^1 z_0^1 + C_{2\Lambda} \mathbb{T}_t^2 z_0^2 = 0, \quad \text{for almost every } t \in [0, T],$$

implies that $(z_0^1, z_0^2) = (0, 0)$.

Proposition II.14. *With the notation of Definition* II.13*, the pairs (A^1, C^1) and (A^2, C^2) are simultaneously exactly observable in time T if and only if (A^{1*}, C^{1*}) and (A^{2*}, C^{2*}) are simultaneously exactly controllable in time T. A similar statement holds for simultaneous approximate observability.*

Theorem II.15. *Denote by A the generator of the strongly continuous semigroup $\mathbb{T}$ on the Hilbert space X. We assume that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$ and that (A, C) is exactly observable in time T_0. Let $a \in \mathbb{C}^{n\times n}$ and $c \in \mathbb{C}^{m\times n}$ be matrices such that (a, c) is observable. Further, assume that A and a have no common eigenvalues. Then the pairs (A, C) and (a, c) are simultaneously exactly observable in any time $T > T_0$.*

Using the duality from Propositions II.11 and II.14, we can easily find the dual version of the above theorem, which we leave to the reader to formulate.

Proposition II.16. *Let A be the generator of a strongly continuous semigroup on X. Let $\rho_\infty(A)$ be the connected component of $\rho(A)$ containing some right half-plane. We assume that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$ and that (A, C) is approximately observable. Let $a \in \mathbb{C}^{n\times n}$ and $c \in \mathbb{C}^{m\times n}$ be such that (a, c) is observable. Assume that $\sigma(a) \subset \rho_\infty(A)$. Then (A, C) and (a, c) are simultaneously approximately observable in some time.*

Again, using the duality from Propositions II.11 and II.14, the reader can easily formulate the dual version of the above proposition.

The following result is taken from Weiss and Zhao [19]:

Proposition II.17. *Let A be the generator of the strongly continuous semigroup $\mathbb{T}$ on X. Let $B \in \mathcal{L}(\mathbb{C}^m, X_{-1})$ be an admissible control operator for $\mathbb{T}$. Let $a \in \mathbb{C}^{n\times n}$ and $b \in \mathbb{C}^{n\times m}$. Suppose that there exists $T > 0$ such that* *the pairs (A, B) and (a, b) are simultaneously approximately controllable in time T.*

Then for every $z \in X$, $q \in \mathbb{C}^n$ and $\varepsilon > 0$ there exists $u \in L^2([0,T];\mathbb{C}^m)$ such that

$$\left\| \int_0^T \mathbb{T}_{T-t} Bu(t)\mathrm{d}t - z \right\| \le \varepsilon, \quad \int_0^T e^{a(T-t)} bu(t)\mathrm{d}t = q.$$

III. The cascaded system

In this section we analyze the well-posedness and controllability of Σ_{casc} introduced in Section I (see Figure 2). Recall that Σ_{casc} is described by:

$$\begin{cases} \dot{q}(t) = aq(t) + bv(t), & (3.1) \\ u(t) = cq(t) + dv(t), & (3.2) \\ \dot{z}(t) = Az(t) + Bu(t), & (3.3) \\ y(t) = C_\Lambda z(t). & (3.4) \end{cases}$$

Here (3.1)–(3.2) describe the finite-dimensional subsystem Σ_f with input space $\mathbb{C}^m$, state space $\mathbb{C}^n$, output space $\mathbb{C}^m$ and matrices $a \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^{n\times m}$, $c \in \mathbb{C}^{m\times n}$, $d \in \mathbb{C}^{m\times m}$. The equations (3.3)–(3.4) describe the well-posed and strictly proper system Σ_d with input space $\mathbb{C}^m$, state space X, output space $\mathbb{C}^m$, semigroup $\mathbb{T}$, semigroup generator A, control operator $B \in \mathcal{L}(\mathbb{C}^m, X_{-1})$ and observation operator $C \in \mathcal{L}(X_1, \mathbb{C}^m)$. C_Λ is the Λ-extension of C, defined in (2.4). $q(t) \in \mathbb{C}^n$ is the state of Σ_f, while $z(t) \in X$ is the state of Σ_d. $\begin{bmatrix} z(t) \\ q(t) \end{bmatrix} \in X \times \mathbb{C}^n$ is the state of Σ_{casc} at the time t. $v \in L^2_{loc}([0,\infty);\mathbb{C}^m)$ is the input signal of both Σ_f and Σ_{casc}. $y \in L^2_{loc}([0,\infty);\mathbb{C}^m)$ is the output signal of Σ_d while $\begin{bmatrix} u \\ y \end{bmatrix}$ is the output signal of Σ_{casc}. The transfer functions of Σ_f and Σ_d are

$$g(s) = c(sI - a)^{-1}b + d, \qquad \mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B.$$

We denote the state of Σ_{casc} by $\vartheta = \begin{bmatrix} z \\ q \end{bmatrix}$. The state space for Σ_{casc} is $X \times \mathbb{C}^n$ with the usual product norm $\|\vartheta(t)\|^2_{X\times\mathbb{C}^n} = \|z(t)\|^2_X + \|q(t)\|^2_{\mathbb{C}^n}$.

Proposition III.1. *Σ_{casc} described by (3.1)–(3.4) is well-posed and strictly proper (hence regular) on the state space $X \times \mathbb{C}^n$ with the input signal v, the state $\vartheta = \begin{bmatrix} z \\ q \end{bmatrix}$ and output signal $\begin{bmatrix} u \\ y \end{bmatrix}$. Its generating operators are*

$$\mathcal{A} = \begin{bmatrix} A & Bc \\ 0 & a \end{bmatrix},$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in X \times \mathbb{C}^n \;\middle|\; Az + Bcq \in X \right\},$$

$$\mathcal{B} = \begin{bmatrix} Bd \\ b \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} 0 & c \\ C & 0 \end{bmatrix}, \qquad \mathcal{D} = \begin{bmatrix} d \\ 0 \end{bmatrix}.$$

The transfer function of Σ_{casc} is defined for $s \in \rho(\mathcal{A}) = \rho(A) \cap \rho(a)$ by

$$\mathbf{G}^{casc} = \begin{bmatrix} g \\ \mathbf{G}g \end{bmatrix}. \tag{3.5}$$

The proof is easy and it can also be derived as a particular case of Lemma 5.1 in Weiss and Curtain [17], so that we omit the details. It can be proved either directly or as a consequence of Lemma 5.2 in [17] that

$$\mathcal{C}_\Lambda = \begin{bmatrix} 0 & c \\ C_\Lambda & 0 \end{bmatrix}.$$

We will need the following fact from finite-dimensional linear systems theory. It concerns flow-inversion, i.e., interchanging the role of input and output (when this is possible). Flow-inversion of infinite-dimensional systems has been investigated in Staffans and Weiss [12].

Lemma III.2. *If Σ_f is a finite-dimensional system described by*

$$\begin{cases} \dot{x} = ax + bu, \\ y = cx + du, \end{cases} \tag{3.6}$$

where a, b, c, d are matrices of appropriate dimensions and d is invertible, then Σ_f is flow-invertible. Its flow-inverse system $\Sigma_f^\times$ is described by

$$\begin{cases} \dot{x} = (a - bd^{-1}c)x + bd^{-1}y, \\ u = -d^{-1}cx + d^{-1}y. \end{cases} \tag{3.7}$$

If u, x, y are functions satisfying (3.6), then the same functions satisfy also (3.7) and vice versa. The system (3.6) is controllable (observable) iff the system (3.7) is controllable (observable).

We omit the simple proof.

Proposition III.3. *With the assumptions (i)–(iv) of Theorem I.1, the cascaded system Σ_{casc} described by (3.1)–(3.4) (with the state space $X \times \mathbb{C}^n$) is exactly controllable in any time $T > T_0$.*

Proof. Set the initial state $z(0) = 0$ and $q(0) = 0$. The exact controllability of Σ_{casc} on the state space $X \times \mathbb{C}^n$ means that for any time $T > T_0$ and for any $\begin{bmatrix} z_1 \\ q_1 \end{bmatrix} \in X \times \mathbb{C}^n$, there exists an input signal $v \in L^2([0,T];\mathbb{C}^m)$ such that the solution of (3.1)–(3.3) satisfies $z(T) = z_1$ and $q(T) = q_1$.

By assumption (iii) and Lemma III.2, we know that Σ_f described by (3.1)–(3.2) is flow-invertible and that its flow-inverse system, denoted by $\Sigma_f^\times$, is described by

$$\begin{cases} \dot{q} = (a - bd^{-1}c)q + bd^{-1}u, & (3.8) \\ v = -d^{-1}cq + d^{-1}u. & (3.9) \end{cases}$$

Recall that $a^\times = a - bd^{-1}c$. From assumption (ii) and Lemma III.2 it follows that $\Sigma_f^\times$ is controllable. Combining this fact with the assumptions (i) and (iv), and the dual version of Theorem II.15, it follows that Σ_d and $\Sigma_f^\times$ (more precisely, the pairs (A, B) and $(a^\times, bd^{-1})$) are simultaneously exactly controllable in any time $T > T_0$. Therefore for any $\begin{bmatrix} z_1 \\ q_1 \end{bmatrix} \in X \times \mathbb{C}^n$, and for the systems (3.3) and (3.8), we can find $u \in L^2([0,T];\mathbb{C}^m)$ such that $z(T) = z_1$ and $q(T) = q_1$.

Let q and v be the state trajectory and the output signal (on the time interval $[0,T]$) of the system (3.8)–(3.9) corresponding to the input signal u found above and $q(0) = 0$. Obviously $v \in L^2([0,T];\mathbb{C}^m)$. By Lemma III.2 these functions also satisfy (3.1) and (3.2) (and $q(T) = q_1$). Let z be the solution of (3.3) with the signal u found above and with $z(0) = 0$, so that $z(T) = q_1$.

Thus we have found $v \in L^2([0,T];\mathbb{C}^m)$ such that the solution of the equations (3.1)–(3.3) satisfies $z(T) = z_1$ and $q(T) = q_1$. ■

Proposition III.4. *With the assumptions* (i)–(iv) *of Proposition* I.3*,* Σ_{casc} *described by* (3.1)–(3.4) *(with state space* $X \times \mathbb{C}^n$*) is approximately controllable.*

Proof. It has to prove the following fact: for any $\left[\begin{smallmatrix} z_1 \\ q_1 \end{smallmatrix}\right] \in X \times \mathbb{C}^n$ and $\delta > 0$, there exists an input function $v \in L^2([0,T];\mathbb{C}^m)$ such that if q, u and z are as in (3.1)–(3.3) with $z(0) = 0$, $q(0) = 0$, then

$$\left\| \begin{bmatrix} z(T) \\ q(T) \end{bmatrix} - \begin{bmatrix} z_1 \\ q_1 \end{bmatrix} \right\|_{X \times \mathbb{C}^n} \leq \delta .$$

If we can achieve $q(T) = q_1$, then (using the definition of the norm on $X \times \mathbb{C}^n$) the above estimate reduces to

$$\|z(T) - z_1\|_X \leq \delta . \tag{3.10}$$

Thus, it will be enough to show that we can find $v \in L^2([0,T];\mathbb{C}^m)$ such that $q(T) = q_1$ and (3.10) holds.

The remaining part of the proof is similar to that of Proposition III.3. First, by Lemma III.2 and assumptions (ii) and (iii), we get that the flow-inverse system of Σ_f, denoted by $\Sigma_f^\times$ (see (3.8)–(3.9)) is controllable. From this fact, assumptions (i) and (iv) and the dual version of Proposition II.16, we get the simultaneous approximate controllability of Σ_d and $\Sigma_f^\times$. Now by Proposition II.17, we can find a suitable $u \in L^2([0,T];\mathbb{C}^m)$ to achieve $q(T) = q_1$ and (3.10). Following the same procedure as at the end of the proof of Proposition III.3, we can show that there exists $v \in L^2([0,T];\mathbb{C}^m)$ such that if q and z are the solutions of (3.1)–(3.3) corresponding to $z(0) = 0$, $q(0) = 0$, then $q(T) = q_1$ and (3.10) holds. ■

Now we consider a new cascaded system Σ_{casco} as shown in Figure 5, to study the observability of the coupled system Σ_{cs} described by (1.1), (1.2) and (1.3) with output u. Since we are only interested in observability, we assume that the external input u_e of Σ_{cs} is zero, so that $v = -y$. The output of Σ_{casco} is u. We can obtain Σ_{cs} from Σ_{casco} via the feedback $u_0 = u$. The system Σ_{casco} is described by:

$$\begin{cases} \dot{z}(t) = Az(t) + Bu_0(t) , & (3.11) \\ v(t) = -C_\Lambda z , & (3.12) \\ \dot{q}(t) = aq(t) + bv(t) , & (3.13) \\ u(t) = cq(t) + dv(t) . & (3.14) \end{cases}$$

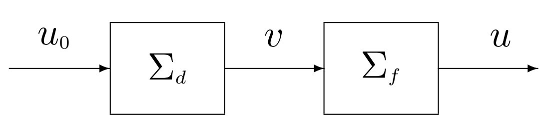

Fig. 5. The cascaded system Σ_{casco} consisting of the well-posed and strictly proper system Σ_d and the finite-dimensional system $\Sigma_f = (a, b, c, d)$.

Proposition III.5. *The cascaded system* Σ_{casco} *described by* (3.11)–(3.14) *with input* u_0 *and output* u *is well-posed and regular on the state space* $X \times \mathbb{C}^n$*. With the assumptions* (i)–(iv) *of Theorem* I.2*,* Σ_{casco} *is exactly observable in any time* $T > T_0$.

Proof. The well-posedness and regularity of Σ_{casco} can be proved similarly as for Σ_{casc}, see the comments after Proposition III.1. Now we show its exact observability.

By assumption (iii) and Lemma III.2, we know that Σ_f is flow-invertible and that its flow-inverse system, denoted by $\Sigma_f^\times$, is described by (3.8)–(3.9). From assumption (ii) and Lemma III.2 it follows that $\Sigma_f^\times$ is observable. This fact and assumptions (i) and (iv) imply, according to Theorem II.15, that (A, C) and $(a^\times, -d^{-1}c)$ are simultaneously exactly observable in any time $T > T_0$.

We denote by g the transfer function of Σ_f, so that $g(s) = c(sI - a)^{-1}b + d$. Since d is invertible by assumption (iii), g has a proper rational inverse g^{-1} which is the transfer function of $\Sigma_f^\times$. For any $T \geq 0$, we denote by $\psi_T^\times$ and $\mathbb{F}_T^{gi}$ the output map and the input-map of $\Sigma_f^\times$ on the time interval $[0,T]$ (see Section II for the terminology). The output function v of $\Sigma_f^\times$ (see (3.9)) can be written as

$$v = \psi_T^\times q_0 + \mathbb{F}_T^{gi} u . \tag{3.15}$$

We denote by Ψ_T the output maps of (A, C). Assuming $u_0 = 0$, the output function v of Σ_d (see (3.12)) can be written as $v = -\Psi_T z_0$. Combining this with (3.15) we obtain

$$\Psi_T z_0 + \psi_T^\times q_0 = -\mathbb{F}_T^{gi} u .$$

The simultaneously exact observability result derived earlier implies that for every $T > T_0$ there exists $k_T > 0$ such that

$$\|\Psi_T z_0 + \psi_T^\times q_0\| \geq k_T \| \left[\begin{smallmatrix} z_0 \\ q_0 \end{smallmatrix}\right] \|$$

(see (2.12)). We also have $\|\mathbb{F}_T^{gi} u\| \leq \|\mathbb{F}_T^{gi}\| \cdot \|u\|$. From the last two estimates we clearly obtain the exact observability inequality (2.11) for Σ_{casco}. ■

Proposition III.6. *With the assumptions* (i)–(iv) *of Proposition* I.4*,* Σ_{casco} *is approximately observable.*

The proof is obtained by adjusting the previous proof. Indeed, if we use Proposition II.16 instead of Theorem II.15, we obtain that (A, C) and $(a^\times, -d^{-1}c)$ are simultaneously approximately observable in some time $T > 0$. Now we proceed as in the previous proof, but we have to modify its last four lines, to show that $u = 0$ implies $z_0 = 0$ and $q_0 = 0$. We omit the details.

IV. Well-posedness, controllability and observability of coupled systems

In this section we prove the main well-posedness and controllability results for the coupled systems Σ_{cs} from Figure 1 and Σ_c from Figure 3. We consider the output of Σ_{cs} and of Σ_c to be $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ (where u is the output of the finite-dimensional subsystem Σ_f and y is the output of the infinite-dimensional subsystem Σ_d). We also prove the exact and approximate observability results for Σ_{cs} with output u only. We continue to use the notation from Section III.

Proposition IV.1. *The coupled system* Σ_{cs} *is well-posed and regular on* $(\mathbb{C}^m, X \times \mathbb{C}^n, \mathbb{C}^{2m})$ *with generating operators* $(\mathcal{A}^{cs}, \mathcal{B}, \mathcal{C}^{cs}, \mathcal{D})$ *and transfer function* $\mathbf{G}^{cs}$*, where*

$$\mathcal{A}^{cs} = \begin{bmatrix} A - BdC_\Lambda & Bc \\ -bC_\Lambda & a \end{bmatrix} ,$$

$$\mathcal{D}(\mathcal{A}^{cs}) = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in X \times \mathbb{C}^n \;\middle|\; \mathcal{A}^{cs} \begin{bmatrix} z \\ q \end{bmatrix} \in X \times \mathbb{C}^n \right\},$$

$$\mathcal{C}^{cs} = \begin{bmatrix} -dC_\Lambda & c \\ C_\Lambda & 0 \end{bmatrix}, \quad \mathbf{G}^{cs} = \begin{bmatrix} g \\ \mathbf{G}g \end{bmatrix} (I + \mathbf{G}g)^{-1}.$$

If B is bounded (i.e., $B \in \mathcal{L}(\mathbb{C}^m, X)$), then $\mathcal{D}(\mathcal{A}^{cs}) = \mathcal{D}(A) \times \mathbb{C}^n$.

Proof. The coupled system Σ_{cs} can be considered as being obtained from Σ_{casc} via output feedback with the feedback operator $K = \begin{bmatrix} 0 & -I \end{bmatrix}$ (as in Figure 4). From Proposition III.1 we know that Σ_{casc} is regular with the state space $X \times \mathbb{C}^n$. From (3.5) we know that the transfer function of Σ_{casc} is $\mathbf{G}^{casc} = \begin{bmatrix} g \\ \mathbf{G}g \end{bmatrix}$, where g is the transfer function of Σ_f, while $\mathbf{G}$ is the transfer function of Σ_d.

Since g is proper and $\mathbf{G}$ is strictly proper, it follows that $(I - K\mathbf{G}^{casc})^{-1} = (I + \mathbf{G}g)^{-1}$ is proper, which means that $K = \begin{bmatrix} 0 & -I \end{bmatrix}$ is an admissible feedback operator for Σ_{casc}. The feedback leading from Σ_{casc} to Σ_{cs} fits into the framework discussed in Proposition II.7. Using the formulas for the closed-loop generating operators from Proposition II.7, we obtain after a short computation that the generating operators of Σ_{cs} are indeed $(\mathcal{A}^{cs}, \mathcal{B}, \mathcal{C}^{cs}, \mathcal{D})$, with $\mathcal{A}^{cs}$ and $\mathcal{C}^{cs}$ as described in the proposition. Using (3.5) and (2.8), we obtain the formula for $\mathbf{G}^{cs}$. It is easy to verify from the formula for $\mathcal{D}(\mathcal{A}^{cs})$ that if B is bounded, then $\mathcal{D}(\mathcal{A}^{cs}) = \mathcal{D}(A) \times \mathbb{C}^n$. ■

Proof of Theorem I.1 *and Proposition* I.3. The first part (the well-posedness and regularity part) of Theorem I.1 is contained in Proposition IV.1.

Now we prove the exact controllability part of Theorem I.1. From Proposition IV.1 and its proof we know that the coupled system Σ_{cs} can be considered as being obtained from Σ_{casc} (which is well-posed and regular) via output feedback with the admissible feedback operator $K = \begin{bmatrix} 0 & -I \end{bmatrix}$.

According to Proposition III.3, the assumptions (i)–(iv) in Theorem I.1 imply that the cascaded system Σ_{casc} (with state space $X \times \mathbb{C}^n$) is exactly controllable in any time $T > T_0$. According to Proposition II.12, it follows that Σ_{cs} is also exactly controllable (with state space $X \times \mathbb{C}^n$) in any time $T > T_0$.

The proof of Proposition I.3 is similar. According to Proposition III.4, the assumptions (i)–(iv) in Proposition I.3 imply that the cascaded system Σ_{casc} (with state space $X \times \mathbb{C}^n$) is approximately controllable. According to Proposition II.12, it follows that Σ_{cs} is also approximately controllable (in the same state space). ■

Proof of Theorem I.2 *and Proposition* I.4. The coupled system Σ_{cs} described by (1.1), (1.2) and (1.3) with external input $u_e = 0$ and output u, can be considered as being obtained from the cascaded system Σ_{casco} described by (3.11)–(3.14) via output feedback with the feedback operator I (as in Figure 4 with $K = I$). The transfer function of Σ_{casco} is $\mathbf{G}^{casco} = g\mathbf{G}$, where g and $\mathbf{G}$ are the transfer functions of Σ_f and Σ_d respectively. From Proposition III.5 we know that Σ_{casco} is well-posed and regular with the state space $X \times \mathbb{C}^n$. Since $\mathbf{G}^{casco}$ is strictly proper, I is an admissible feedback operator for Σ_{casco}. From Proposition III.5 we also know that under the assumptions (i)–(iv) of Theorem I.2, Σ_{casco} is exactly observable in any time $T > T_0$. This observability is preserved under admissible feedback, see Proposition II.12. For approximate observability (Proposition I.4) the proof is similar, but now we use Proposition III.6 to show that Σ_{casco} is approximately observable. ■

Now we analyze the well-posedness and controllability of the more general coupled system Σ_c from Figure 3. We denote by Σ_{ca} the corresponding cascaded system shown in Figure 6. This system is very similar to Σ_{casc}, but now we have (1.4) and (1.5) instead of (1.1) and (1.2), so that the input signal is $\left[\begin{smallmatrix} u_e \\ u_y \end{smallmatrix}\right]$ (with values in $\mathbb{C}^{m+p}$). The semigroup generator $\mathcal{A}$ and the observation operator $\mathcal{C}$ are the same as for Σ_{casc} (see Proposition III.1) while the control operator and the feedthrough operator of Σ_{ca} are given by

$$\mathcal{B}^{ca} = \begin{bmatrix} Bd & -Bd_f \\ b & -b_f \end{bmatrix}, \qquad \mathcal{D}^{ca} = \begin{bmatrix} d & -d_f \\ 0 & 0 \end{bmatrix}.$$

The transfer function of Σ_{ca} is

$$\mathbf{G}^{ca} = \begin{bmatrix} g & -g_f \\ \mathbf{G}g & -\mathbf{G}g_f \end{bmatrix}, \tag{4.1}$$

where

$$g(s) = c(sI - a)^{-1}b + d, \qquad g_f(s) = c(sI - a)^{-1}b_f + d_f,$$

and $\mathbf{G}$ is the transfer function of Σ_d, which is strictly proper.

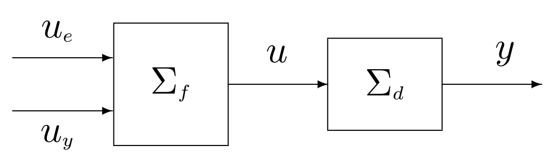

Fig. 6. The cascaded system Σ_{ca} corresponding to Σ_c, consisting of a well-posed and strictly proper system Σ_d and a finite-dimensional system $\Sigma_f = (a, b, b_f, c, d, d_f)$. To obtain from here Σ_c, we have to close the feedback $u_y = y$.

Proposition IV.2. *Σ_c is well-posed and regular on $(\mathbb{C}^m, X \times \mathbb{C}^n, \mathbb{C}^{m+p})$ with generating operators $(\mathcal{A}^c, \mathcal{B}, \mathcal{C}^c, \mathcal{D})$ and transfer function $\mathbf{G}^c$, where*

$$\mathcal{A}^c = \begin{bmatrix} A - Bd_f C_\Lambda & Bc \\ -b_f C_\Lambda & a \end{bmatrix},$$

$$\mathcal{D}(\mathcal{A}^c) = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in X \times \mathbb{C}^n \;\middle|\; \mathcal{A}^c \begin{bmatrix} z \\ q \end{bmatrix} \in X \times \mathbb{C}^n \right\},$$

$$\mathcal{C}^c = \begin{bmatrix} -d_f C_\Lambda & c \\ C_\Lambda & 0 \end{bmatrix}, \quad \mathbf{G}^c = \begin{bmatrix} I \\ \mathbf{G} \end{bmatrix} (I + g_f \mathbf{G})^{-1} g.$$

If B is bounded (i.e., $B \in \mathcal{L}(\mathbb{C}^m, X)$), then $\mathcal{D}(\mathcal{A}^c) = \mathcal{D}(A) \times \mathbb{C}^n$. The operators $\mathcal{B}$ and $\mathcal{D}$ mentioned above are as defined in Proposition III.1.

Proof. The coupled system Σ_c can be considered as being obtained from Σ_{ca} via output feedback with the feedback operator $K = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ and then ignoring the second input of the resulting closed-loop system (i.e., setting it to be zero).

We have already seen that Σ_{ca} is a regular system with state space $X \times \mathbb{C}^n$, generating operators $(\mathcal{A}, \mathcal{B}^{ca}, \mathcal{C}, \mathcal{D}^{ca})$ and transfer function $\mathbf{G}^{ca}$ from (4.1). From the properness of g and g_f and the strict properness of $\mathbf{G}$ it is easy to see that $(I - K\mathbf{G}^{ca})^{-1} = \begin{bmatrix} I & 0 \\ -\mathbf{G}g & I + \mathbf{G}g_f \end{bmatrix}^{-1}$ is proper, which means that $K = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ is an admissible feedback operator for Σ_{ca}. The feedback leading from Σ_{ca} to Σ_c fits into the framework discussed in Proposition II.7. The remaining part of the proof is practically the same as for Proposition IV.1. ■

Proof of Theorem I.5. In the *first step* we point out that the well-posedness part of Theorem I.5 is contained in Proposition IV.2. Let $\left[\begin{smallmatrix} z \\ q \end{smallmatrix}\right]$ be a state trajectory of Σ_c corresponding to the input signal $u_e \in L^2([0, \infty); \mathbb{C}^m)$, with initial conditions $z(0) = 0$, $q(0) = 0$. Due to the well-posedness of Σ_c, z and q are continuous functions of t (with values in X and in $\mathbb{C}^n$). Another consequence of the well-posedness is that the signals u and y appearing in Figure 3 are in L^2_{loc} and have Laplace transforms.

The *second step* is to show that on any finite time interval, u_e (and hence all the above signals) can be expressed in a continuous way from u. From $\dot{q}(t) = aq(t) + bu_e(t) - b_f y(t)$ we get, applying the Laplace transformation,

$$\hat{q}(s) = (sI - a)^{-1}[b\hat{u}_e(s) - b_f \hat{y}(s)], \qquad (4.2)$$

for all s in some right half-plane. Since $u(t) = cq(t) + du_e(t) - d_f y(t)$, we obtain

$$\begin{aligned} \hat{u}(s) &= c(sI - a)^{-1}[b\hat{u}_e(s) - b_f \hat{y}(s)] + d\hat{u}_e(s) - d_f \hat{y}(s) \\ &= g(s)\hat{u}_e(s) - g_f(s)\hat{y}(s), \end{aligned} \qquad (4.3)$$

where the rational transfer function g and g_f are defined as after (4.1).

Using $\hat{y}(s) = \mathbf{G}(s)\hat{u}(s)$, we get

$$[I + g_f(s)\mathbf{G}(s)]\,\hat{u}(s) = g(s)\hat{u}_e(s). \qquad (4.4)$$

Note that $\lim_{s \to \infty} g(s) = d$. Since d is invertible by assumption (iii), it follows that g has a proper rational inverse. Now (4.4) can be rewritten in the form

$$\hat{u}_e(s) = g^{-1}(s)\,[I + g_f(s)\mathbf{G}(s)]\,\hat{u}(s). \qquad (4.5)$$

As $\mathbf{G}$ is strictly proper, $g_f\mathbf{G}$ is strictly proper too, so that $\hat{u}_e$ is obtained from $\hat{u}$ via a proper transfer function. It is known (see Subsection II-B) that this implies that on any finite time interval $[0, T]$, the mapping from the restriction $u|_{[0,T]}$ to the restriction $u_e|_{[0,T]}$ is continuous (in the L^2 norm).

The *third step* is to express $q(\tau)$ from u (where $\tau > 0$ is fixed). If we substitute (4.5) into (4.2) and use again that $\hat{y}(s) = \mathbf{G}(s)\hat{u}(s)$, we get

$$\begin{aligned} \hat{q}(s) &= (sI - a)^{-1} b g^{-1}(s)\,[I + g_f(s)\mathbf{G}(s)]\,\hat{u}(s) \\ &\quad - (sI - a)^{-1} b_f \mathbf{G}(s)\hat{u}(s) \\ &= (sI - a)^{-1} b g^{-1}(s)\hat{u}(s) \\ &\quad + (sI - a)^{-1}\left[b g^{-1}(s) g_f(s) - b_f\right]\mathbf{G}(s)\hat{u}(s). \end{aligned} \qquad (4.6)$$

We remark that if $p = m$, $b_f = b$, $d_f = d$, then $bg^{-1}(s)g_f(s) - b_f = 0$, so that (4.6) becomes much simpler. This is the situation discussed in Theorem I.1.

Denote by $\mathbb{F}^{gi}_\tau$ $(\tau \geq 0)$ the (bounded) input-output maps corresponding to the proper transfer function g^{-1}, see Subsection II-B for the meaning of this concept. Similarly, let $\mathbb{F}^{gf}_\tau$ and $\mathbb{F}^{G}_\tau$ $(\tau \geq 0)$ be the (bounded) input-output maps corresponding to the proper transfer functions g_f and $\mathbf{G}$, respectively. We denote by ϕ_τ and ϕ^f_τ $(\tau \geq 0)$ the input maps of (a, b) and (a, b_f), respectively (see Subsection II-A for this concept). Then (4.6) shows that we have

$$q(\tau) = \phi_\tau \mathbb{F}^{gi}_\tau u + \left[\phi_\tau \mathbb{F}^{gi}_\tau \mathbb{F}^{gf}_\tau - \phi^f_\tau\right]\mathbb{F}^G_\tau u.$$

If we combine the above formula with (2.2), we obtain that

$$\begin{bmatrix} z(\tau) \\ q(\tau) \end{bmatrix} = \begin{bmatrix} \Phi_\tau \\ \phi_\tau \mathbb{F}^{gi}_\tau \end{bmatrix} u + \begin{bmatrix} 0 \\ \phi_\tau \mathbb{F}^{gi}_\tau \mathbb{F}^{gf}_\tau - \phi^f_\tau \end{bmatrix} \mathbb{F}^G_\tau u, \qquad (4.7)$$

where Φ_τ $(\tau \geq 0)$ are the input maps of (A, B).

In the *fourth step* we show that if $\tau > T_0$, then the operators $\begin{bmatrix} \Phi_\tau \\ \phi_\tau \mathbb{F}^{gi}_\tau \end{bmatrix}$ appearing above are onto $X \times \mathbb{C}^n$. During this step we assume that $p = m$, $b_f = b$ and $d_f = d$. Since Φ_τ, ϕ_τ and $\mathbb{F}^{gi}_\tau$ depend on A, B, a, b, c, d but not on b_f or on d_f or on C, this assumption does not entail any loss of generality. According to the remark after (4.6), if $p = m$, $b_f = b$ and $d_f = d$, then the second term on the right-hand side of (4.7) is zero, so that

$$\begin{bmatrix} z(\tau) \\ q(\tau) \end{bmatrix} = \begin{bmatrix} \Phi_\tau \\ \phi_\tau \mathbb{F}^{gi}_\tau \end{bmatrix} u. \qquad (4.8)$$

Denoting $v = u_e - y$, from (4.3) we have $\hat{u}(s) = g(s)\hat{v}(s)$. Since g is proper, we can associate to it bounded input-output maps $\mathbb{F}^g_\tau$ (note that $\mathbb{F}^g_\tau = [\mathbb{F}^{gi}_\tau]^{-1}$). Thus, $u|_{[0,\tau]} = \mathbb{F}^g_\tau v|_{[0,\tau]} = \mathbb{F}^g_\tau v$ for every $\tau \geq 0$. Combining this with (4.8), we obtain

$$\begin{bmatrix} z(\tau) \\ q(\tau) \end{bmatrix} = \begin{bmatrix} \Phi_\tau \\ \phi_\tau \mathbb{F}^{gi}_\tau \end{bmatrix} \mathbb{F}^g_\tau v. \qquad (4.9)$$

Let $z_1 \in X$, $q_1 \in \mathbb{C}^n$ and $\tau > T_0$. According to Proposition III.3 there exists $v \in L^2([0, \tau]; \mathbb{C}^m)$ such that $z(\tau) = z_1$ and $q(\tau) = q_1$, in other words, the operator on the right-hand side of (4.9) is surjective. This implies that $\begin{bmatrix} \Phi_\tau \\ \phi_\tau \mathbb{F}^{gi}_\tau \end{bmatrix}$ is surjective.

In the *fifth step* we show that if $T > T_0$, $b_1 \in \mathbb{C}^{n \times p}$, $d_1 \in \mathbb{C}^{m \times p}$ then there exists a finite set $F_T \subset \mathbb{C}$ (which depends on T and the pair (b_1, d_1)) such that the operator

$$\tilde{\Phi}_T = \begin{bmatrix} \Phi_T \\ \phi_T \mathbb{F}^{gi}_T \end{bmatrix} + \begin{bmatrix} 0 \\ \phi_T \mathbb{F}^{gi}_T \mathbb{F}^{gf}_T - \phi^f_T \end{bmatrix} \mathbb{F}^G_T,$$

corresponding to $b_f = \lambda b_1$ and $d_f = \lambda d_1$ is onto $X \times \mathbb{C}^n$ for each $\lambda \in \mathbb{C} \setminus F_T$. Since $\begin{bmatrix} \Phi_T \\ \phi_T \mathbb{F}^{gi}_T \end{bmatrix}$ is surjective (see the fourth step), it has a bounded right inverse R_T. Then

$$\tilde{\Phi}_T R_T = I + \begin{bmatrix} 0 \\ \phi_T \mathbb{F}^{gi}_T \mathbb{F}^{gf}_T - \phi^f_T \end{bmatrix} \mathbb{F}^G_T R_T. \qquad (4.10)$$

Let $\mathbb{F}^{gf1}_T$ be the input-output map $\mathbb{F}^{gf}_T$ corresponding to $\lambda = 1$, and similarly, let ϕ^{f1}_T be the input map ϕ^f_T corresponding to $\lambda = 1$. Then from the definitions

$$\mathbb{F}^{gf}_T = \lambda \mathbb{F}^{gf1}_T, \qquad \phi^f_T = \lambda \phi^{f1}_T,$$

while the other operators on the right-hand side of (4.10) are independent of λ. Thus, (4.10) becomes

$$\tilde{\Phi}_T R_T = I + \lambda \begin{bmatrix} 0 \\ \phi_T \mathbb{F}_T^{gi} \mathbb{F}_T^{gf1} - \phi_T^{f1} \end{bmatrix} \mathbb{F}_T^G R_T .$$

The second term on the right-hand side above is a finite-rank operator of rank $\leq n$. It follows that there exists a set $F_T \subset \mathbb{C}$ with at most n elements such that for $\lambda \notin F_T$, $\tilde{\Phi}_T R_T$ is invertible. Hence, for all such λ, $\tilde{\Phi}_T$ is onto $X \times \mathbb{C}^n$.

The *sixth step* is to notice that whenever $\tilde{\Phi}_T$ is onto, the system Σ_c is exactly controllable in time T. Indeed, $\tilde{\Phi}_T$ maps u into $\begin{bmatrix} z(T) \\ q(T) \end{bmatrix}$ (see (4.7)), and from the second step we know that on any finite time interval, u_e can be expressed in a continuous way from u, meaning that the operator F from u_e to u is invertible. Thus, the operator $\tilde{\Phi}_T F$ that maps u_e to $\begin{bmatrix} z(T) \\ q(T) \end{bmatrix}$ is onto.

The last sentence of Theorem I.5 clearly follows from what we have shown in the fifth and sixth steps. From the last sentence of Theorem I.5 it follows that for each $T > T_0$, the set $\mathcal{O}_T \subset \left(\mathbb{C}^{n\times p} \times \mathbb{C}^{m\times p}\right)$ of all those pairs (b_f, d_f) for which Σ_c is exactly controllable in time T, is dense. The operator $\tilde{\Phi}_T$ depends in an affine and hence continuous way on the pair (b_f, d_f) (using the operator norm). Since the set of surjective operators in $\mathcal{L}(L^2([0,T];\mathbb{C}^m), X \times \mathbb{C}^n)$ is open, it follows that the set $\mathcal{O}_T$ is also open. The case $d_f = 0$ follows by taking d_1=0. ■

V. ILLUSTRATIVE EXAMPLE

Consider the system Σ_{cs} which consists of a flexible shaft Σ_f with one end connected to the rigid body of a non-uniform SCOLE beam system Σ_d, and the other end receiving the control signal: the angular velocity u_e. The flexible shaft can be modelled as a torsional spring in parallel with a torsional damper while the SCOLE system is a well-known model for a system consisting of a flexible beam with one end clamped and the other end linked to a rigid body. The SCOLE model has two possible inputs: the torque and the force acting on the rigid body, see Guo [3], Littman and Markus [6], [7]. Here we use only its torque input.

We take the angular velocity of the rigid body, denoted by y, and the torque acting on the rigid body from the shaft, denoted by u, as the outputs of Σ_{cs}. From physical considerations, we get the dynamic equations of Σ_{cs} as follows:

$$\begin{cases} \rho(x) w_{tt}(x,t) + (EI(x) w_{xx}(x,t))_{xx} = 0, & (5.1) \\ (x,t) \in (0,l) \times [0,\infty), & \\ w(0,t) = 0, \quad w_x(0,t) = 0, & (5.2) \\ m w_{tt}(l,t) - (EI w_{xx})_x(l,t) = 0, & (5.3) \\ J w_{xtt}(l,t) + EI(l) w_{xx}(l,t) = u(t), & (5.4) \\ y(t) = w_{xt}(l,t), & (5.5) \\ q_t(t) = u_e(t) - y(t), & (5.6) \\ u(t) = K_s q(t) + C_s q_t(t), & (5.7) \end{cases}$$

where (5.1)–(5.5) describe the SCOLE model Σ_d. The subscripts t and x denote derivatives with respect to the time t and the position x. l is the length of the beam, w is its transverse displacement, while EI and ρ are its flexural rigidity and mass density. m and J are the mass and the moment of inertia of the rigid body (these are positive constants). We assume that $\rho, EI \in C^4[0,l]$, $0 < \rho_0 \leq \rho(x) < \rho_1$ and $0 < EI_0 \leq EI(x) < EI_1$ where $\rho_0, \rho_1, EI_0, EI_1$ are positive constants. (5.6)–(5.7) describe the flexible shaft Σ_f. q is the angular difference between the two ends of the flexible shaft. The parameter $K_s > 0$ is the torsional stiffness of the shaft while $C_s > 0$ is its torsional damping.

From this description, it is easy to see that Σ_{cs}, Σ_d and Σ_f fit the framework of coupled systems with the special structure shown in Figure 1. It is clear that Σ_f is a one-dimensional linear system with state $q(t) \in \mathbb{C}$ and its matrices are

$$a = 0, \quad b = 1, \quad c = K_s, \quad d = C_s.$$

We define the norm on $\mathbb{C}$ by $\|q(t)\|^2 = K_s |q(t)|^2$, which is twice the physical energy in Σ_f.

Now we analyze the SCOLE model Σ_d. We introduce the following auxiliary functions: $z_1(x,t) = w(x,t)$, $z_2(x,t) = w_t(x,t)$, $z_3(t) = w_t(l,t)$, $z_4(t) = w_{xt}(l,t)$. We define $z(t) = [z_1(\cdot,t), z_2(\cdot,t), z_3(t), z_4(t)]^T$ (the superscript T means transpose) to be the state of Σ_d at the time t. The natural energy state space of Σ_d is

$$X = \mathcal{H}_l^2(0,l) \times L^2[0,l] \times \mathbb{C}^2 ,$$

where

$$\mathcal{H}_l^2(0,l) = \left\{ h \in \mathcal{H}^2(0,l) \mid h(0) = h_x(0) = 0 \right\}$$

and $\mathcal{H}^n$ ($n \in \mathbb{N}$) denote the usual Sobolev spaces. We define the norm on X as follows: For any $\xi = [\xi_1 \ \xi_2 \ \xi_3 \ \xi_4]^T \in X$,

$$\|\xi\|^2 = \int_0^l EI(x) |\xi_{1xx}(x)|^2 \mathrm{d}x + \int_0^l \rho(x) |\xi_2(x)|^2 \mathrm{d}x + m|\xi_3|^2 + J|\xi_4|^2 .$$

It is clear that $\|z(t)\|^2$ represents twice the physical energy in Σ_d at the time t. Of course, the formulas $z_3(t) = w_t(l,t)$ and $z_4(t) = w_{xt}(l,t)$ do not make sense for $z(t) \in X$, only for smoother $z(t)$ (for example, for $z(t) \in \mathcal{D}(A)$, defined below).

We define the generating operators of Σ_d as follows:

$$A\xi = \begin{bmatrix} \xi_2 \\ -\rho^{-1}(x) \big(EI(x) \xi_{1xx}(x) \big)_{xx} \\ m^{-1} \big(EI \xi_{1xx} \big)_x (l) \\ -J^{-1} EI(l) \xi_{1xx}(l) \end{bmatrix} \qquad \forall\, \xi \in \mathcal{D}(A),$$

$$\mathcal{D}(A) = \left\{ \xi \in \left[\mathcal{H}^4 \cap \mathcal{H}_l^2\right] \times \mathcal{H}_l^2 \times \mathbb{C}^2 \;\middle|\; \begin{matrix} \xi_3 = \xi_2(l) \\ \xi_4 = \xi_{2x}(l) \end{matrix} \right\},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \dfrac{1}{J} \end{bmatrix}^T, \qquad C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} .$$

Note that we have suppressed the interval $(0,l)$ for $\mathcal{H}^4$ and $\mathcal{H}_l^2$ in the the definition of $\mathcal{D}(A)$. We get the following state space formulation of Σ_d from (5.1)–(5.5):

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ y(t) = Cz(t). \end{cases} \tag{5.8}$$

From Guo and Ivanov [4, Proposition 1.1] we know that A is skew-adjoint on X, so that it is the generator of a unitary group

$\mathbb{T}$. Clearly B and C are bounded on X (i.e., $B \in \mathcal{L}(\mathbb{C}, X)$, $C \in \mathcal{L}(X, \mathbb{C})$), and the feedthrough operator is zero, so that Σ_d is strictly proper (hence, regular). By Theorem I.1, Σ_{cs} (with input u_e, state $\left[\begin{smallmatrix} z \\ q \end{smallmatrix}\right]$ and output $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$) is well-posed and regular with the state space $X \times \mathbb{C}$. The restriction of A to $\mathcal{D}(A^2)$ is skew-adjoint on $X_1 = \mathcal{D}(A)$. From Guo [3, Proposition 4.2] we know that B is admissible for $\mathbb{T}$ restricted to X_1. Clearly $C \in \mathcal{L}(X_1, \mathbb{C})$, so that Σ_d is well-posed and strictly proper on X_1 as well, according to Propositions II.2 and II.5. By Theorem I.1, Σ_{cs} (with input u_e, state $\left[\begin{smallmatrix} z \\ q \end{smallmatrix}\right]$ and output $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$) is well-posed and regular with the state space $X_1 \times \mathbb{C}$ as well.

Now we prove the exact controllability of Σ_{cs} on $X_1 \times \mathbb{C}$ using Theorem I.1, and we also prove its approximate observability on $X \times \mathbb{C}$ using Proposition I.4.

From [3, Theorem 4.3] we know that Σ_d described by (5.8) is exactly controllable on X_1. From [4, Corollary 2.2] we know that Σ_d is approximately observable on X. Therefore assumptions (i) of Theorem I.1 and of Proposition I.4 hold.

Clearly (a,b) is controllable, (a,c) is observable and $d = C_s$ is invertible. Hence assumptions (ii) and (iii) of Theorem I.1 and of Proposition I.4 are satisfied. By computation, we get

$$a^\times = -\frac{K_s}{C_s}.$$

From [4, Propositions 1.1 and 1.2] we know that $\sigma(A)$ consists of simple eigenvalues that isolated, purely imaginary and nonzero. Therefore $\rho(A) = \rho_\infty(A)$. Since A and $a^\times$ have no common eigenvalues, we have $\sigma(a^\times) \subset \rho_\infty(A)$. Thus, all the four assumptions of Theorem I.1 and of Proposition I.4 are satisfied. Therefore Σ_{cs} is exactly controllable on $X_1 \times \mathbb{C}$, and it is approximately observable on $X \times \mathbb{C}$ using only the output u. Obviously, Σ_{cs} is approximately controllable on $X \times \mathbb{C}$.

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