# The state feedback regulator problem for regular linear systems

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***Abstract*—This paper is about the state feedback regulator problem for infinite-dimensional linear systems. The plant, assumed to be an exponentially stable regular linear system, is driven by a linear (possibly infinite-dimensional) exosystem via a disturbance signal. The exosystem has its spectrum in the closed right half-plane and also generates the reference signal for the plant output. The regulator problem is to design a controller that, while guaranteeing the stability of the closed-loop system without the exosystem, drives the tracking error to zero. A particular version of this problem is the state feedback regulator problem in which the states of the exosystem and the plant are known to the controller. Under suitable assumptions, we show that the latter problem is solvable if and only if a pair of algebraic equations, called the regulator equations, is solvable. We derive conditions, in terms of the transfer function of the plant and eigenvalues of the exosystem, for the solvability of the regulator equations. Three examples illustrating the theory are presented.**

## I. INTRODUCTION

This paper is devoted to the tracking and disturbance rejection problem, also called the *regulator problem*, for a linear infinite-dimensional plant from the special class of regular linear systems, when the reference and disturbance signals are produced by a linear unstable signal generator called the *exosystem*. *Regular systems* model many physical systems involving waves, beams, plates, shells, elastic media, heat propagation, etc, see [3], [8], [9], [10], [20], [21], [22], [44], [47], and they usually have unbounded control and observation operators. However, in the literature on the regulator problem, in order to avoid technical difficulties, it is usually assumed that these operators are bounded. (A notable exception is [41], on which we shall comment at the end of Section IV.) In this paper we overcome this limitation.

There are two standard versions of the regulator problem: In the first, called the *state feedback regulator problem*, the controller is provided with full information of the state of the plant and the exosystem, while in the second version, called the *error feedback regulator problem*, only the tracking error is available to the controller. In this work we will focus on the state feedback version alone, and under the assumption that the plant is exponentially stable. Indeed, we think that stabilizing the plant and solving the regulator problem are two distinct issues, and it would only obfuscate the theory to present them mixed together. The exponential stability implies that for the state feedback we actually only need the state of the exosystem (this is explained in detail in Remark IV.6). We plan to address the error feedback regulator problem in a follow-up paper. We mention that it is easy, in principle, to design an error feedback controller if the plant together with the exosystem are detectable via the tracking error. Indeed, the straightforward approach is to use a full state observer, which of course is infinite-dimensional. This is the approach taken in several references, for example, in Byrnes, Lauko, Gilliam and Shubov [4] or in Immonen and Pohjolainen [28]. For plants that are already stable, the real challenge (that we shall address in our follow-up paper) is to design a finite-dimensional error feedback controller.

Pioneering work on the regulator problems for linear finite-dimensional systems is in Francis [17], where the solvability of these problems is shown to be equivalent to the solvability of a pair of linear matrix equations called the *regulator equations*. Similar results have been established for finite-dimensional nonlinear systems in Byrnes and Isidori [6] (see also [7]) under the assumption that the plant is locally exponentially stabilizable and the exosystem has a Lyapunov stable equilibrium at the origin with each initial condition in a neighborhood of this origin being Poisson stable. It is shown in [6] that the solvability conditions given in [17] can be generalized naturally in terms of the solvability of a pair of nonlinear equations – still called the regulator equations. These equations express the existence of a manifold in the state space on which the actual and reference outputs coincide and which can be rendered attracting and invariant using feedback, see also Knobloch, Isidori and Flockerzi [33]. A passivity-based approach to the nonlinear regulator problem has been explored in Jayawardhana and Weiss [29], [30]. Nonlinear regulator theory has led to the immersion and invariance approach to the stabilization and adaptive control of nonlinear plants, see Astolfi and Ortega [1] and others.

In Byrnes *et al* [4], building on the results in [17], a geometric theory of output feedback regulation for infinite-dimensional linear plants with bounded control and observation operators driven by finite-dimensional exosystems has been developed. In particular in [4] the solvability of both the state and error feedback regulator problems has been characterized in terms of the solvability of certain equations referred to as the regulator equations. Also, simple criteria for the solvability of the regulator equations have been derived. Under similar assumptions as in [4], but with finite-dimensional input and output spaces, an output regulation problem is addressed in Deutscher [15], where the measured and regulated outputs are allowed to be different. The proposed solution involves solving

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certain Sylvester and regulator equations and remarkably the proposed controller is finite-dimensional. A relevant earlier work in this direction is Schumacher [43] in which, under rather strong assumptions on the infinite-dimensional plant, a finite-dimensional controller has been proposed to solve the error feedback regulator problem.

Regulator theory for infinite-dimensional linear systems with bounded control and observation operators has been significantly advanced by a group of researchers at Tampere University of Technology (Finland) who have developed a sophisticated theory of infinite-dimensional exosystems, see for instance [24], [27], [28], [26], [39], [40]. The state feedback regulator problem for exponentially stabilizable linear plants driven by infinite-dimensional exosystems generating periodic signals was addressed in [27]. The results in [27] were generalized in [28] by considering strongly stabilizable plants and a broader class of exosystems, and addressing both the state and error feedback regulator problems. The robustness of solutions to the error feedback regulator problem was characterized in [26] in terms of the controller having the *Internal Model Structure* (IMS) and in [24] in terms of the controller satisfying the $\mathcal{G}$*-conditions* (as it is called in [39]). The recent paper Boulite *et al* [2] builds on the above works to address the state feedback regulator problem for polynomially stabilizable linear plants driven by infinite-dimensional exosystems. By introducing a new characterization for the solvability of a Sylvester equation, alternate conditions for the solvability of the state feedback regulator problem are presented in [2].

In this work we restrict the state operator $S$ of the linear, unstable and possibly infinite-dimensional exosystem to be bounded. Our reason for imposing this restriction is to avoid the following robustness problem: according to the internal model principle due to Davison, or Wonham and Francis [13], [18], all the unstable eigenvalues of $S$ will be poles of the controller, if we use an error feedback controller. Thus, in the case of an exosystem with an unbounded set of unstable eigenvalues, the closed-loop system cannot be robustly stable with respect to small delays in the feedback loop, see Logemann *et al* [34, Theorem 1.2]. Closely related negative results are in Georgiou and Smith [19], and this issue is explained in more detail in Weiss and Hafele [52, Sec. 4]. There is also a positive result in this context: if the error feedback controller is strictly proper (which can only happen when the spectrum of $S$ is bounded), then the closed-loop system is robustly stable with respect to small delays, as follows from [34, Theorem 1.1].

We extend the key results in [4] on the state feedback regulator problem to plants with unbounded control and observation operators. There is considerable interest in plants with boundary control and/or boundary observation, for which the control and/or observation operators are unbounded, see for instance Staffans [44], Tucsnak and Weiss [46], [47]. Probably the most general class of distributed parameter systems for which there is a well established and relatively simple representation and feedback theory, are the regular linear systems (see [44], [48], [49]). In this work we have chosen to formulate the state feedback regulator problem for regular plants. We also assume that the plant is exponentially stable and not just stabilizable, the latter assumption being customary. This is not limiting, since in regulator theory the problems of stabilization and regulation can be decoupled and addressed sequentially. Hence we shall assume that the plant has been stabilized via a suitable feedback and we shall solve the regulator problem for the exponentially stable plant.

A parallel approach to the regulator problem for stable finite-dimensional plants was first developed in Davison [14]. This approach was extended to exponentially stable uncertain regular plants in Logemann and Townley [37], but considering only constant references and disturbances (see also [35], [36] and [38] for relevant results), to plants in the Callier-Desoer algebra in Hämäläinen and Pohjolainen [23] and to exponentially stable well-posed plants in Rebarber and Weiss [42]. The finite-dimensional controller proposed in [42] solves the error feedback regulator problem. Sampled-data versions of this controller can be found in Ke, Logemann and Rebarber [31], [32]. In spite of the results in the cited papers, in this work we pursue the state feedback regulator problem (and will address the error feedback regulator problem in a future work) in the hope that, as in finite-dimensions, the state space approach developed here will be more suitable (in comparison to the approach in [23], [42], [32]) for a generalization addressing the regulator problem for non-linear infinite-dimensional plants.

In the last section we show how the regulator theory developed in this work can be applied to some systems described by PDEs, all with unbounded control and observation operators. The first example is a one-dimensional heat equation with Robin boundary control at one end of the interval, a disturbance entering through the other end, and observation at an interior point. The exosystem is 4-dimensional, has a nontrivial Jordan block decomposition and the disturbance generated may grow linearly. In the second example we consider the heat equation on the two-dimensional unit rectangle, with boundary control through a part of the boundary. The outputs are averages over some boundary regions, the references are constant and there are no disturbances. In the third example we consider tracking for a Rayleigh beam with structural damping. The control is the torque applied at one end-point and the output is the angular velocity at the same point. This output is required to track a sinusoidal reference signal.

## II. Background on regular linear systems

This section is a very brief overview of regular systems theory, mostly following [45], [46], [48]. For a Hilbert space $Y$ and $\alpha \in \mathbb{R}$ we define the weighted function space

$$L^2_\alpha([0,\infty);Y) = \left\{ \phi \in L^2_{loc}([0,\infty);Y) \,\middle|\, \int_0^\infty e^{-2\alpha t}\|\phi(t)\|^2 \mathrm{d}t < \infty \right\},$$

with the norm being the square-root of the integral appearing above. For any $a \in \mathbb{R}$ we define the open and closed right half-planes bounded by $a$, by

$$\mathbb{C}^+_a = \left\{ s \in \mathbb{C} \,\middle|\, \operatorname{Re} s > a \right\}, \qquad \overline{\mathbb{C}^+_a} = \left\{ s \in \mathbb{C} \,\middle|\, \operatorname{Re} s \geq a \right\}.$$

Let $Z$ be a Hilbert space and $A$ the generator of an operator semigroup (also called strongly continuous semigroup of operators) $\mathbb{T}$ on $Z$. We denote by $\rho(A)$ the resolvent set of $A$. We define two new Hilbert spaces as follows: for $\beta \in \rho(A)$,

$$Z_1 = \mathcal{D}(A) \quad \text{with} \quad \|z\|_1 = \|(\beta I - A)z\|$$

and $Z_{-1}$ is the completion of $Z$ with respect to the norm

$$\|z\|_{-1} = \|(\beta I - A)^{-1} z\|\,.$$

These spaces are independent of the choice of $\beta$ and we have the dense embeddings

$$Z_1 \hookrightarrow Z \hookrightarrow Z_{-1}\,. \tag{2.1}$$

Let $Z_1^d$ be the analogue of the space $Z_1$ but for the adjoint semigroup generator $A^*$. Then $Z_{-1}$ may also be regarded as the dual of $Z_1^d$ with respect to the pivot space $Z$.

The operators $\mathbb{T}_t$ extend to $Z_{-1}$, and the generator of the extended semigroup is an extension of $A$ to an operator in $\mathcal{L}(Z, Z_{-1})$. We use the same notation $\mathbb{T}_t$ and $A$ for these extended operators. It will be useful to note that for $\beta \in \rho(A)$,

$$(\beta I - A)^{-1} \in \mathcal{L}(Z_{-1}, Z)\,, \qquad (\beta I - A)^{-1} \in \mathcal{L}(Z, Z_1)\,.$$

We refer to [46] for more details about these spaces and extensions. We denote by $\omega_0(\mathbb{T})$ the *growth bound* of the semigroup $\mathbb{T}$ (thus $\omega_0(\mathbb{T}) = \lim\limits_{t\to\infty} \frac{1}{t} \log \|\mathbb{T}_t\|$). Recall that $\mathbb{T}$ (or $A$) is called *exponentially stable* if $\omega_0(\mathbb{T}) < 0$.

If $C \in \mathcal{L}(Z_1, Y)$, where $Y$ is another Hilbert space, then the $\Lambda$-*extension* of $C$ (with respect to $A$), denoted $C_\Lambda$, is defined as follows (see [49]):

$$C_\Lambda z = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} z \tag{2.2}$$

and its domain $\mathcal{D}(C_\Lambda)$ consists of those $z \in Z$ for which the above limit exists.

We call $C$ an *admissible observation operator* for $\mathbb{T}$ if for some (hence, for every) $\tau > 0$ there exists $m_\tau > 0$ such that

$$\int_0^\tau \|C\mathbb{T}_t z\|^2 \mathrm{d}t \le m_\tau \|z\|^2 \qquad \forall\, z \in \mathcal{D}(A)\,. \tag{2.3}$$

In this case, for every $z \in Z$, the formula $y(t) = C_\Lambda \mathbb{T}_t z$ makes sense for almost every $t \ge 0$ and it defines a function $y \in L^2_\alpha([0,\infty); Y)$, for every $\alpha > \omega_0(\mathbb{T})$. Also, (2.3) becomes valid for all $z \in Z$ if we replace $C$ with $C_\Lambda$. $C$ is called *bounded* if it can be extended such that $C \in \mathcal{L}(Z, Y)$ and *unbounded* otherwise. The dual of the above admissibility concept can be expressed as follows: if $U$ is a Hilbert space and $B \in \mathcal{L}(U, Z_{-1})$, then $B$ is called an *admissible control operator* for $\mathbb{T}$ if for some (hence, for every) $\tau > 0$ and for every $u \in L^2([0,\infty); U)$,

$$\int_0^\tau \mathbb{T}_{\tau-\sigma} Bu(\sigma) \mathrm{d}\sigma \in Z\,.$$

Note that this integral gives the strong solution of $\dot{z}(t) = Az(t) + Bu(t)$ at time $\tau$, if $z(0) = 0$. In this case, $z(\tau)$ depends continuously on $u$ and on $\tau$, hence there exists $\kappa_\tau > 0$ such that

$$\left\| \int_0^\tau \mathbb{T}_{\tau-\sigma} Bu(\sigma) \mathrm{d}\sigma \right\| \le \kappa_\tau \|u\|_{L^2([0,\tau];U)}\,.$$

$B$ is called *bounded* if $B \in \mathcal{L}(U, Z)$, and *unbounded* otherwise. It can be shown that if $B$ is admissible and $\alpha > \omega_0(\mathbb{T})$ then there exists $M \ge 0$ such that for all $s \in \mathbb{C}^+_\alpha$,

$$\|(sI - A)^{-1} B\| \le \frac{M}{\sqrt{\operatorname{Re} s - \alpha}}\,. \tag{2.4}$$

*Definition* II.1. Consider the generator $A$ of a strongly continuous semigroup $\mathbb{T}$ on $Z$, an admissible control operator $B \in \mathcal{L}(U, Z_{-1})$ and an admissible observation operator $C \in \mathcal{L}(Z_1, Y)$, as defined earlier. The triple $(A, B, C)$ is called *regular*, in the sense of [48], [49], if in addition the following conditions hold:
(1) $C_\Lambda(sI - A)^{-1}B$ exists for some (hence, for every) $s \in \rho(A)$ (this means that we have $(sI - A)^{-1}BU \subset \mathcal{D}(C_\Lambda)$).
(2) The mapping $\mathbf{G}_0(s) = C_\Lambda(sI - A)^{-1}B$, called the *transfer function* associated to the triple $(A, B, C)$, is bounded on some right half-plane.

The above assumptions imply that $\mathbf{G}_0(s)v \to 0$, as $s \to +\infty$ along the real axis, for every $v \in U$. The fact that $(A, B, C)$ is a regular triple is equivalent to the fact that for some (hence, for every) $D \in \mathcal{L}(U, Y)$, the equations

$$\dot{z}(t) = Az(t) + Bu(t)\,, \qquad y(t) = C_\Lambda z(t) + Du(t)\,, \tag{2.5}$$

define a *regular linear system* $\Sigma$. This system has input space $U$, state space $Z$ and output space $Y$. The signals $u, z$ and $y$ are called the *input, state trajectory* and *output* of $\Sigma$. $A$ is called the *semigroup generator* of $\Sigma$, $B$ is called the *control operator* of $\Sigma$, $C$ is called the *observation operator* of $\Sigma$ and $D$ is called the *feedthrough operator* of $\Sigma$. For any initial state $z(0) = z_0 \in Z$ and for any $u \in L^2_\alpha([0,\infty); U)$, the equations (2.5) describing $\Sigma$ have unique solutions $z$ and $y$ such that $z$ is continuous, $y \in L^2_\gamma([0,\infty); Y)$ for all $\gamma \ge \alpha$ with $\gamma > \omega_0(\mathbb{T})$ and both equations hold for almost every $t \ge 0$. The *transfer function* of $\Sigma$ is $\mathbf{G}(s) = \mathbf{G}_0(s) + D$, which means that

$$\hat{y}(s) = C(sI - A)^{-1} z_0 + \mathbf{G}(s)\hat{u}(s)\,,$$

where a hat is used to denote the Laplace transformation, and this formula holds for all $s$ in the right half-plane $\mathbb{C}^+_\gamma$. The *generating operators* of $\Sigma$ are $(A, B, C, D)$ and every regular linear system is determined by its four generating operators.

An operator-valued analytic function defined on a domain containing a right half-plane is called *proper* if it is bounded on some right half-plane $\mathbb{C}^+_a$, and it is called *regular* if it is proper and if it has a strong limit at $+\infty$ along the real axis. If the input space is finite-dimensional, "strong limit" simply means "limit". It is clear that the transfer function of a regular linear system is regular. In recent years, many systems described by partial differential equations have been proven to be regular, especially by B.Z. Guo and his collaborators, see [8], [9], [10], [20], [21], [22] and also [3].

## III. The plant, the exosystem and the error

In this section we describe the basic assumptions about the plant to be controlled and the exosystem, and we derive some simple consequences of these assumptions. The *plant* is described by the following equations (for $t \ge 0$):

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) + B^1 d(t), \text{ (state equation)} \\ y(t) = C_\Lambda z(t) + Du(t) + D^1 d(t). \text{ (output)} \end{cases} \tag{3.1}$$

The state of this system is $z(t)$, its input signal is $\left[\begin{smallmatrix} u \\ d \end{smallmatrix}\right]$ and its output signal is $y$. We regard $u$ as the control input (to be generated by a controller) while $d$ is a disturbance. For each $t \ge 0$ we have $z(t) \in Z$, where the state space $Z$ is assumed to be a Hilbert space, $u(t) \in U$, $d(t) \in U^1$ and $y(t) \in Y$, where $U$, $U^1$ and $Y$ are Hilbert spaces. The operator $A$ is the generator of an *exponentially stable* operator semigroup $\mathbb{T}$ on $Z$. The control operator $B \in \mathcal{L}(U, Z_{-1})$ is admissible for $\mathbb{T}$, while $B^1 \in \mathcal{L}(U^1, Z_{-1})$ (not necessarily admissible). The observation operator $C \in \mathcal{L}(Z_1, Y)$ is admissible for $\mathbb{T}$, $D \in \mathcal{L}(U,Y)$ and $D^1 \in \mathcal{L}(U^1, Y)$. We assume that the triple $(A,B,C)$ is regular and for some (hence, for every) $s \in \rho(A)$, the product $C_\Lambda(sI-A)^{-1}B^1$ exists (which is weaker than demanding $(A, B^1, C)$ to be regular). The assumption that the triple $(A,B,C)$ is regular can be relaxed (see Remark V.6).

We assume that there exists a linear system with no input, referred to as the *exosystem* (sometimes called the exogenous system), that produces both the reference output $r$ and the disturbance signal $d$: for all $t \geq 0$,

$$\dot{w}(t) = Sw(t), \quad r(t) = Q^1 w(t), \quad d(t) = C^1 w(t). \quad (3.2)$$

Here $S \in \mathcal{L}(W)$, where $W$ is a Hilbert space, and its spectrum $\sigma(S)$ is a subset of $\overline{\mathbb{C}_0^+}$, i.e., the exosystem is completely unstable. In the applications that we have in mind, $\sigma(S)$ is on the imaginary axis. We have $Q^1 \in \mathcal{L}(W,Y)$ and $C^1 \in \mathcal{L}(W, U^1)$. We refer to the difference between the measured and reference outputs as the *error*:

$$\begin{aligned} e(t) = y(t) - r(t) &= C_\Lambda z(t) + Du(t) + D^1 d(t) - Q^1 w(t) \\ &= C_\Lambda z(t) + Du(t) + Qw(t), \end{aligned}$$

where $Q \in \mathcal{L}(W,Y)$ is defined by $Q = D^1C^1 - Q^1$.

We will also need to consider the *combined plant* $\Sigma_p$ representing the plant and the exosystem together, on the combined state space $X = Z \times W$, with the state

$$x(t) = \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \in X = Z \times W,$$

input space $U$ and output space $Y$, described by the equations

$$\dot{x}(t) = A_p x(t) + B_p u(t), \; A_p = \begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \; B_p = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad (3.3)$$

$$e(t) = C_{p\Lambda} x(t) + D_p u(t), \quad C_p = \begin{bmatrix} C_\Lambda & Q \end{bmatrix}, \quad D_p = D, \quad (3.4)$$

where $P = B^1 C^1$ and

$$\mathcal{D}(A_p) = \mathcal{D}(C_p) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in X \;\middle|\; Az + Pw \in Z \right\}. \quad (3.5)$$

*Remark* III.1. We define a subspace of $Z$ as follows:

$$\begin{aligned} \underset{\sim}{Z} = \mathcal{D}(A) &+ (\lambda I - A)^{-1} BU \\ &+ (\lambda I - A)^{-1} B^1 U^1 \subset \mathcal{D}(C_\Lambda), \quad (3.6) \end{aligned}$$

where $\lambda \in \rho(A)$. It is easy to see that $\underset{\sim}{Z}$ is independent of $\lambda$ and the inclusion $\underset{\sim}{Z} \subset \mathcal{D}(C_\Lambda)$ follows from the regularity of $(A,B,C)$ and the assumption (made a little earlier) that $C_\Lambda(\lambda I - A)^{-1}B^1$ exists. For any $z_0 \in Z$ we have

$$z_0 \in \underset{\sim}{Z} \Leftrightarrow \exists\, u_0 \in U, \; d_0 \in U^1 \text{ so that } \; Az_0 + Bu_0 + B^1 d_0 \in Z,$$

where $A$ is regarded as an operator from $Z$ to $Z_{-1}$ (this is easy to verify). Hence, $\mathcal{D}(A_p) \subset \underset{\sim}{Z} \times W$, so that there is indeed no problem defining $C_p$ on $\mathcal{D}(A_p)$.

*Lemma* III.2. $A_p$ defined in (3.3), (3.5) generates an operator semigroup $\mathbb{T}^p$ on $X$.

*Proof:* First we write down the formula for $\mathbb{T}_t^p$ and then we check that this formula indeed defines an operator semigroup on $X$:

$$\mathbb{T}_t^p \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} \mathbb{T}_t z + \int_0^t \mathbb{T}_\sigma P e^{S(t-\sigma)} w \,\mathrm{d}\sigma \\ e^{St} w \end{bmatrix}. \quad (3.7)$$

To see that this is in $X$, we integrate by parts. We rewrite the integral term in (3.7):

$$\begin{aligned} \int_0^t \mathbb{T}_\sigma P e^{S(t-\sigma)} w \,\mathrm{d}\sigma = &\; A^{-1} \mathbb{T}_\sigma P e^{S(t-\sigma)} w \Big|_0^t \\ &- \int_0^t A^{-1} \mathbb{T}_\sigma P \frac{\mathrm{d}}{\mathrm{d}\sigma} \left[ e^{S(t-\sigma)} w \right] \mathrm{d}\sigma \\ = \mathbb{T}_t A^{-1} P w - A^{-1} P e^{St} w & \\ &+ \int_0^t \mathbb{T}_\sigma A^{-1} P S e^{S(t-\sigma)} w \,\mathrm{d}\sigma. \quad (3.8) \end{aligned}$$

Since $A^{-1}P \in \mathcal{L}(W,Z)$, the integral term in (3.7) is a continuous $Z$-valued function of $t$. This shows that $\mathbb{T}^p$ is a strongly continuous family of operators in $\mathcal{L}(X)$. Obviously $\mathbb{T}_0^p = I$ and the semigroup property is easy to verify. A short computation shows that the generator of this operator semigroup is $A_p$ defined in (3.3) and (3.5). ■

Consider the spaces $X_1$ and $X_{-1}$ introduced in Section II. We have $X_1 = \mathcal{D}(A_p)$, of course, and $X_{-1} = Z_{-1} \times W$, which is easy to verify. The domain of $C_p$ is (by definition) $\mathcal{D}(A_p)$ and $C_{p\Lambda}$ in (3.4) is the $\Lambda$-extension of $C_p$, as defined in Section II, but of course with $A_p$ in place of $A$. We shall now prove that $\Sigma_p$ is a regular linear system. This combined plant is partially stable (since $A$ is stable) but not stabilizable, because there is no way to influence the component $w$ of the state. The problem we want to solve in this paper is to make the output signal $e$ of $\Sigma_p$ small, meaning that it belongs to a weighted $L^2$ space, see Section IV for details.

*Proposition* III.3. The combined plant $\Sigma_p$ from (3.3)–(3.5) is regular. In particular, $B_p$ and $C_p$ are admissible for $\mathbb{T}^p$ and the transfer function of $\Sigma_p$ is

$$\begin{aligned} \mathbf{G}_p(s) &= C_{p\Lambda}(sI - A_p)^{-1} B_p + D_p \\ &= C_\Lambda (sI - A)^{-1} B + D. \quad (3.9) \end{aligned}$$

The operator $C_{p\Lambda}$ can be described as follows:

$$\mathcal{D}(C_{p\Lambda}) = \mathcal{D}(C_\Lambda) \times W \text{ and } C_{p\Lambda} \begin{bmatrix} z \\ w \end{bmatrix} = C_\Lambda z + Qw. \quad (3.10)$$

*Proof:* The admissibility of $B_p$ follows from

$$\int_0^t \mathbb{T}_{t-\sigma}^p B_p u(\sigma) \,\mathrm{d}\sigma = \begin{bmatrix} \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) \,\mathrm{d}\sigma \\ 0 \end{bmatrix},$$

together with the admissibility of $B$ for $\mathbb{T}$. To show that $C_p$ is admissible, we adopt an unusual approach: we find it easier to prove the version of (2.3) with the $\Lambda$-extension $C_{p\Lambda}$ in place of $C$ and $\mathbb{T}^p$ in place of $\mathbb{T}$, and using an arbitrary initial state $\left[ \begin{smallmatrix} z \\ w \end{smallmatrix} \right] \in X$, rather than an initial state in $\mathcal{D}(A_p)$. First we claim that for every $s \in \rho(A)$, $C_\Lambda(sI-A)^{-1}B^1 \in \mathcal{L}(U^1, Y)$. Indeed, as assumed at the beginning of Section III, $C_\Lambda(sI - A)^{-1}B^1$ exists and, according to (2.2), it is the strong limit of the family of operators $C\lambda(\lambda I - A)^{-1}(sI-A)^{-1}B^1 \in \mathcal{L}(U^1, Y)$, as $\lambda \to +\infty$. Our claim now follows from the uniform boundedness principle. Since $P = B^1C^1$, it follows that

$$C_\Lambda (sI - A)^{-1} P \in \mathcal{L}(W, Y). \quad (3.11)$$

We integrate again by parts in (3.8), obtaining

$$\int_0^t \mathbb{T}_\sigma P e^{S(t-\sigma)} w \,\mathrm{d}\sigma = \mathbb{T}_t A^{-1} P w - A^{-1} P e^{St} w \\ + \mathbb{T}_t A^{-2} P S w - A^{-2} P S e^{St} w \\ + A^{-1} \int_0^t \mathbb{T}_\sigma A^{-1} P S^2 e^{S(t-\sigma)} w \,\mathrm{d}\sigma. \quad (3.12)$$

Notice that the last integral is a continuous $Z$-valued function of $t$, hence the last term (which is $A^{-1}$ applied to the integral) is a continuous $Z_1$-valued function of $t$.

We claim that for every $w \in W$, the formula

$$f(t) = C_\Lambda \int_0^t \mathbb{T}_\sigma P e^{S(t-\sigma)} w \,\mathrm{d}\sigma$$

makes sense for almost every $t \geq 0$, and for every $\tau > 0$ there is a $k_\tau \geq 0$ such that

$$\int_0^\tau \|f(t)\|^2 \mathrm{d}t \leq k_\tau \|w\|^2. \quad (3.13)$$

Indeed, this follows from (3.12), (3.11) (with $s = 0$) and the admissibility of $C$ for $\mathbb{T}$. To complete the proof of the admissibility of $C_p$, we have to verify (3.10) first.

To compute $C_{p\Lambda}$, note that (3.3) implies that for all $\lambda \in \rho(A) \cap \rho(S)$,

$$(\lambda I - A_p)^{-1} = \begin{bmatrix} (\lambda I - A)^{-1} & (\lambda I - A)^{-1} P (\lambda I - S)^{-1} \\ 0 & (\lambda I - S)^{-1} \end{bmatrix}. \quad (3.14)$$

Hence, for all $z \in Z$ and $w \in W$,

$$\begin{aligned} & C_p \lambda (\lambda I - A_p)^{-1} \begin{bmatrix} z \\ w \end{bmatrix} \\ &= C\lambda(\lambda I - A)^{-1} z + Q\lambda(\lambda I - S)^{-1} w \\ &\quad + C_\Lambda \lambda (\lambda I - A)^{-1} P (\lambda I - S)^{-1} w \\ &= C\lambda(\lambda I - A)^{-1} z + Q\lambda(\lambda I - S)^{-1} w \\ &\quad + C_\Lambda (\lambda I - A)^{-1} A A^{-1} P \left[I + (\lambda I - S)^{-1} S\right] w \\ &= C\lambda(\lambda I - A)^{-1} z + Q\lambda(\lambda I - S)^{-1} w \\ &\quad + \left[C\lambda(\lambda I - A)^{-1} - C_\Lambda\right] A^{-1} P \left[I + (\lambda I - S)^{-1} S\right] w. \end{aligned}$$

From here, using (3.11) and the fact that $\lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} A^{-1} P = C_\Lambda A^{-1} P$, it is easy to derive that (3.10) holds.

Now we can check that $C_p$ is admissible. For this, we will check that for all $\tau > 0$, there exists $m_\tau \geq 0$ such that for all $z \in Z$ and all $w \in W$,

$$\int_0^\tau \left\| C_{p\Lambda} \mathbb{T}_t^p \begin{bmatrix} z \\ w \end{bmatrix} \right\|^2 \mathrm{d}t \leq m_\tau \left(\|z\|^2 + \|w\|^2\right),$$

which implies the condition (2.3). It is easy to see that this follows from (3.10), (3.7), (3.13) and the admissibility of $C$ for $\mathbb{T}$. The formula (3.9) follows from (3.10) and (3.14) by a simple computation. Now conditions (1) and (2) in Definition II.1 follow from (3.9) and the regularity of $(A, B, C)$. ∎

The *Sylvester* equation

$$\Pi S = A\Pi + P + BL, \quad (3.15)$$

which must be solved for $\Pi$, when $L \in \mathcal{L}(W, U)$ is given, will play an important role in the sequel. The intuitive meaning of this equation is as follows: Consider the combined plant $\Sigma_p$ from (3.3)–(3.5) with the linear state feedback $u = Lw$, as shown in Figure 1. Then it can be shown (using the exponential stability of $A$) that in steady state we have $z(t) = \Pi w(t)$, i.e., $\lim_{t\to\infty} \|z(t) - \Pi w(t)\| = 0$.

*Lemma* III.4. The Sylvester equation (3.15) has a unique solution $\Pi \in \mathcal{L}(W, Z)$, moreover $\mathrm{Ran}\,\Pi \subset \underset{\sim}{Z}$, so that the product $C_\Lambda \Pi$ exists and is in $\mathcal{L}(W, Y)$.

*Proof:* Suppose that (3.15) has a solution $\Pi$. Then for each $t \geq 0$ and $w \in W$,

$$\mathbb{T}_t \Pi S e^{-St} w - \mathbb{T}_t A \Pi e^{-St} w = \mathbb{T}_t (P + BL) e^{-St} w,$$

which is equivalent to

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{T}_t \Pi e^{-St} w) = \mathbb{T}_t (P + BL) e^{-St} w.$$

Integrating the above equation in $Z_{-1}$ on the interval $[0, \infty)$, we get

$$\Pi w = \int_0^\infty \mathbb{T}_t (BL + P) e^{-St} w \mathrm{d}t \quad (3.16)$$

which shows that $\Pi$ is unique. Via integration by parts, as in (3.8), we get

$$\Pi w = -A^{-1}(BL + P) w + \\ \int_0^\infty \mathbb{T}_t A^{-1} (BL + P) e^{-St} S w \,\mathrm{d}t. \quad (3.17)$$

Since $A^{-1}(P + BL) \in \mathcal{L}(W, Z)$, we can conclude from (3.17) that $\Pi \in \mathcal{L}(W, Z)$. Multiplying both sides of (3.17) by $A \in \mathcal{L}(Z, Z_{-1})$, we can verify that $\Pi$ as defined in (3.16) solves (3.15). From (3.15) we obtain that

$$\Pi = A^{-1} \Pi S - A^{-1} P - A^{-1} BL,$$

which implies that $\mathrm{Ran}\,\Pi \subset \underset{\sim}{Z}$ and therefore $C_\Lambda \Pi$ exists (see (3.6)). This operator is the strong limit of the operators $C\lambda(\lambda I - A)^{-1} \Pi \in \mathcal{L}(W, Y)$ as $\lambda \to +\infty$. According to the uniform boundedness principle we have $C_\Lambda \Pi \in \mathcal{L}(W, Y)$. ∎

*Remark* III.5. Sometimes instead of the first equation of the plant from (3.1), the evolution of the state $z$ of the plant is determined via a *boundary control system* as follows:

$$\dot{z}(t) = \tilde{A} z(t), \qquad G z(t) = \mathcal{B} u(t) + \mathcal{B}^1 d(t). \quad (3.18)$$

Here, for some Hilbert spaces $\underset{\sim}{Z}$ and $\tilde{U}$ such that $\underset{\sim}{Z} \subset Z$ with continuous embedding,

$$\tilde{A} \in \mathcal{L}(\underset{\sim}{Z}, Z), \; G \in \mathcal{L}(\underset{\sim}{Z}, \tilde{U}), \; \mathcal{B} \in \mathcal{L}(U, \tilde{U}), \; \mathcal{B}^1 \in \mathcal{L}(U^1, \tilde{U}).$$

The operators $\tilde{A}$ and $G$ define a boundary control system in the sense of [46, Section 10.1] (with input space $\tilde{U}$ and state space $Z$), if the following two assumptions hold:

(i) $G$ is onto,

(ii) $A = \tilde{A}|_{\mathrm{Ker}\,G}$ generates an operator semigroup on $Z$.

If these assumptions hold, and using the notation $Z_{-1}$ from Section II, then there is a unique $\tilde{B} \in \mathcal{L}(\tilde{U}, Z_{-1})$ such that $\tilde{A} = A + \tilde{B} G$, where $A$ is regarded as an operator in $\mathcal{L}(Z, Z_{-1})$. Moreover, for every $s \in \rho(A)$ we have $(sI - A)^{-1} \tilde{B} \in \mathcal{L}(\tilde{U}, \underset{\sim}{Z})$ and $G(sI - A)^{-1} \tilde{B} = I$. Notice that we have $G\phi = 0$ for all $\phi \in \mathcal{D}(A)$. Moreover, we have

$$\underset{\sim}{Z} = Z_1 + (sI - A)^{-1} \tilde{B} \tilde{U}. \quad (3.19)$$

Indeed, all this follows from Proposition 10.1.2 in [46] and the text around it, if we use the following correspondence of the notation: what is called $X, Z, U, L, G, A, B$ in [46] is called here (in the same order) $Z, \underset{\sim}{Z}, \tilde{U}, \tilde{A}, G, A, \tilde{B}$. Now we can rewrite (3.18) exactly as the first equation in (3.1), if we denote $B = \tilde{B}\mathcal{B}$ and $B^1 = \tilde{B}\mathcal{B}^1$. It is easy to see from (3.19) that if $\operatorname{Ran}[\mathcal{B}\ \mathcal{B}^1] = \tilde{U}$, then $\underset{\sim}{Z}$ as defined in Remark III.1 is the same as $\underset{\sim}{Z}$ introduced above.

We shall see at the end of Section IV that our main result has a neat version for boundary control systems, and this version is much easier to apply when we design a state feedback for a boundary controlled beam equation at the end of Section VI.

## IV. The state feedback regulator problem

We continue to use the assumptions and the notation from Section III. In particular, recall that $Q = D^1C^1 - Q^1$ and $P = B^1C^1$. In the state feedback regulator problem, stated below, we consider the state of the combined plant $\Sigma_p$ to be accessible to the controller, which is a static linear feedback. Our terminology and notation are consistent with the treatment of the finite-dimensional case in Knobloch, Isidori and Flockerzi [33], but we assume that $A$ is exponentially stable, since (as explained in Section I) we do not want to discuss here the stabilization problem.

*Problem* IV.1. **The linear state feedback regulator problem:** For the combined plant $\Sigma_p$ from (3.3)–(3.5), find a feedback control law in the form $u = Lw$, with $L \in \mathcal{L}(W,U)$, such that for the resulting closed-loop system with no input, described by

$$\begin{bmatrix} \dot z \\ \dot w \end{bmatrix} = \begin{bmatrix} A & P+BL \\ 0 & S \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} := A_p^L \begin{bmatrix} z \\ w \end{bmatrix}, \tag{4.1}$$

$$e = \begin{bmatrix} C_\Lambda & Q+DL \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}, \tag{4.2}$$

we have $e \in L^2_\alpha([0,\infty);Y)$ for some $\alpha < 0$ and for all initial conditions $z(0) = z_0 \in Z$ and $w(0) = w_0 \in W$ (i.e., for any initial state in $X$).

To clarify that the equations (4.1), (4.2) are correctly formulated, we investigate the closed-loop system obtained by subjecting $\Sigma_p$ to state feedback via $[0\ \ L]$.

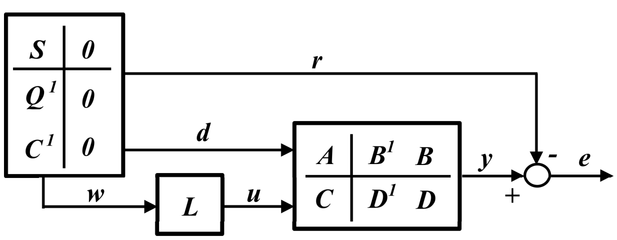


Figure 1. The closed-loop system corresponding to the state feedback regulator problem. The closed-loop system is not asymptotically stable, but the error is in $L^2_\alpha([0,\infty);Y)$ with $\alpha < 0$, like the output of an exponentially stable system.

*Proposition* IV.2. For every $L \in \mathcal{L}(W,U)$, if we define $F_p \in \mathcal{L}(Z \times W, U)$ by $F_p = \begin{bmatrix} 0 & L \end{bmatrix}$, then the following holds:

(1) The triple $(A_p, B_p, F_p)$ is regular (with state space $X$).

(2) The transfer function $\left[I - F_p(sI - A_p)^{-1}B_p\right]^{-1}$ is proper (actually, it is $I$ because $F_p(sI - A_p)^{-1}B_p = 0$).

(3) The operator $A_p^L = A_p + B_pF_p$ (see (4.1)) with its natural domain,
$$\mathcal{D}(A_p^L) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in X \,\middle|\, Az + (P+BL)w \in Z \right\},$$
is the generator of an operator semigroup $\mathbb{T}^{p,L}$ on $X$. For each $t \geq 0$, $\mathbb{T}_t^{p,L}$ has the upper triangular form $\begin{bmatrix} \mathbb{T}_t & * \\ 0 & e^{St} \end{bmatrix}$.

(4) If we define $C_p^L$ to be the restriction of $[C_\Lambda\ \ Q+DL]$ to $\mathcal{D}(A_p^L)$, then $C_{p\Lambda}^L$ (the $\Lambda$-extension of $C_p^L$) satisfies
$$C_{p\Lambda}^L = \begin{bmatrix} C_\Lambda & Q+DL \end{bmatrix}$$
and $(A_p^L, B_p, C_p^L)$ is regular (with state space $X$).

*Proof:* From Proposition III.3 we know that $(A_p, B_p, C_p, D_p)$ are the generating operators of a regular linear system with state $x = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$ and output $e$. We build an extension $\widetilde{\Sigma}_p$ of this system by adding to it a second output $v = F_px = Lw$. Since $F_p$ is bounded, using (2.4) we get that claim (1) holds and $\widetilde{\Sigma}_p$ is a regular linear system with the generating operators $(A_p, B_p, \widetilde{C}_p, \widetilde{D}_p)$, where

$$\widetilde{C}_p = \begin{bmatrix} C_\Lambda & Q \\ 0 & L \end{bmatrix}, \qquad \widetilde{D}_p = \begin{bmatrix} D \\ 0 \end{bmatrix}.$$

We are interested in the closed-loop system $\widetilde{\Sigma}_p^K$ obtained from $\widetilde{\Sigma}_p$ by applying to it the output feedback operator $K = [0\ \ I]$. For $\widetilde{\Sigma}_p^K$ to be well-posed, according to [49, Proposition 3.6 and Theorem 6.1], we need that $I - K\widetilde{\mathbf{G}}$ has a proper inverse, where $\widetilde{\mathbf{G}}$ is the transfer function of $\widetilde{\Sigma}_p$. It is easy to check that $K\widetilde{\mathbf{G}} = F_p(sI - A_p)^{-1}B_p = 0$, so that indeed $\widetilde{\Sigma}_p^K$ is well-posed and also our claim (2) holds. According to [49, Proposition 4.6 and Theorem 4.7] $\widetilde{\Sigma}_p^K$ is regular, and its feedthrough operator is $\widetilde{D}_p$. From [49, Proposition 5.3 and Theorem 7.2] we get that the semigroup generator of $\widetilde{\Sigma}_p^K$ is $A_p^L$, which confirms the first part of our claim (3). The upper triangular form of $\mathbb{T}_t^{p,L}$ follows directly from the upper triangular form of $A_p^L$. Using again [49, Theorem 7.2] we see that the observation operator of $\widetilde{\Sigma}_p^K$ is

$$\widetilde{C}_p^K = \begin{bmatrix} C_\Lambda & Q+DL \\ 0 & L \end{bmatrix},$$

with domain $\mathcal{D}(A_p^L)$. According to [49, Proposition 7.1] and (3.10), its $\Lambda$-extension is $\begin{bmatrix} C_\Lambda & Q+DL \\ 0 & L \end{bmatrix}$, with domain $\mathcal{D}(C_\Lambda) \times W$. Looking at the first line of $\widetilde{C}_p^K$ and its $\Lambda$-extension, we see that claim (4) is also true. ∎

*Lemma* IV.3. If $R \in \mathcal{L}(W,Y)$ is such that the function $m(t) = Re^{St}w_0$ belongs to $L^2_\alpha([0,\infty);Y)$ for some $\alpha < 0$ and all $w_0 \in W$, then $R = 0$.

*Proof:* Fix $0 < \beta < |\alpha|$ and $w_0 \in W$ and define the function $v(t) = e^{\beta t}m(t)$. We can factor $v(t) = e^{(\alpha+\beta)t}[e^{-\alpha t}m(t)]$. Since $\alpha + \beta < 0$ and $m \in L^2_\alpha([0,\infty);Y)$, both factors are in $L^2$, so that $v \in L^1([0,\infty);Y)$ and hence

$$\hat v(s) = R((s-\beta)I - S)^{-1}w_0, \qquad \hat v \in H^\infty(\mathbb{C}_0^+;Y). \tag{4.3}$$

This needs some explanation. The above formula for $\hat v$ only holds for $s - \beta$ in some right half-plane contained in $\rho(S)$. Nevertheless, $\hat v$ is defined on all of $\mathbb{C}_0^+$ and it is an analytic continuation of the function defined by the above formula.

Let $\omega = \beta/2$. Since $\sigma(S) \subset \overline{\mathbb{C}_0^+}$, we have the following bound for the semigroup $e^{-St}$: $\|e^{-St}\| \leq M_\omega e^{\omega t}$ for some $M_\omega \geq 1$ and all $t \geq 0$. This implies that

$$\|(sI+S)^{-1}\| \leq \frac{M_\omega}{\operatorname{Re} s - \omega} \qquad \forall\, s\in\mathbb{C}^+_\omega,$$

or equivalently that

$$\|((s-\beta)I-S)^{-1}\| \leq \frac{M_\omega}{\omega - \operatorname{Re} s} \qquad \text{for } \operatorname{Re} s < \omega.$$

This, along with (4.3), shows that $\hat{v}$ has a bounded analytic continuation to the left half-plane where $\operatorname{Re} s < \omega$. Since we already know that $\hat{v}$ is bounded and analytic on $\mathbb{C}^+_0$, it follows that it is a bounded entire function. By Liouville's theorem, $\hat{v}$ is constant. It is easy to see that the limit of $\hat{v}$ at infinity is 0, and hence $\hat{v}=0$. Therefore, for every $t\geq 0$ we have $v(t)=0$ and hence $m(t)=0$. Since this is true for every $w_0\in W$, we get that $R=0$. ■

The next theorem is the main result of this section and it gives necessary and sufficient conditions for the solvability of the state feedback regulator problem.

*Theorem* IV.4. Suppose that there exist operators $\Pi\in\mathcal{L}(W,Z)$ and $\Gamma\in\mathcal{L}(W,U)$ satisfying the *regulator equations*

$$\Pi S = A\Pi + B\Gamma + P, \tag{4.4}$$
$$0 = C_\Lambda \Pi + D\Gamma + Q. \tag{4.5}$$

The first regulator equation holds in $\mathcal{L}(W,Z)$ and the second holds in $\mathcal{L}(W,Y)$. In this case a feedback law solving the linear state feedback regulator problem is

$$u(t) = \Gamma w(t). \tag{4.6}$$

Conversely, if an operator $L\in\mathcal{L}(W,U)$ solves the linear state feedback regulator problem, then there exists $\Pi\in\mathcal{L}(W,Z)$ such that, taking $\Gamma=L$, the equations (4.4)–(4.5) are satisfied.

We mention that (in the context of the above theorem) if $L$ exists, it may be non-unique, as explained in Remark V.4.

*Proof:* We start by proving the second (converse) part of the theorem. Suppose that $u(t)=Lw(t)$ solves the linear state feedback regulator problem. According to Lemma III.4 the Sylvester equation (3.15) has a unique solution $\Pi\in\mathcal{L}(W,Z)$ for this $L$. Hence $\Gamma=L$ and $\Pi$ satisfy the first regulator equation (4.4). We want to show that this $\Gamma$ and $\Pi$ solve the second regulator equation (4.5) as well. For any $w_0\in W$, we claim that $\left[\begin{smallmatrix}\Pi w_0\\ w_0\end{smallmatrix}\right]\in\mathcal{D}(A^L_p)$ and

$$\mathbb{T}^{p,L}_t \begin{bmatrix}\Pi w_0\\ w_0\end{bmatrix} = \begin{bmatrix}\Pi e^{St} w_0\\ e^{St} w_0\end{bmatrix} \qquad \forall\, t\geq 0. \tag{4.7}$$

Indeed, if we differentiate the right-hand side, using (3.15) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\Pi e^{St} w_0\\ e^{St} w_0\end{bmatrix} = \begin{bmatrix}(A\Pi+BL+P)e^{St}w_0\\ Se^{St}w_0\end{bmatrix} = A^L_p \begin{bmatrix}\Pi e^{St} w_0\\ e^{St} w_0\end{bmatrix}.$$

Recall that $\operatorname{Ran}\Pi \subset \underset{\sim}{Z} \subset \mathcal{D}(C_\Lambda)$. Thus we can apply $\begin{bmatrix}C_\Lambda & Q+DL\end{bmatrix}$ to (4.7) and use (4.2) to obtain that

$$e(t) = (C_\Lambda\Pi + DL + Q)e^{St}w_0 \qquad \forall\, t\geq 0, \tag{4.8}$$

when the initial condition for (4.1) is $\left[\begin{smallmatrix}\Pi w_0\\ w_0\end{smallmatrix}\right]$. By assumption, for some $\alpha<0$, $e\in L^2_\alpha([0,\infty);Y)$ for all $w_0\in W$ and $\sigma(S)\subset\overline{\mathbb{C}^+_0}$. According to Lemma IV.3 (with $R=C_\Lambda\Pi+DL+Q$) we get that $C_\Lambda\Pi+DL+Q=0$. Thus, (4.5) holds with $\Gamma=L$.

Now we prove the first part of the theorem. Suppose that the regulator equations (4.4) and (4.5) are satisfied by $\Pi\in\mathcal{L}(W,Z)$ and $\Gamma\in\mathcal{L}(W,U)$. Define $L=\Gamma$, then the first regulator equation becomes (3.15). We define a closed subspace $X^+$ of $X=Z\times W$ as the graph of the mapping $\Pi$, i.e.,

$$X^+ = \left\{ \begin{bmatrix}\Pi w\\ w\end{bmatrix} \,\middle|\, w\in W\right\}.$$

As already mentioned at (4.7), $X^+\subset\mathcal{D}(A^L_p)$ and $X^+$ is $\mathbb{T}^{p,L}_t$-invariant. For any initial condition $\left[\begin{smallmatrix}z_0\\ w_0\end{smallmatrix}\right]\in X$ we have

$$\mathbb{T}^{p,L}_t\begin{bmatrix}z_0\\ w_0\end{bmatrix} = \mathbb{T}^{p,L}_t\begin{bmatrix}\Pi w_0\\ w_0\end{bmatrix} + \mathbb{T}^{p,L}_t\begin{bmatrix}z_0-\Pi w_0\\ 0\end{bmatrix}.$$

Formula (4.7) and the upper triangular form of $\mathbb{T}^{p,L}_t$ (claim (3) in Proposition IV.2) imply that

$$\mathbb{T}^{p,L}_t\begin{bmatrix}z_0\\ w_0\end{bmatrix} = \begin{bmatrix}\Pi e^{St}w_0\\ e^{St}w_0\end{bmatrix} + \begin{bmatrix}\mathbb{T}_t(z_0-\Pi w_0)\\ 0\end{bmatrix}.$$

Applying $[C_\Lambda \quad Q+DL]$ to the above equation, using (4.2) and $C_\Lambda\Pi+Q+DL=0$, we get that

$$e(t) = C_\Lambda \mathbb{T}_t(z_0-\Pi w_0).$$

Since $\mathbb{T}$ is exponentially stable and $C_\Lambda$ is an admissible observation operator for $\mathbb{T}$, it follows that $e\in L^2_\alpha([0,\infty);Y)$ for some $\alpha<0$. Therefore, the linear state feedback regulator problem is solved by $u=\Gamma w$. ■

*Remark* IV.5. In the first part of the above proof, where the necessity of the regulator equations (4.4) and (4.5) is established, the existence of the mapping $\Pi$ that solves (4.4) for the given $L$ follows from Lemma III.4. The existence of such a $\Pi$ can also be inferred using the notions of spectral decomposition and spectral projections, see [12, Lemma 2.5.7]. The rest of the proof of necessity remains the same.

*Remark* IV.6. This long remark is about a more general version of the state feedback regulator problem. The feedback usually encountered in the state feedback regulator theory for possibly unstable plants with bounded control and observation operators is of the form $u=Fz+Lw$, where $F\in\mathcal{L}(Z,U)$ and $L\in\mathcal{L}(W,U)$, see for instance [4], [28]. In this paper, where we generalize the theory to regular linear systems, it would seem natural to consider in Problem IV.1 a possibly unstable $A$ and a feedback control law of the form

$$u(t) = F_\Lambda z(t) + Lw(t),$$

where $F\in\mathcal{L}(Z_1,U)$ is a stabilizing state feedback operator for the plant. This means (see [50]) that $(A,B,F)$ is regular (see Definition II.1), the transfer function $[I-F_\Lambda(sI-A)^{-1}B]^{-1}$ is proper and $A+BF_\Lambda$ is exponentially stable. Of course, the requirement on $e$ would remain the same. This is a generalization of the state feedback regulator problem, as we allow unstable plants and we have a larger class of feedback laws to choose from, and for certain additional control objectives this might be desirable. In this case, in Proposition IV.2 we would have to take $F_p=[F_\Lambda \quad L]$, with $\mathcal{D}(F_p)=\mathcal{D}(A_p)$. With this more general $F_p$, claims (1) and (2) in Proposition IV.2 would remain unchanged (but now they would be less trivial to prove, and the transfer function in (2) would not be $I$). In claim (3) we would have to modify the domain to

$$\mathcal{D}(A_p^L) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in X \,\middle|\, (A+BF_\Lambda)z + (P+BL)w \in Z \right\}$$

and the restriction of $\mathbb{T}_t^{p,L}$ to $Z$ would be $\mathbb{T}_t^F$. Here $\mathbb{T}_t^F$ is the exponentially stable operator semigroup generated by $A + BF_\Lambda$. In claim (4) $[C_\Lambda \;\; Q + DL]$ would have to be replaced with $[C_\Lambda + DF_\Lambda \;\; Q + DL]$.

If we adopt the above more general statement of the state feedback regulator problem, with a possibly unstable $A$ for which a stabilizing state feedback operator $F$ exists, then Theorem IV.4 remains valid, except that its last line (equation (4.6)) has to be replaced with

$$u(t) = F_\Lambda z(t) + (\Gamma - F_\Lambda \Pi) w(t) \,.$$

The proof of this more general version of Theorem IV.4 is similar to the proof given above. If we assume that $A$ is exponentially stable (for reasons mentioned at the beginning of Section I), then comparing the two versions of Theorem IV.4 we see that the solvability of Problem IV.1 is equivalent to the solvability of its more general version. This is one reason why we have adopted the simpler version of the problem in this paper. The other reason is that when we shall solve the error feedback regulator problem in a follow-up paper, we shall use the feedback law $u = L\hat{w}$, where $L$ solves the state feedback regulator problem as stated in this paper, while $\hat{w}$ is an estimate of $w$ obtained by a finite-dimensional estimator.

When solving the regulator problem for a plant that is a boundary control system, it may be advantageous to use an alternative form of (4.4), as described below.

*Proposition* IV.7. Suppose that the first plant equation from (3.1) can be written alternatively as a boundary control system, as in (3.18) (with the spaces and operators as in Remark III.5). Then the first regulator equation (4.4) can be rewritten equivalently as the following two equations:

$$\Pi S = \tilde{A}\Pi, \qquad G\Pi = \mathcal{B}\Gamma + \mathcal{B}^1 C^1 \,. \tag{4.9}$$

*Proof:* Assume that the first regulator equation holds. We have explained in Remark III.5 that the plant can be written in the standard form (3.1) if we denote $B = \tilde{B}\mathcal{B}$ and $B^1 = \tilde{B}\mathcal{B}^1$. Hence the first regulator equation is

$$\Pi S = A\Pi + \tilde{B}\left(\mathcal{B}\Gamma + \mathcal{B}^1 C^1\right) \,. \tag{4.10}$$

We apply $GA^{-1}$ to both sides and use the fact that $GA^{-1}$ is zero on $Z$:

$$G\Pi + GA^{-1}\tilde{B}\left(\mathcal{B}\Gamma + \mathcal{B}^1 C^1\right) = 0 \,.$$

Now we recall from Remark III.5 that for every $s \in \rho(A)$, $G(sI - A)^{-1}\tilde{B} = I$, so that $GA^{-1}\tilde{B} = -I$. Hence, the first regulator equation implies that $G\Pi = \mathcal{B}\Gamma + \mathcal{B}^1 C^1$, as claimed in the second part of (4.9). If we substitute this formula into (4.10), we get $\Pi S = A\Pi + \tilde{B}G\Pi$. Since (as mentioned in Remark III.5) we have $\tilde{A} = A + \tilde{B}G$ (where $A \in \mathcal{L}(Z, Z_{-1})$), we obtain from here the first part of (4.9).

Conversely, suppose that (4.9) holds. From the first equation we obtain, using that $\tilde{A} = A + \tilde{B}G$, that $\Pi S = A\Pi + \tilde{B}G\Pi$. Express here $G\Pi$ using the second equation from (4.9), to obtain $\Pi S = A\Pi + \tilde{B}\left(\mathcal{B}\Gamma + \mathcal{B}^1 C^1\right)$. Using that $\tilde{B}\mathcal{B} = B$, $\tilde{B}\mathcal{B}^1 = B^1$ and $P = B^1 C^1$, we get (4.4). ■

*Remark* IV.8. The recent paper Paunonen and Pohjolainen [41] explores the error feedback regulator problem for a plant of the type (3.1), with bounded $B^1$, $D^1 = 0$ and with unbounded $B \in \mathcal{L}(U, Z_{-1})$ and $C \in \mathcal{L}(Z_1, Y)$ but without any admissibility or well-posedness assumptions. The error feedback controller is possibly infinite-dimensional as well, but its control operator is bounded. The authors assume that the interconnection of the two systems leads to a strongly continuous and strongly stable semigroup on the product state space. Moreover, they assume that a certain Sylvester equation has a solution in the right space. The exosystem operator $S$ is allowed to be unbounded, with imaginary eigenvalues and Jordan blocks. Under these assumptions, they show that the controller solves a version of the error feedback problem iff certain equations, formulated in terms of the operators of the closed loop system, are solvable. (They call these equations the regulator equations.) They also explore a robust version of the output regulation problem. We think that there is no overlap at all between our results and those of [41].

## V. Solvability of the regulator equations

In Section IV we have characterized the solvability of the state feedback regulator problem in terms of the solvability of the regulator equations. In this section, following Byrnes *et al* [4], we characterize the solvability of the regulator equations in terms of the nonresonance condition between the system transmission zeros and the natural frequencies of the exosystem. Under some reasonable additional assumptions on $S$, we also give an explicit formula for the feedback operator $L$ that solves the state feedback regulator problem.

**Assumptions.** We continue to use the assumptions and the notation of Section III. Thus, the plant to be controlled is described by (3.1) and the exosystem by (3.2). In addition, *we assume that $W$ is finite-dimensional and certain eigenvectors of $S$ are an (algebraic) basis in $W$* (i.e., $S$ has no Jordan blocks). The basis assumption is made to simplify our presentation and can be dropped (see Remark V.3).

Recall that since $A$ is exponentially stable and $S$ is completely unstable, $\sigma(A) \cap \sigma(S) = \emptyset$. Denote $\omega_0 = \omega_0(\mathbb{T}) < 0$. We denote $\mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B + D$, defined on $\mathbb{C}^+_{\omega_0}$, so that $\mathbf{G}$ is the transfer function of the plant from $u$ to $y$.

*Definition* V.1. $s_0 \in \mathbb{C}^+_{\omega_0}$ is a *transmission zero* of $\mathbf{G}$ if $\mathbf{G}(s_0)$ is not onto.

The following theorem is the main result of this section.

*Theorem* V.2. The regulator equations (4.4) and (4.5) are solvable for any $P \in \mathcal{L}(W, Z_{-1})$ and any $Q \in \mathcal{L}(W, Y)$ such that $C_\Lambda(sI - A)^{-1}P$ exists for some (hence, for every) $s \in \rho(A)$, if and only if each $\lambda \in \sigma(S)$ is not a transmission zero of $\mathbf{G}$.

In this case, a feedback operator $L$ that solves Problem IV.1 is defined by its action on a basis of eigenvectors $w_i$ of $S$ as follows:

$$\begin{aligned} Lw_i = & -\mathbf{G}^*(\lambda_i)\left[\mathbf{G}(\lambda_i)\mathbf{G}^*(\lambda_i)\right]^{-1} \\ & \cdot \left[C_\Lambda(\lambda_i I - A)^{-1}Pw_i + Qw_i\right] , \end{aligned} \tag{5.1}$$

where $\lambda_i$ is the eigenvalue corresponding to $w_i$.

*Proof:* Suppose that the regulator equations are solvable for any $P$ and $Q$ such that $C_\Lambda(sI - A)^{-1}P$ exists for some

$s \in \rho(A)$. Then from Theorem IV.4, for each such $P$ and $Q$, there exists a feedback law $u = Lw$ that solves Problem IV.1. With this feedback, the equations of the closed-loop system are (4.1) and (4.2). It is easy to see that

$$\hat{e}(s) = C(sI - A)^{-1}z(0) + \mathbf{H}(s)(sI - S)^{-1}w(0), \quad (5.2)$$

where

$$\mathbf{H}(s) = C_\Lambda(sI - A)^{-1}(P + BL) + Q + DL \quad \forall\, s \in \mathbb{C}^+_{\omega_0}, \quad (5.3)$$

and the formula for $\hat{e}(s)$ holds on any right half-plane to the right of $\sigma(S)$. Choosing $z(0) = 0$ and $w(0) = w_i$, we obtain

$$\hat{e}(s) = \mathbf{H}(s)\frac{w_i}{s - \lambda_i}. \quad (5.4)$$

By analytic continuation, this remains valid on $\mathbb{C}^+_{\omega_0}$, except at the point $\lambda_i$. Since the feedback $u = Lw$ solves the state feedback regulator problem, $e \in L^2_\alpha([0,\infty); Y)$ for some $\alpha < 0$, so that $\hat{e}$ is analytic on $\mathbb{C}^+_\alpha$. Comparing this with (5.4), we get

$$\mathbf{H}(\lambda_i)w_i = 0 \qquad \forall\, \lambda_i \in \sigma(S). \quad (5.5)$$

Since this equality must hold for any $P$ and $Q$ as in the theorem, we get

$$\operatorname{Ran}\mathbf{G}(\lambda_i) = Y \qquad \forall\, \lambda_i \in \sigma(S). \quad (5.6)$$

Indeed, this follows from the fact that when $P = 0$, then $\mathbf{H} = \mathbf{G}L + Q$.

Conversely, suppose that the condition (5.6) holds, so that $\mathbf{G}(\lambda_i)\mathbf{G}^*(\lambda_i)$ is bijective and therefore, using the bounded inverse theorem, invertible. On the set of eigenvectors of $S$, which is a basis in $W$, define $L$ using (5.1). It then follows that

$$-\mathbf{G}(\lambda_i)Lw_i = C_\Lambda(\lambda_i I - A)^{-1}Pw_i + Qw_i$$

and from here we can easily derive (5.5). From (5.2) we see that the component of $e$ due to $z(0)$ is in $L^2_\alpha([0,\infty); Y)$ for any $\alpha$ such that $\omega_0 < \alpha < 0$. By using superposition in (5.2) we see that it is enough to verify that $e \in L^2_\alpha([0,\infty); Y)$ (with $\alpha < 0$) when $z(0) = 0$ and $w(0) = w_i$. In this case,

$$\hat{e}(s) = \mathbf{H}(s)\frac{w_i}{s - \lambda_i} \qquad \forall\, s \in \mathbb{C}^+_\gamma, \quad (5.7)$$

for some $\gamma > 0$ with $\mathbb{C}^+_\gamma \cap \sigma(S) = \emptyset$. Using (5.5) and analytic continuation, we can rewrite (5.7):

$$\hat{e}(s) = [\mathbf{H}(s) - \mathbf{H}(\lambda_i)]\frac{w_i}{s - \lambda_i} \qquad \forall\, s \in \mathbb{C}^+_{\omega_0}.$$

Using (5.3) and the resolvent identity, this becomes

$$\hat{e}(s) = -C(sI - A)^{-1}(\lambda_i I - A)^{-1}(P + BL)w_i.$$

Notice that the vector $z_i = -(\lambda_i I - A)^{-1}(P + BL)w_i$ is in $Z$. Therefore, for almost every $t \geq 0$, $e(t) = C_\Lambda \mathbb{T}_t z_i$, which shows (as explained in Section II after (2.3)) that $e \in L^2_\alpha([0,\infty); Y)$ for any $\alpha$ such that $\omega_0 < \alpha < 0$.

Thus, the linear state feedback regulator problem, and consequently also the regulator equations (see Theorem IV.4) can be solved using $L$ defined in (5.1). ∎

*Remark* V.3. All the conclusions of Theorem V.2, except the formula for $L$, remain valid when the eigenvectors of $S$ do not span $W$. In this case, $L$ is given by more complicated formulas. For example, assume that some of the eigenvalues $\lambda_i$ correspond to Jordan blocks of order 2, which means that we can find for them an eigenvector $w_i$ and a generalized eigenvector $\tilde{w}_i$ such that $S\tilde{w}_i = \lambda_i\tilde{w}_i + w_i$. We assume for simplicity that there are no Jordan blocks of higher order for $S$. Any vector in $W$ has a unique representation as a linear combination of the eigenvectors $w_i$, $i \in \{1, 2, \ldots\, p\}$, and the generalized eigenvectors $\tilde{w}_i$, which exist for a subset $\mathcal{J} \subset \{1, 2, \ldots\, p\}$ of the indices $i$. Thus, $L$ is completely defined by its action on these vectors.

Only the "conversely" part of the proof of Theorem V.2 needs to be adjusted in this more general situation. If $\mathbf{G}(\lambda_i)$ is onto for each eigenvalue, we have to construct $L \in \mathcal{L}(W, U)$ that solves the state feedback regulator problem.

We define $L$ as follows: $Lw_i$ is defined by (5.1), while

$$\begin{aligned} L\tilde{w}_i = &-\mathbf{G}^*(\lambda_i)\left[\mathbf{G}(\lambda_i)\mathbf{G}^*(\lambda_i)\right]^{-1}\big[C_\Lambda(\lambda_i I - A)^{-1}P\tilde{w}_i \\ &+ Q\tilde{w}_i - C_\Lambda(\lambda_i I - A)^{-2}(P + BL)w_i\big]. \end{aligned} \quad (5.8)$$

By using superposition in (5.2) it is enough to verify $e \in L^2_\alpha([0,\infty); Y)$ for $z(0) = 0$ and $w(0) = w_i$ or $w(0) = \tilde{w}_i$. But for $w(0) = w_i$ we have already verified this in the proof of Theorem V.2. Thus, it remains to verify the case $z(0) = 0$ and $w(0) = \tilde{w}_i$. Then, using (5.5),

$$\begin{aligned} \hat{e}(s) &= \mathbf{H}(s)\left(\frac{w_i}{(s - \lambda_i)^2} + \frac{\tilde{w}_i}{s - \lambda_i}\right) \\ &= \frac{1}{s - \lambda_i}\left(\frac{\mathbf{H}(s) - \mathbf{H}(\lambda_i)}{s - \lambda_i}\, w_i + \mathbf{H}(s)\tilde{w}_i\right). \end{aligned} \quad (5.9)$$

Due to (5.5), the expression in the large parentheses above is analytic for $s$ in a neighborhood of $\lambda_i$, regardless of the choice of $L\tilde{w}_i$. Our choice in (5.8) is dictated by the need to ensure that this large parentheses converges to zero when $s \to \lambda_i$ (so that $\hat{e}$ is analytic on $\mathbb{C}_{\omega_0}$). Indeed, the choice (5.8) implies that

$$\mathbf{H}'(\lambda_i)w_i + \mathbf{H}(\lambda_i)\tilde{w}_i = 0.$$

Hence, we can subtract the above zero expression from the expression in the large parentheses in (5.9), obtaining that

$$\begin{aligned} \hat{e}(s) = \frac{1}{s - \lambda_i}&\left(\frac{\mathbf{H}(s) - \mathbf{H}(\lambda_i)}{s - \lambda_i} - \mathbf{H}'(\lambda_i)\right)w_i \\ &+ \frac{\mathbf{H}(s) - \mathbf{H}(\lambda_i)}{s - \lambda_i}\tilde{w}_i. \end{aligned}$$

Using (5.3) and twice the resolvent identity, this becomes

$$\begin{aligned} \hat{e}(s) = &\ C(sI - A)^{-1}(\lambda_i I - A)^{-2}(P + BL)w_i \\ &- C(sI - A)^{-1}(\lambda_i I - A)^{-1}(P + BL)\tilde{w}_i. \end{aligned}$$

From this it follows that $e \in L^2_\alpha([0,\infty); Y)$ for any $\alpha \in (\omega_0, 0)$, as at the end of the proof of Theorem V.2.

*Remark* V.4. The feedback operator $L$ in Theorem V.2 (and also in Remark V.3) is not unique, in general. Indeed, everything in this theorem and its proof (and also in the remark) remains valid if we replace $\mathbf{G}^*(\lambda_i)[\mathbf{G}(\lambda_i)\mathbf{G}^*(\lambda_i)]^{-1}$ with some other right inverse of $\mathbf{G}(\lambda_i)$.

From Theorem V.2 it follows that for a given pair $P$ and $Q$, (5.6) is a sufficient condition for the regulator equations to be solvable. When $U = \mathbb{C}$ and $Y = \mathbb{C}$, under an additional hypothesis, this condition also becomes necessary.

*Corollary* V.5. Let $U = Y = \mathbb{C}$. Assume that the pair $(A_p, C_p)$ is detectable in the sense of [50] and $H_p \in \mathcal{L}(U, X_{-1})$ detects this pair (this implies that $(A_p, H_p, C_p)$ is a regular triple and $A_p + H_p C_{p\Lambda}$ generates an exponentially stable semigroup). Then the regulator equations (4.4) and (4.5) have a solution for a given $P \in \mathcal{L}(W, Z_{-1})$ and $Q \in \mathcal{L}(W, Y)$ such that $C_\Lambda(sI - A)^{-1}P$ exists for some (hence, for every) $s \in \rho(A)$, if and only if for each $\lambda_i \in \sigma(S)$, $\mathbf{G}(\lambda_i) \neq 0$.

*Proof:* The sufficiency of the condition $\mathbf{G}(\lambda_i) \neq 0$ follows from Theorem V.2. To establish its necessity assume that a feedback law $u = Lw$ solves the state feedback regulator problem. Each $\lambda_i \in \sigma(S)$ is also an eigenvalue of $A_p$ with the corresponding eigenvector being

$$v_i = \begin{bmatrix} (\lambda_i I - A)^{-1} P w_i \\ w_i \end{bmatrix},$$

where $w_i$ is the eigenvector of $S$ corresponding to $\lambda_i$. It follows from (3.6) and (3.10) that $v_i \in \mathcal{D}(C_{p\Lambda})$. We will show that $C_{p\Lambda} v_i \neq 0$ for each $i \in \{1, 2, \ldots, k\}$.

Fix $i$ and consider the exponentially stable system

$$\dot{\Theta} = (A_p + H_p C_{p\Lambda})\Theta, \quad \Theta(0) = v_i\,.$$

Assume that $C_{p\Lambda} v_i = 0$. Then $C_{p\Lambda} \mathbb{T}^p_t v_i = e^{\lambda_i t} C_{p\Lambda} v_i = 0$. Clearly the function $\Theta(t) = \mathbb{T}^p_t v_i$ is the unique classical solution to the above exponentially stable system and $\mathbb{T}^p_t v_i \to 0$ as $t \to \infty$. Since $A_p$ is upper triangular, so is $\mathbb{T}^p_t$ (claim (3) in Proposition IV.2 with $L = 0$). In particular, $\mathbb{T}^p_t v_i \to 0$ implies that $e^{St} w_i \to 0$, as $t \to \infty$, which is impossible. Therefore $C_{p\Lambda} v_i \neq 0$ for all $i$ and for each $\lambda_i \in \sigma(S)$,

$$C_\Lambda(\lambda_i I - A)^{-1} P w_i + Q w_i \neq 0\,.$$

This fact along with (5.5), which holds here for reasons similar to those in the proof of Theorem V.2, implies that $C_\Lambda(\lambda_i I - A)^{-1}B + D \neq 0$, i.e., $\mathbf{G}(\lambda_i) \neq 0$ for each $\lambda_i \in \sigma(S)$. ∎

*Remark* V.6. At the beginning of Section III we have assumed that $(A, B, C)$ is regular. This will be convenient for our planned follow-up paper about the error feedback regulator problem. However, all our results in Sections III, IV and V except Propositions III.3 and IV.2 remain valid when we replace the assumption that $(A, B, C)$ is regular with the weaker assumption that $C$ is admissible for the semigroup generated by $A$, $B$ maps into $Z_{-1}$ and for some (hence, for every) $s \in \rho(A)$, the product $C_\Lambda(sI - A)^{-1}B$ exists. In Proposition III.3, the claim that the triple $(A_p, B_p, C_p)$ is regular must be replaced with the claim that $B_p \in \mathcal{L}(U, X_{-1})$, $C_p \in \mathcal{L}(\mathcal{D}(C_\Lambda) \times W, Y)$ is admissible for $\mathbb{T}^p$ and for some (hence, for every) $s \in \rho(A_p)$, the product $C^p_\Lambda(sI - A_p)^{-1}B_p$ exists. The claims regarding triples of operators in Proposition IV.2 (claims (1) and (4)) must be similarly modified.

## VI. Examples of state feedback regulation

We present three examples of state feedback regulation which illustrate our theory. In these examples the plant is an infinite-dimensional control system governed by either a parabolic or hyperbolic partial differential equation with the controls acting via the boundary conditions (in fact, we consider one- and two-dimensional heat equations on the domains $[0, 1]$ and $[0, 1] \times [0, 1]$, respectively, and a Rayleigh beam on $[0, \pi]$). In each example we solve the regulator equations to obtain the desired feedback control law and we illustrate its performance via numerical simulation.

*Example* VI.1. Consider a one-dimensional heat equation on the interval $[0, 1]$ with a Robin boundary control $u(t)$ at the right end point ($x = 1$) and a Neumann boundary disturbance $d(t)$ acting at the left end:

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t), \quad x \in (0,1), \quad z(x,0) = \varphi(x), \tag{6.1}$$

$$-\frac{\partial z}{\partial x}(0,t) = d(t), \quad \frac{\partial z}{\partial x}(1,t) + kz(1,t) = u(t), \tag{6.2}$$

where $\varphi(x) \in L^2(0,1)$ and $k > 0$ is a constant. Assume that the output $y(t)$ is obtained via point evaluation of the state $z(x,t)$ at a prescribed point $x_1 \in [0,1]$:

$$y(t) = Cz(t) = z(x_1, t)\,. \tag{6.3}$$

Our objective is to design a state feedback law which guarantees that the output (6.3) tracks a given sinusoidal reference $r(t) = M\sin(\omega t + \psi)$, where $M, \omega, \psi \in \mathbb{R}$, while rejecting a known disturbance $d(t) = c_1 + c_2 t$, where $c_1, c_2 \in \mathbb{R}$. For this, we construct the exosystem as in (3.2), with

$$S = \begin{bmatrix} 0 & \omega & 0 & 0 \\ -\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{array}{l} Q^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \\ C^1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}. \end{array} \tag{6.4}$$

It is clear that if we choose a suitable initial state $w(0)$, this exosystem will generate the given signals $r$ and $d$. We intend to apply Theorem IV.4 to find a feedback law of the form $u(t) = \Gamma w(t)$, where $\Gamma \in \mathbb{R}^4$ ensures that for some $\alpha < 0$, the error $e = y - r$ is in $L^2_\alpha([0, \infty); \mathbb{C})$ for every initial condition of the plant and the exosystem.

The system (6.1)–(6.3) can be reformulated in the abstract form (3.1) (see also [3] where a similar heat equation is considered in higher dimensions). Here we merely list the relevant spaces and operators, working with real-valued functions only:

1) $Z = L^2[0,1]$, $U = Y = \mathbb{R}$, $A = \mathrm{d}^2/\mathrm{d}x^2$ with $\mathcal{D}(A) = \{\varphi \in \mathcal{H}^2(0,1) \,|\, \varphi'(0) = \varphi'(1) + k\varphi(1) = 0\}$.
2) We have $A^* = A$, so that (as explained after (2.1)) we may regard $Z_{-1}$ as the dual of $Z_1$ with respect to the pivot space $Z$. In particular, the distributions $\delta_\xi$ (Dirac pulse at the point $\xi$) are in $Z_{-1}$ for any $\xi \in [0,1]$.
3) The operators $B, B^1 \in \mathcal{L}(U, Z_{-1})$ are $B = \delta_1$, $B^1 = \delta_0$.
4) The operator $C \in \mathcal{L}(Z_1, Y)$ is defined by $C\phi = \phi(x_1)$. It can be verified that $\mathcal{D}(C_\Lambda) \supset \mathcal{H}^1(0,1)$.

Thus we can replace the original plant (6.1)–(6.3) with the following system:

$$\dot{z} = Az + B^1 d + Bu, \quad y = C_\Lambda z\,. \tag{6.5}$$

The well-posedness and regularity of the system (6.5) can be established via trivial modifications to the results in [3]. It is easy to verify that $A$ is strictly negative and generates an exponentially stable analytic semigroup.

A straightforward calculation shows that for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$ and $x_1 \in [0,1]$, the transfer function for the system (6.5) is (see [5])

$$\mathbf{G}(s) = C_\Lambda (sI - A)^{-1} B = \frac{\cosh(x_1\sqrt{s})}{\sqrt{s}\sinh(\sqrt{s}) + k\cosh(\sqrt{s})}\,.$$

It is easy to see that for each $k > 0$, $\mathbf{G}$ is regular with feedthrough zero.

For the plant (6.5) driven by the exosystem determined in (6.4), we seek a control law in the form $u = \Gamma w$ that solves the regulator problem. Thus we seek mappings $\Pi \in \mathcal{L}(\mathbb{R}^4, L^2(0,1))$ and $\Gamma \in \mathcal{L}(\mathbb{R}^4, \mathbb{R})$ which satisfy the regulator equations

$$\Pi S w = A \Pi w + B \Gamma w + B^1 C^1 w\,, \tag{6.6}$$

$$0 = C_\Lambda \Pi w - Q^1 w\,, \tag{6.7}$$

for all $w \in \mathbb{R}^4$. Since $\mathbf{G}(0) \neq 0$, it follows from Theorem V.2 and Remark V.3 that such mappings exist if $\mathbf{G}(i\omega) \neq 0$. Note that these equations consist of a coupled system of one dimensional elliptic boundary value problems (6.6) subject to the algebraic constraint (6.7) and can be easily solved using elementary techniques. With the notation $\Gamma = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix}$ where $\gamma_j \in \mathbb{R}$, we obtain

$$\gamma_1 = \frac{\mathrm{Re}\,(\mathbf{G}(i\omega))}{|\mathbf{G}(i\omega)|^2}\,, \qquad \gamma_2 = \frac{\mathrm{Im}\,(\mathbf{G}(i\omega))}{|\mathbf{G}(i\omega)|^2}\,,$$

$$\gamma_3 = k x_1 - k - 1\,, \quad \gamma_4 = -\frac{k}{3} x_1^3 + \frac{k}{2} x_1 + x_1 - \frac{k}{6} - \frac{1}{2}\,.$$

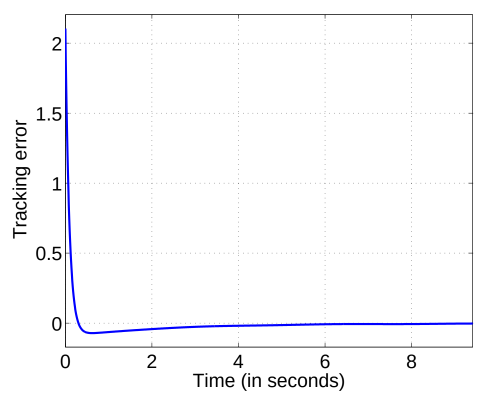


Figure 2. Plot of the tracking error in Example VI.1. The error tends to zero.

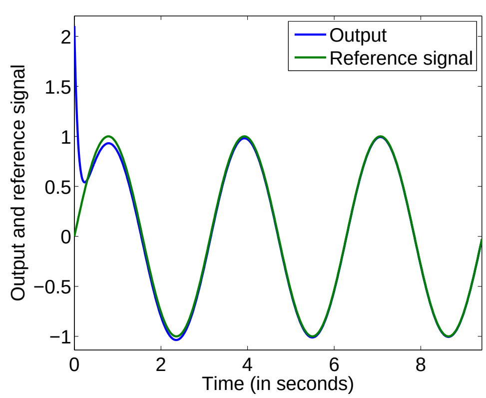


Figure 3. Plot of the reference signal $\sin(2t)$ and of the plant output.

For the numerical simulation we take the initial condition $\varphi(x) = 3\cos(\pi x)$, $k = 0.5$, disturbance signal $d(t) = 0.1(t+1)$, reference signal $r(t) = \sin(2t)$ and observation point $x_1 = 0.25$. The results are presented in Figures 2 and 3.

*Example* VI.2. We consider a set-point problem for the heat equation on the rectangular domain $\Omega = \{(x_1, x_2) : 0 < x_1 < 1,\ 0 < x_2 < 1\} \subset \mathbb{R}^2$. The plant has boundary controls $(u_1(t), u_2(t))$ imposed on two subsets of its boundary:

$$\frac{\partial z}{\partial t}(x,t) = \Delta z(x,t)\,, \qquad x = (x_1, x_2) \in \Omega\,, \tag{6.8}$$

$$\frac{\partial z}{\partial n}(x,t)\big|_{\partial\Omega_1} + 2k \int_0^{\frac12} z(0, x_2, t)\,\mathrm{d}x_2 = u_1(t)\,, \tag{6.9}$$

$$\frac{\partial z}{\partial n}(x,t)\big|_{\partial\Omega_2} + 2k \int_{\frac12}^1 z(1, x_2, t)\,\mathrm{d}x_2 = u_2(t)\,, \tag{6.10}$$

$$\frac{\partial z}{\partial n}(x,t)\big|_{\partial\Omega_0} = 0, \qquad z(x,0) = \varphi(x)\,, \tag{6.11}$$

where $n$ denotes the outward normal, $k > 0$ is a constant and

$$\partial\Omega_1 = \left\{ x = (x_1, x_2) \,\middle|\, x_1 = 0,\ 0 \le x_2 \le 1/2 \right\}\,,$$

$$\partial\Omega_2 = \left\{ x = (x_1, x_2) \,\middle|\, x_1 = 1,\ 1/2 \le x_2 \le 1 \right\}\,,$$

$$\partial\Omega_0 = \partial\Omega - \partial\Omega_1 - \partial\Omega_2\,.$$

The domain $\Omega$ and its boundary are shown in Figure 4. The plant outputs are

$$y_1(t) = 2\int_0^{\frac12} z(0, x_2, t)\,\mathrm{d}x_2, \quad y_2(t) = 2\int_{\frac12}^1 z(1, x_2, t)\,\mathrm{d}x_2\,.$$

Let $u = \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right]$ and $y = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$. Then the plant (6.8)–(6.11) can be rewritten in the abstract form (2.5), with state space $Z = L^2(\Omega)$. Here $A = \Delta$ with

$$\mathcal{D}(A) = \left\{ \varphi \in \mathcal{H}^2(\Omega) \;\middle|\; \frac{\partial\varphi}{\partial n}(x,t)\big|_{\partial\Omega_0} = 0, \right.$$

$$\frac{\partial\varphi}{\partial n}(x,t)\big|_{\partial\Omega_1} + 2k \int_0^{\frac12} \varphi(0, x_2, t)\,\mathrm{d}x_2 = 0,$$

$$\left. \frac{\partial\varphi}{\partial n}(x,t)\big|_{\partial\Omega_2} + 2k \int_{\frac12}^1 \varphi(1, x_2, t)\,\mathrm{d}x_2 = 0 \right\}.$$

For each $\phi \in \mathcal{D}(A)$, we have

$$\langle A\phi, \phi\rangle_Z = -\int_\Omega \left[ \left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi}{\partial x_2}\right)^2 \right] \mathrm{d}x$$

$$-2k \left( \int_0^{\frac12} \phi(0, x_2)\,\mathrm{d}x_2 \right)^2 - 2k \left( \int_{\frac12}^1 \phi(1, x_2)\,\mathrm{d}x_2 \right)^2,$$

which implies that for some small $\delta > 0$, $A + \delta I$ is dissipative. Hence the semigroup associated with $A$ is exponentially stable.

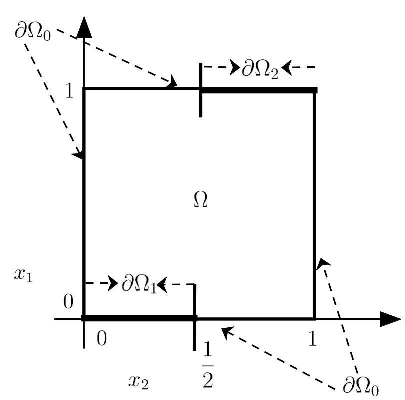


Figure 4. The rectangular domain $\Omega$. The part of the boundary $\partial\Omega_0$ is insulated. The control signal is applied on the rest of the boundary.

The control operator is defined by

$$Bu = B_1 u_1 + B_2 u_2 = \delta_{\partial\Omega_1} u_1 + \delta_{\partial\Omega_2} u_2 \in \widetilde{\mathcal{H}^{-2}}(\Omega) \subset Z_{-1}.$$

For the definition of the space $\widetilde{\mathcal{H}^{-2}}(\Omega)$, and further details, see [3]. The definition of the operator $C$ follows directly. The well-posedness and regularity of the above system can be established as in Byrnes *et al* [3].

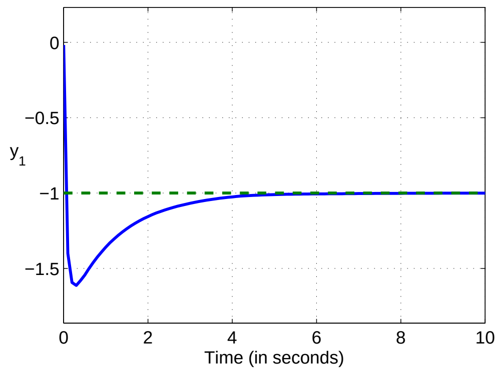


Figure 5. The constant reference signal of magnitude $-1$ and the plant output $y_1$ (average temperature on $\partial\Omega_1$).

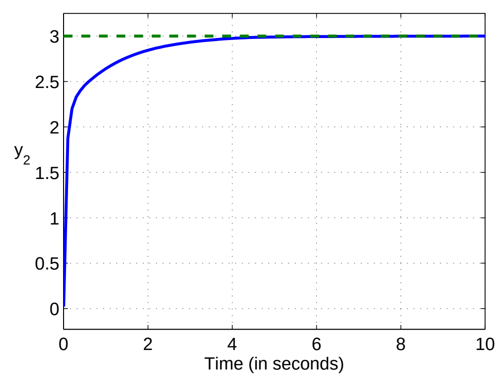


Figure 6. The constant reference signal of magnitude 3 and the plant output $y_2$ (average temperature on $\partial\Omega_2$).

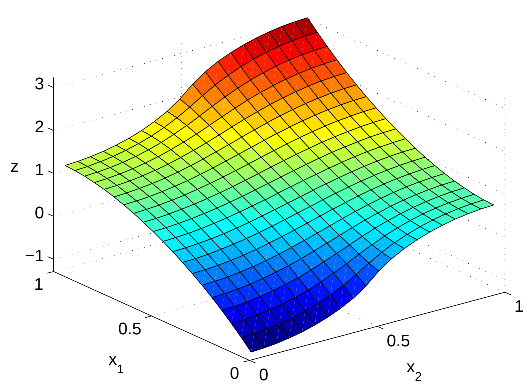


Figure 7. The temperature profile on the square domain $\Omega$ at time $t = 5$.

Our objective in this example is to drive the average temperature on $\partial\Omega_j$ to a given constant $\mathcal{M}_j$ for $j = 1, 2$, i.e., $y_j(t) - \mathcal{M}_j \in L^2_\alpha([0,\infty);\mathbb{R})$ for some $\alpha < 0$. Therefore, in the exosystem (3.2), we choose

$$S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

According to our theory, we have to solve the regulator equations

$$0 = \Pi S = A\Pi + B\Gamma, \quad C\Pi - Q^1 = 0. \tag{6.12}$$

We denote by $\mathbf{G}$ the transfer function of the plant. It can be checked that $\mathbf{G}(0)$ has the following structure: $\mathbf{G}(0) = \left[\begin{smallmatrix} g & h \\ h & g \end{smallmatrix}\right]$, where $g > h > 0$, so that $\mathbf{G}(0)$ is onto. According to Theorem V.2 we can solve (6.12) and

$$\Gamma = \mathbf{G}(0)^{-1} Q^1 = -(C_\Lambda A^{-1} B)^{-1}.$$

In this problem we approximate $\Gamma$ numerically by solving the elliptic boundary value problem defined by the first equation in (6.12) under the constraint imposed by the second equation. For the simulations we set $k = 1$, $\mathcal{M}_1 = -1$, $\mathcal{M}_2 = 3$ and $\varphi(x) = 0$. The numerical results are shown in Figures 5, 6 and 7.

*Example* VI.3. We consider a harmonic tracking problem for a damped Rayleigh beam, in the presence of structural damping (see [25]). We denote the transverse displacement of the beam at the position $x \in [0,\pi]$ and the time $t \geq 0$ by $q(x,t)$. The beam equation, influenced by a boundary control $u(t)$, is:

$$\frac{\partial^2 q(x,t)}{\partial t^2} - \alpha \frac{\partial^4 q(x,t)}{\partial x^2 \partial t^2} - a \frac{\partial^3 q(x,t)}{\partial x^2 \partial t} + \frac{\partial^4 q(x,t)}{\partial x^4} = 0, \tag{6.13}$$

$$q(0,t) = q(\pi,t) = \frac{\partial^2 q}{\partial x^2}(\pi,t) = 0, \quad -\frac{\partial^2 q}{\partial x^2}(0,t) = u(t), \tag{6.14}$$

$$y(t) = \frac{\partial^2 q}{\partial x \partial t}(0,t). \tag{6.15}$$

Here $\alpha > 0$ is proportional to the moment of inertia of the cross section of the beam and $a > 0$ is the damping coefficient. This equation models a single-input-single-output boundary control system with $u$ being the torque applied at $x = 0$ and the output $y$ being the angular velocity at the same point.

We now briefly discuss the state space formulation for the Rayleigh beam and refer to Weiss and Curtain [51] for more details. Let $H = \mathcal{H}^1_0(0,\pi)$ and $V = \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_0(0,\pi)$. Define the inner product on $H$ such that

$$\langle \varphi, \psi \rangle_H = \left\langle \left( I - \alpha \frac{\mathrm{d}^2}{\mathrm{d}x^2} \right) \varphi, \psi \right\rangle_{L^2(0,\pi)} \qquad \forall\, \varphi, \psi \in V.$$

Consider the operator $\mathcal{R} : L^2[0,\pi] \to V$ defined as

$$\mathcal{R} = \left( I - \alpha \frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{-1}.$$

As a bounded operator on $L^2[0,\pi]$, $\mathcal{R}$ is strictly positive and it leaves both $H$ and $V$ invariant. We define the operator $A_0 : \mathcal{D}(A_0) \to H$ by

$$\mathcal{D}(A_0) = \left\{ \varphi \in \mathcal{H}^3(0,\pi) \,\middle|\, \varphi(0) = \varphi(\pi) = 0, \right. \\ \left. \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(0) = \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(\pi) = 0 \right\},$$

$$A_0\varphi = \frac{\mathrm{d}^4}{\mathrm{d}x^4}(\mathcal{R}\varphi) \qquad \forall\, \varphi \in \mathcal{D}(A_0)\,.$$

The operator $A_0$ is strictly positive, self-adjoint and commutes with $\mathcal{R}$. We will use the following notation: $H_1 = \mathcal{D}(A_0)$, $H_{\frac{1}{2}} = V$, $H_{-\frac{1}{2}} = L^2[0,\pi]$ and $H_{-1} = \mathcal{H}^{-1}(0,\pi)$. $A_0$ can be extended to a bounded operator from $H_{\frac{1}{2}}$ to $H_{-\frac{1}{2}}$ that commutes with $\mathcal{R}$ (hence, also with $\mathcal{R}^{-1}$).

We now rewrite (6.13)–(6.14) as a boundary control system as in Remark III.5. We consider the transverse displacement $q$ and the velocity $\dot{q}$ to be the state variables. Let $Z = H_{\frac{1}{2}} \times H$ and $U = \mathbb{C}$ be the state space and the input space. It is now easy to see that the following operator $A : \mathcal{D}(A) \to Z$ is m-dissipative, hence a generator:

$$\mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}\,, \quad A = \begin{bmatrix} 0 & I \\ -A_0 & -A_1 \end{bmatrix},$$

where $A_1 = -a\mathcal{R}\frac{\mathrm{d}^2}{\mathrm{d}x^2} \in \mathcal{L}(V)$. Note that $A_1 \geq 0$ on $V$ and on $H$. Define $\tilde{A}_0$, the obvious extension of $A_0$ to

$$\mathcal{D}(\tilde{A}_0) = \left\{ \varphi \in \mathcal{H}^3(0,\pi) \,\middle|\, \varphi(0) = \varphi(\pi) = 0,\ \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(\pi) = 0 \right\},$$

and we define $\tilde{A} : \underset{\sim}{Z} \to Z$, an extension of $A$, and $G : \underset{\sim}{Z} \to U$ by

$$\underset{\sim}{Z} = \mathcal{D}(\tilde{A}_0) \times V,\ \ \tilde{A} = \begin{bmatrix} 0 & I \\ -\tilde{A}_0 & -A_1 \end{bmatrix},\ \ G \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = -\frac{\mathrm{d}^2 z_1}{\mathrm{d}x^2}(0).$$

Then the equations (6.13)–(6.14) can be written exactly as in (3.18), with $\mathcal{B} = I$ and $\mathcal{B}^1 = 0$. Notice that the restriction of $\tilde{A}$ to $\operatorname{Ker} G$ is $A$, as required. Moreover, $\underset{\sim}{Z}$ defined above (as in Remark III.5) coincides with $\underset{\sim}{Z}$ from Remark III.1.

Define the operator $C_0 \in \mathcal{L}(H_{\frac{1}{2}}, \mathbb{C})$ by $C_0\varphi = \frac{\mathrm{d}\varphi}{\mathrm{d}x}(0)$ and the observation operator $C : \mathcal{D}(A) \to \mathbb{C}$ by $C = \begin{bmatrix} 0 & C_0 \end{bmatrix}$, which corresponds to the output equation (6.15). When $a = 0$ (no structural damping), it is established in [51] that the above boundary control system with the observation operator $C$ is regular. Since $A_1$ is a bounded operator, it follows from [49], that the same is true when $a \neq 0$. In order to see that $A$ is exponentially stable one can, with straightforward modifications, apply Proposition 3.14 and Theorem 3.18 in [16] (in their notation, set $A = -A_0$ and $B = -A_1$). We mention that the control operator is $B = C^*$.

We denote the two components of the state $z$ by $z_1$ (displacement) and $z_2$ (velocity). We want to design a control $u$ such that the output $y$ in (6.15) tracks a prescribed sinusoidal trajectory $r(t) = M\sin(\omega t + \psi)$ of known frequency $\omega > 0$, amplitude $M$, and phase $\psi$, i.e., the error $e = y - r$ is in $L^2_\delta[0,\infty)$ for some $\delta < 0$. For any $M$ and $\psi$, the signal $r$ can be generated by the exosystem in (3.2) with

$$S = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \qquad Q^1 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Using the notation $\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix}$ and $\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix}$, the first regulator equation (4.4) rewritten in the equivalent form (4.9) becomes

$$-\omega^2\Pi_1 + \tilde{A}_0\Pi_1 - \omega A_1\Pi_2 = 0\,, \tag{6.16}$$

$$-\omega^2\Pi_2 + \tilde{A}_0\Pi_2 + \omega A_1\Pi_1 = 0\,, \tag{6.17}$$

$$\omega\Pi_1 = \Pi_4\,, \qquad -\omega\Pi_2 = \Pi_3\,, \tag{6.18}$$

$$-\frac{\mathrm{d}^2\Pi_1}{\mathrm{d}x^2}(0) = \Gamma_1\,, \qquad -\frac{\mathrm{d}^2\Pi_2}{\mathrm{d}x^2}(0) = \Gamma_2\,. \tag{6.19}$$

From (6.16) and (6.17), using the definitions of $\tilde{A}_0$, $A_1$ and $\mathcal{R}$, we get that

$$\Pi_1'''' + \alpha\omega^2\Pi_1'' + a\omega\Pi_2'' - \omega^2\Pi_1 = 0\,, \tag{6.20}$$

$$\Pi_2'''' + \alpha\omega^2\Pi_2'' - a\omega\Pi_1'' - \omega^2\Pi_2 = 0\,. \tag{6.21}$$

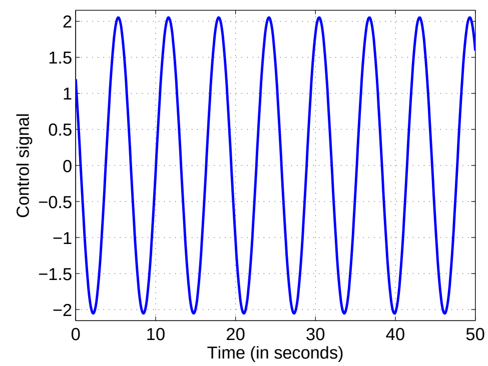


Figure 8. The control signal $u = \Gamma w$ for the Rayleigh beam in Example VI.3.

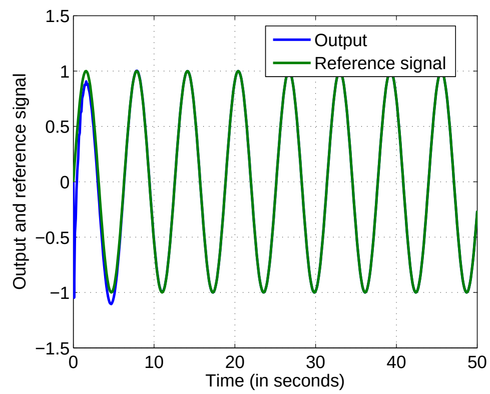


Figure 9. The sinusoidal reference signal and the plant output.

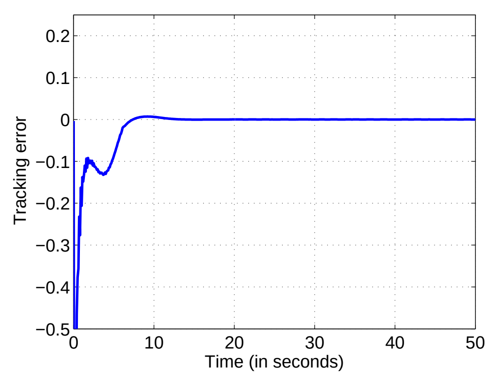


Figure 10. The tracking error in Example VI.3. This error tends to zero.

Since $\operatorname{Ran}\Pi \subset \underset{\sim}{Z}$ (see Lemma III.4), we have $\Pi_1, \Pi_2 \in \mathcal{D}(\tilde{A}_0)$ and therefore

$$\Pi_1(0) = \Pi_2(0) = \Pi_1(\pi) = \Pi_2(\pi) = 0\,, \tag{6.22}$$

$$\Pi_1''(\pi) = \Pi_2''(\pi) = 0\,. \tag{6.23}$$

The second regulator equation (4.5) and (6.18) give

$$\Pi_1'(0) = 0, \quad \Pi_2'(0) = -1/\omega. \tag{6.24}$$

The ordinary differential equations in (6.20), (6.21), along with the boundary conditions (6.22), (6.23) and (6.24), are first solved for $\Pi_1$ and $\Pi_2$. The functions $\Pi_3$, $\Pi_4$, $\Gamma_1$ and $\Gamma_2$ can then be computed from (6.18) and (6.19).

For our numerical simulations we choose $\alpha = 1$, $a = 2$, $M = 1$ and $\omega = 1$. Hence the signal to be tracked is $r(t) = \sin(t)$. We set the initial conditions to be $z_1(x, 0) = 0$, $z_2(x, 0) = 0$. The system (6.19)–(6.24) was solved using the finite element package COMSOL [11] on the time interval $0 < t < 50$. Our choice of using COMSOL is motivated by its flexibility for solving coupled multi-physics problems. The simulation results are presented in Figures 8, 9, 10 and 11.

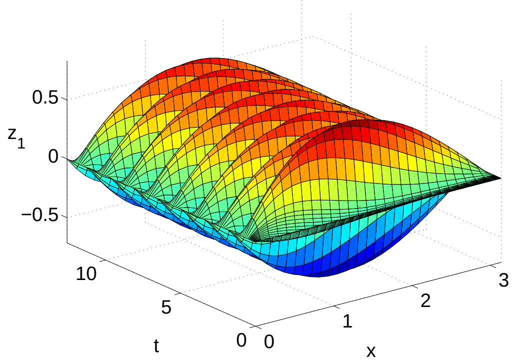


Figure 11. Displacement profile on the interval $[0, \pi]$ as a function of time.

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