Bucket Brigade with Stochastic Worker Pace

Yossi Bukchin* Eran Hanany* Eugene Khmelnitsky*

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Abstract

We study Bucket Brigade (BB) production systems under the assumption of stochastic worker speeds. Our analysis provides interesting and counter-intuitive insights into realistic production environments. We analyze the following three systems: the traditional BB of Bartholdi & Eisenstein (1996), BB with overtaking allowed (BBO), and a benchmark system of parallel workers. After formulating the dynamic equations for all systems, we solve them analytically when possible and numerically in general. We identify settings in which conclusions that emerge from deterministic analysis fail to hold when speeds are stochastic, in particular relating to worker order assignment. Specifically, a fastest to slowest order with respect to expected speeds may be optimal as long as the standard deviation of the fastest worker is large enough. Significantly, in a stochastic environment the BB can improve the throughput rate compared to parallel workers, despite the fact that no blockage or starvation may occur in the latter. The BBO setting, which is relevant in a stochastic environment, and can be sometimes implemented in practice, provides an upper bound on the throughput rate of parallel workers, and is shown numerically to significantly improve upon BB.

Key words: Work sharing, line balancing, stochastic process.

1 Introduction

Mass production environments have gone through major changes due to the implementation of work-sharing. Traditionally, such systems were based on division of work among multiple workers, each trained to repeatedly perform a small segment of the work. Education and training improvements have provided incentives for workers to handle more complex requirements, thus providing the potential for enhancing system performance. Modern environments have applied work-sharing, whereby multiple cross-trained workers are able to perform the same task. The implementation of work-sharing has improved the ability to achieve balanced lines (Anuar and Bukchin 2006, Bukchin and Sofer 2011), leading to fewer blockages and starvation in the system, and as a result, to a higher throughput rate. Ostolaza et al. (1990) were the first to coin the term Dynamic Line Balancing (DLB). This term refers to the operational side of work-sharing as “allowing tasks to be assigned

*Department of Industrial Engineering, Tel Aviv University. E-mail: bukchin@tau.ac.il, hananye@tau.ac.il, xmel@tau.ac.il
on the fly based on the current state of the system”. According to DLB, some shared tasks can be shifted between adjacent stations/workers in order to better balance the flow along the line given prespecified control rules.

When a serial system is concerned, possibly the most common work-sharing approach is the Bucket-Brigade (BB) proposed by Bartholdi and Eisenstein (1996).¹ This approach initially assumed full cross-training of the workers, and suggested conditions ensuring a self-balancing line. In the basic model, they also assumed that a task can be handed from one worker to another at any point, and that the system is deterministic in all respects. The basic rule of BB is that whenever the last downstream worker completes an item, the worker returns and takes over the item of the next upstream worker; the preempted worker continues similarly with the next upstream worker, where this procedure continues until the first worker is preempted and starts processing a new item. This procedure prevents starvation, however the authors show that when the workers are located from slowest-to-fastest, the line does not suffer from blockage as well, thus achieving the maximal theoretical throughput rate. Moreover, they show that in steady-state the ‘work taking over’ between any two adjacent workers always occurs at the same point, meaning that each worker always executes exactly the same work segment on each item. Further analysis of the BB system dynamics for two and three workers was presented in Bartholdi et al. (1999), while the chaotic behavior of the hand-off point when the convergence condition does not hold was studied in Bartholdi et al. (2009).

In this paper we analyze the BB model under stochastic worker speeds, a realistic assumption when considering human workers. Workers may vary in their expected working speeds, whereas actual speeds may vary across cycles due to the specific worker-item interaction. We also propose an extension for this model, allowing workers to pass each other. We use a dynamic Markov process to evaluate the throughput rate (TR) of the system and of each worker, and compare these values to the deterministic model and to a model in which each worker completes independently the whole work (parallel workers). Interestingly, in a stochastic environment the BB can significantly improve the TR compared to parallel workers, despite the fact that no blockage or starvation occurs in the latter. We analyze the limit cases of no blockage and full blockage, but also consider the general case of partial blockage, while providing analytical results when possible along with numerical results. In particular we show that in the limit case where one worker dominates the other, the TR of BB can be up to twice the TR of parallel workers. For the general case we demonstrate the effect of the worker speed parameters on the TR, and show that in some cases the fastest-to-slowest assignment in terms of expected speeds provides higher TR than the reverse order. We also analyze a system with overtaking allowed (see Bartholdi et al. 2009, 2010). This setting (named BBO) is relevant in a stochastic environment, and can be sometimes implemented in practice. Note that we assume that when overtaking occurs, each worker continues working on the same item. Clearly, when one worker dominates the other (no blockage), the BBO system performs the same as BB with workers sequenced from slowest to fastest. In a partial blockage situation, BBO outperforms

¹A variety of real-world uses of Bucket Brigade can be found in http://www.bucketbrigades.com/
parallel workers, and is shown numerically to significantly improve upon BB.

In the remainder of the introduction we review some of the relevant literature. The fundamental decision underlying work-sharing is the level of cross-training: no cross-training exists when there is no overlapping among the capabilities of the workers, whereas full cross-training holds when all workers are capable of performing the whole work. This issue is investigated in Hopp et al. (2004), in order to achieve a low ratio of work-in-process (WIP) to throughput. Besides the level of cross-training, they determine the line strategy, while studying D-Skilled Chaining (DSC), proposed by Jordan and Graves (1995), and Cherry Picking (CP). In CP, only the bottleneck worker is assisted by the other workers, while the cross-training under DSC is symmetric among the workers, as each worker can help the adjacent upstream/downstream workers. A special case of DSC worth mentioning is the 2-skill chaining, which was found as a robust and highly effective strategy. Other variations of work-sharing models have been addressed in the literature, such as preemptions, namely the ability to split a specific task between workers (McClain et al. 2000), line operations (Gel et al. 2007, Anuar and Bukchin 2006), and the analysis when machines are involved (Zavadlav et al. 1996).

Many extensions have been suggested to the basic BB (Bratcu and Dolgui 2005), mostly in the deterministic domain, and some are mentioned herein. Armbruster and Gel (2006) relax the uniform dominance assumption of the basic model, and study a 2-worker line in which one worker may be faster than his neighbor in some part of the line, and slower in the other part. They provide insights and operation principles for various scenarios. Gurumoorthy et al. (2009) analyze the system’s dynamics of a 2-worker line with arbitrary worker speed, however constant at each station. Another extension refers to the effect of discrete stations on the system’s performance. The results of the basic continuous model continue to hold when the number of stations is much larger than the number of workers, however this is not necessarily the case for a small number of stations. This issue is considered in Lim and Yang (2009), who find conditions under which 2-worker and 3-station lines maximize throughput when the work-content distribution over the stations is given. In such systems, each station can accommodate at most one worker, consequently blocking may occur even when workers are sequenced from slowest-to-fastest and preemption is allowed. Bartholdi et al. (2006) applied the BB principles in an in-tree network of sub-assembly lines. The convergence of BB in a tree-shaped picking system with two trucks was analyzed by Xu et al. (2014). Bratcu and Dolgui (2009) analyzed the basic model while relaxing the assumption of infinite speed for worker walk-back. A BB line with a finite walk-back speed was first rigorously analyzed by Bartholdi et al. (2009). The unproductive walk-back travel was then addressed by Lim (2011), who suggested a new design of a cellular configuration in which the work content is distributed on both sides of a central aisle, also analyzed with a case study in Lim (2012) and further developed in Lim (2017) to incorporate positive hand-off times. Lim and Wu (2013) analyzed the system dynamics for a U-line with discrete stations. McClain et al. (2000) simulated machine lines under various assumptions relating to processing times, machine-to-worker ratios, preemptions and levels of cross-training. The worker learning curve in BB was studied in Armbruster et al. (2007)
and Montano et al. (2007), where the latter suggested an alternative control rule, named modified-work-sharing (MWS). Some case studies of implementing BB in industrial environments have been presented by Bartholdi and Eisenstein (2005), Lim (2005) and De Carlo et al. (2013).

Bartholdi et al. (2001) relaxed the deterministic work content assumption, and examined the effectiveness of BB assuming stochastic work times under exponential distribution. They show analytically that, as the number of stations increases, the model converges to the deterministic case, with the optimal throughput obtained when workers are assigned from slowest-to-fastest. Hong (2014) also considered stochastic work content and suggested a closed-form expression for the 2-worker blocking congestion in a circular-passage system, for constant workers’ speed and stochastic pattern of picks. He showed via simulation that the analytical expression provides a good approximation for the blockage probability of a BB picking system, when the worker speed and hand-off time are constant and the picking pattern is stochastic. Since the stochastic effects vanish in the limit as the number of stations increases to infinity, the models proposed in both these papers are less suitable for the modeling of stochastic behavior in continuous lines. Bratcu and Dolgui (2009) studied stochastic speeds via simulations, while assuming that both working and walk back speeds are normally distributed. Our modeling approach provides theoretical results for a two-worker system focusing on the comparison between parallel workers and BB with and without overtaking.

The rest of this paper is organized as follows. A formulation of the dynamic BB and BBO lines with stochastic worker speeds is given in Section 2, concentrating on a two worker system. The case of dominance, in which the speed of one worker is almost surely higher or equal to the other, is discussed in Section 3. A generalization of the dominance properties for the case of multiple workers is given in Section 3.3. In Section 4 we analyze the BB and BBO lines for two workers and general speed distributions. The final section concludes with managerial implications of our findings.

2 Model

We study a stochastic production process controlled by a decentralized dynamic protocol, which allows coordinating the efforts of several workers along the line. Each worker moves down the line with an item until the item is completed or preempted by a downstream worker. Two system types are analyzed: the first is the traditional Bucket Brigade (BB) and the second is a modification of such a system which allows overtaking downstream workers. In both systems the most downstream worker, upon completion of an item at the end of the line, returns to take the item from the immediate upstream worker (a hand-off event), who does the same, until the first worker takes a new item from the start of the line. We assume that the time required to return upstream is insignificant compared with the time required to work downstream. When overtaking downstream workers is not allowed, each worker either proceeds at their own pace or at a reduced speed when blocked by a downstream worker. As a result, the order of the workers along the line does not change.
over time. When overtaking is allowed, the order of the workers may change. It follows that the issue of overtaking is mostly irrelevant under our assumption of negligible walk-back time when the process evolves deterministically, but becomes important in the case of a stochastic environment.

When modeling the stochastic behavior of a worker in production systems, one may imagine a continuously changing pace for any worker. Variation and stochasticity can result from instability of human workers with respect to work rate, skill and motivation, as well as sensitivity to failure of complex processes, learning effects and fatigue (see for example Becker and Scholl 2006). We ignore learning effects and other causes of correlation among the workers. On top of the inherent variation in the worker pace, we assume that the pace is also affected by the interaction with a specific item. Whereas different workers may have different expected working speeds, thus representing identifiable between-workers variability, the actual speed in each cycle is affected by within-worker unexplained variability stemming from the worker-item interaction. We assume that items are a-priori homogeneous, namely, each item requires performing the same total amount of work, uniformly distributed along the line. However, since the work is performed by different workers, the worker-item interaction generates variability in actual speeds. Hence, as is common in the modeling of semi-Markov processes, we isolate the discrete hand-off events, i.e. when each worker starts processing a new item. At each such point, a new speed is independently generated for the worker from a given and known distribution and the worker proceeds with this speed until switching to a different item (see also Bratcu and Dolgui, 2009).

As an example for such an environment, consider pickers in a fast picking aisle of a textile warehouse, where each order includes some clothing mix, e.g., shirts, pants, socks, belts, etc., each of which is distributed uniformly along the picking aisle. Assume that a worker may be fast in picking and folding some orders, and may be slower in picking and folding other orders, but there is no good way to predict the speed (namely, in a statistical regression, the clothing mix or any other practically measurable variables are found to be insignificant as explanatory variables of the speed). Hence, the speed generated in each cycle, when starting the work on a new order, captures the consequences of the interaction between the worker and the new order, modeled as the unexplained component of the speed variability. The formal model is given next.

2.1 Bucket Brigade dynamics

This section considers a two-worker system and describes its dynamics under both BB and BBO rules, i.e. when overtaking is forbidden or allowed, respectively. As explained above, hand-off \( n = 0, 1, 2, \ldots, \) refers to the event by which the downstream worker has reached the end of the line completing an item, and returns to receive the item from the upstream worker. A state of the system, \( x_n \), defines the position along the line of hand-off \( n \), i.e., the position of the upstream worker when the downstream worker has reached the end of the line. Without loss of generality, we assume that the length of the line is 1, so that \( 0 \leq x_n \leq 1 \). At each hand-off \( n \), the worker speeds, \( v_{i,n} \), \( i = 1, 2 \), are randomly and independently generated according to known, stationary cumulative distribution functions (cdf) \( F_i(\xi) \), \( \xi \in [0, \infty) \). Thus, \( v_{i,n} = E[v_{i,n}] + \xi_{i,n} \), where \( \xi_{i,n} \) is
the unexplained noisy component of the worker’s speed stemming from the interaction of worker \( i \) with the item in cycle \( n \).

With no overtaking allowed, i.e. the BB rule, the order of the workers, say \( 1 \rightarrow 2 \), is constant and worker 1 can be blocked by worker 2. In such a case, the two workers reach the end of the line concurrently, and the hand-off takes place at the position \( x_n = 1 \). Following this rule, the next hand-off takes place immediately and with certainty at the position \( x_{n+1} = 0 \). Since worker 2 cannot be blocked, and always reaches the end of the line, the cycle time, i.e. the time elapsed between hand-off \( n - 1 \) and hand-off \( n \), is always

\[
CT_n = \frac{1 - x_{n-1}}{v_{2,n}}. \tag{2.1}
\]

Thus, the position of hand-off \( n \) is calculated recursively by,

\[
x_n = \min \left\{ 1, \frac{v_{1,n}}{v_{2,n}} (1 - x_{n-1}) \right\}, \text{ where } x_0 \text{ is given.} \tag{2.2}
\]

Analyzing this stochastic system as a Markov chain, we are interested in determining the steady-state distribution of hand-off position along the line (i.e., the limit distribution of \( x_n \) as \( n \rightarrow \infty \)), the expected cycle time, and the throughput rate.

When overtaking is allowed, i.e. the BBO rule, a hand-off occurs each time the downstream worker reaches the end of the line. Note that due to the possibility of overtaking, the order of the workers changes each time overtaking occurs. A state of the system is defined by the hand-off position, \( x_n \), and by the upstream worker in cycle \( n \), denoted by \( r(n) \in \{1, 2\} \). The dynamics of the two state variables, \( x_n \) and \( r(n) \), satisfy

\[
(x_n, r(n)) = \begin{cases} 
(x_{n-1} + \frac{v_{r(n-1),n}}{v_{r(n-1),n}} (3 - r(n-1)), n) & \text{if } x_{n-1} < 1 - \frac{v_{r(n-1),n}}{v_{r(n-1),n}}, \\
(\frac{v_{r(n-1),n}}{v_{r(n-1),n}} (1 - x_{n-1}), r(n-1)) & \text{if } x_{n-1} \geq 1 - \frac{v_{r(n-1),n}}{v_{r(n-1),n}}, 
\end{cases} \tag{2.3}
\]

where \( x_0 \) and \( r(0) \) are given, and \( 3 - r(n-1) \) is the worker other than \( r(n-1) \) during cycle \( n - 1 \). The cycle time is determined by the worker who first reaches the end of the line, i.e. equals

\[
CT_n = \min \left\{ \frac{1 - x_{n-1}}{v_{(3-r(n-1)),n}}, \frac{1}{v_{r(n-1),n}} \right\}. \tag{2.4}
\]

2.2 Throughput rate

Let us define the throughput rate, \( TR \), of a production process as the total amount of work carried out by the process, \( W(T) \), divided by the total production time, \( T \),

\[
TR \equiv \lim_{T \to \infty} \frac{W(T)}{T}. \tag{2.5}
\]

6
In case the process produces distinct, identical items, the amount of work is measured in the number of produced items, \( N(T) \), i.e.,

\[
W(T) = N(T).
\] (2.6)

In such a case, the cycle time of cycle \( n \), \( CT_n \), is defined as the time elapsed between the deliveries of subsequent items. We use \( x \) and \( r \) and \( CT \) as the steady-state counterparts of \( x_n \), \( r(n) \) and \( CT_n \), respectively, and use \( v_i \) instead of \( v_{i,n} \). In steady-state, the expected cycle time is equal to

\[
E[CT] = \lim_{T \to \infty} \frac{T}{N(T)}.
\] (2.7)

By combining (2.5), (2.6) and (2.7), when \( 0 < E[CT] < \infty \), one obtains that the TR is the inverse of the expected cycle time,

\[
TR = \frac{1}{E[CT]}.
\] (2.8)

Given (2.1), the expected cycle time of the line is

\[
E[CT] = E\left[\frac{1-x}{v_2}\right] = (1 - E[x]) E\left[\frac{1}{v_2}\right],
\] (2.9)

where the second equality follows because, for each cycle, \( v_2 \) is drawn independently from the hand-off position, \( x \). The TR of the BB line when the workers are assigned in the order \( 1 \to 2 \), is denoted by \( TR^{1\to2} \). By (2.8) and (2.9),

\[
TR^{1\to2} = (1 - E[x])^{-1} E\left[\frac{1}{v_2}\right]^{-1}.
\] (2.10)

We now compare the TR of a BB line to the TR of parallel workers. Worker \( i \), proceeding along the entire line with speed \( v_i \), completes an item over the cycle time of \( CT_{i,n} = \frac{1}{v_{i,n}} \) for each cycle \( n \). By (2.8), the TR equals

\[
TR_i = E\left[\frac{1}{v_i}\right]^{-1}.
\] (2.11)

We call this the single worker TR.

The TR of a BB line, as shown in (2.10), depends on \( TR_2 \), i.e., \( E\left[\frac{1}{v_2}\right]^{-1} \), and on the expected hand-off position, \( E[x] \). Thus \( TR^{1\to2} \) combines the efforts of the two workers. Define \( TR^{1\to2}_i \) for \( i = 1, 2 \) as the TR of worker \( i \) when working in a BB line consisting of workers 1 and 2 in this order. The next proposition shows that the TR of worker 1 in a BB line is bounded from above by this worker’s expected speed, and the TR of worker 2 equals the corresponding single worker TR, i.e., does not depend on the distributions of \( v_1 \) and \( x \).

**Proposition 2.1.** In a BB line,

\[
TR^{1\to2}_1 \leq E[v_1]
\] (2.12)
and
\[ TR_2^{1\to2} = TR_2. \]

**Proof.** Denote by \( v_{1,n}^{av} \) the actual average speed of worker 1 in cycle \( n \). When no blockage occurs in cycle \( n \), \( v_{1,n}^{av} \) is equal to \( v_{1,n} \), worker 1’s speed drawn at the beginning of cycle \( n \), otherwise it is strictly lower than \( v_{1,n} \). Thus, from (2.5),

\[
TR_1^{1\to2} = \lim_{T \to \infty} \frac{\sum_{n=1}^{N(T)} CT_n \cdot v_{1,n}^{av}}{\sum_{n=1}^{N(T)} CT_n} \leq \lim_{T \to \infty} \frac{1}{N(T)} \sum_{n=1}^{N(T)} CT_n v_{1,n} = E[CT v_1] = E[v_1],
\]

where the last equality holds since, for each cycle, \( CT \) and \( v_1 \) are independent.

The amount of work performed by worker 2 in cycle \( n \) is \( CT_n v_{2,n} = 1 - x_n \). Therefore the throughput associated with worker 2 is

\[
TR_2^{1\to2} = \lim_{T \to \infty} \frac{\sum_{n=1}^{N(T)} CT_n v_{2,n}}{\sum_{n=1}^{N(T)} CT_n} = \lim_{T \to \infty} \frac{1}{N(T)} \sum_{n=1}^{N(T)} CT_n v_{2,n} = E[1-x] = E\left[\frac{1}{v_2}\right]^{-1} = TR_2.
\]

Contrary to worker 2, the TR of worker 1 depends on all parameters of the BB line. It is obtained by subtracting \( TR_2^{1\to2} \) from the TR of the line,

\[
TR_1^{1\to2} = TR^{1\to2} - TR_2^{1\to2} = E\left[\frac{1}{v_2}\right]^{-1} \frac{1}{1 - E[x]} - E\left[\frac{1}{v_2}\right]^{-1} = E\left[\frac{1}{v_2}\right]^{-1} \frac{E[x]}{1 - E[x]}, \tag{2.13}
\]

Note that \( TR_1^{1\to2} \) in (2.10) and \( TR_1^{1\to2} \) in (2.13) are fully determined by \( E\left[\frac{1}{v_2}\right]^{-1} \) and \( E[x] \), and do not depend on higher moments of the distribution function of the hand-off position. The higher is \( E[x] \), the greater are \( TR_1^{1\to2} \) and \( TR_1^{1\to2} \). Equation (2.13) is equivalently re-written as

\[
\frac{TR_1^{1\to2}}{TR_2^{1\to2}} = \frac{E[x]}{1 - E[x]}. \tag{2.14}
\]

This result is similar to the deterministic no blockage BB, in which the line segments covered by the workers are proportional to their speeds.

When the line is managed by the BBO rule, the throughput of each worker, \( TR_i^{1\leftrightarrow2} \), is bounded from above by worker’s expected speed and from below by the single worker TR, as proven in the following

**Proposition 2.2.** In a BBO line,

\[
E\left[\frac{1}{v_i}\right]^{-1} \leq TR_i^{1\leftrightarrow2} \leq E[v_i].
\]

**Proof.** For each worker \( i = 1, 2 \),
\[ TR^{1\leftrightarrow2}_i = \lim_{T \to \infty} \sum_{n=1}^{N(T)} CT_n \cdot v_{i,n} = \frac{E[CTv_i]}{E[CT]} = \frac{E[E[CTv_i|(x,r),v_{3-i}]]}{E[CT]}, \]

where the last equality uses double expectation in which the outer expectation is taken with respect to the state \((x,r)\) and the speed of the other worker \(v_{3-i}\), and the inner expectation is taken with respect to \(v_i\) conditional on \((x,r)\) and \(v_{3-i}\). Then,

\[ \frac{E[E[CTv_i|(x,r),v_{3-i}]]}{E[CT]} = \frac{E[\text{Cov}[CT,v_i|(x,r),v_{3-i}]] + E[CT]E[v_i]}{E[CT]} \leq E[v_i], \]

where the second equality uses the definition of covariance and the fact that, for each cycle, \(v_i\) is independent of \((x,r)\) and \(v_{3-i}\), and the inequality follows from applying Chebyshev’s algebraic inequality (see, e.g., Mitrović et al., 1993) for each realization of \((x,r)\) and \(v_{3-i}\) to conclude that \(\text{Cov}[CT,v_i|(x,r),v_{3-i}] \leq 0\) since \(CT\) is a non-increasing function of \(v_i\) (see expression (2.4)).

Similarly,

\[ TR^{1\leftrightarrow2}_i = E \left[ \frac{1}{v_i} \right]^{-1} E \left[ \frac{1}{v_i} \frac{E[CTv_i]}{E[CT]} \right] = E \left[ \frac{1}{v_i} \right]^{-1} E \left[ \frac{1}{v_i} CTv_i \right] - \frac{\text{Cov}[\frac{1}{v_i},CTv_i]}{E[CT]} \geq E \left[ \frac{1}{v_i} \right]^{-1}, \]

where the second equality uses the definition of covariance. To understand the inequality, note first that \(CTv_i\) is a non-increasing function of \(\frac{1}{v_i}\) because (2.4) implies that \(CTv_i\) is the minimum between a constant and an increasing function of \(v_i\) (intuitively, \(CTv_i\) is the work load worker \(i\) covers in a cycle, so it is non-decreasing in \(v_i\), thus non-increasing in \(\frac{1}{v_i}\)). Then the inequality follows from applying Chebyshev’s algebraic inequality to conclude that \(\text{Cov}[\frac{1}{v_i},CTv_i] \leq 0\) since \(CTv_i\) is a non-increasing function of \(\frac{1}{v_i}\).

**Proposition 2.3.** The throughput rate of BBO, \(TR^{1\leftrightarrow2}_i\), dominates that of parallel workers:

\[ TR^{1\leftrightarrow2}_i \geq TR_1 + TR_2. \]

**Proof.** Follows from (2.11) and Proposition 2.2. \(\square\)

### 2.3 Bucket Brigade effect

Section 2.2 has studied the properties of workers’ TR under both BB and BBO rules. It was shown that in a BB line, the TR of worker 2 depends only on \(v_2\) and equals his single worker TR, as if there is no interaction with worker 1. The TR of worker 1, however, depends on both \(v_1\) and \(v_2\). The interaction with worker 2 can substantially increase or decrease \(TR^{1\rightarrow2}_1\) as compared to the single worker TR. The negative effect of worker 2 on \(TR^{1\rightarrow2}_1\) is intuitively caused by possible blockage in the BB line, which does not occur in a parallel worker environment. In this section we
show that the BB line can also increase $TR_{1 \rightarrow 2}$, as compared with the parallel worker environment. Consequently, the cumulative TR of the two workers can also increase, as $TR_{2 \rightarrow 1}$ remains the same. The relative additional TR of worker $i$ in a BB line compared to the single worker TR is called here the *bucket brigade effect* (BBE). Formally, \[
BBE_{i}^{1 \rightarrow 2} \equiv \frac{TR_{1 \rightarrow 2}^{i} - TR_{i}}{TR_{i}}, \quad i = 1, 2. \tag{2.15}
\]
Note that by Proposition 2.1, $BBE_{2}^{1 \rightarrow 2} = 0$ for any BB line. The BBE of the line $1 \rightarrow 2$ is defined similarly, as the BB line’s extra TR divided by the sum of single worker TRs, \[
BBE^{1 \rightarrow 2} \equiv \frac{TR_{1 \rightarrow 2} - (TR_{1} + TR_{2})}{TR_{1} + TR_{2}} = \frac{TR_{1 \rightarrow 2}^{1} - TR_{1}}{TR_{1} + TR_{2}}, \tag{2.16}
\]
where the second equality follows from Proposition 2.1. Note that the BBE can be either positive or negative, depending on the TR of worker 1 in the BB line as compared to the case of parallel workers.

The following is a general upper bound on the BBE, obtained by substituting the inequality (2.12) in (2.16): \[
BBE^{1 \rightarrow 2} = \frac{TR_{1 \rightarrow 2}^{1} - TR_{1}}{TR_{1} + TR_{2}} \leq E[v_{1}] - TR_{1}. \tag{2.17}
\]
The $BBE^{1 \rightarrow 2}$ stands in a certain proportion with the BBE of worker 1, as given by \[
\frac{BBE^{1 \rightarrow 2}}{BBE_{1}^{1 \rightarrow 2}} = \frac{TR_{1 \rightarrow 2}^{1} - TR_{1}}{TR_{1} + TR_{2}} \cdot \frac{TR_{1}}{TR_{1 \rightarrow 2}^{1} - TR_{1}} = \frac{TR_{1}}{TR_{1} + TR_{2}}. \tag{2.18}
\]
Finally, we define the BBE associated with the team of two workers as the best one among the two lines, $1 \rightarrow 2$ and $2 \rightarrow 1$, \[
BBE = \max (BBE^{1 \rightarrow 2}, BBE^{2 \rightarrow 1}). \tag{2.19}
\]
The best effect among the two determines the best assignment of the workers.

The bucket brigade effect can be similarly defined for a BBO line, \[
BBOE^{1 \rightarrow 2} = \frac{TR_{1 \rightarrow 2} - (TR_{1} + TR_{2})}{TR_{1} + TR_{2}}. \tag{2.20}
\]
Given Proposition 2.3, BBOE is non-negative for any BBO line.

### 3 The case of dominance

In an instance of a BB line with stochastic speeds, worker 1 may be blocked by worker 2 in some iterations, resulting in subsequent hand-offs at $x_{n} = 1$ and $x_{n+1} = 0$. In other iterations, where worker 1 is not blocked, the hand-offs occur in some $0 < x_{n} < 1$. This section deals with particular
cases of no blockage, where worker 1 is never blocked, and full blockage, where worker 1 is blocked at each hand-off. These cases, called cases of dominance, come up when one of the workers in the team is almost surely faster than the other, i.e. with probability 1 has a higher or equal speed at each hand-off. Although the cases of dominance are less likely to occur in practice, their investigation aims to provide some interesting analytic results. These results can be used as a good approximation for situations, where blockage occurrences are relatively rare. This occurs when the workers have different expected speeds and the variability of the speeds is much smaller relative to the speed expectations. This section focuses only on BB lines, since under dominance, BBO is the special case of BB where the workers are sequenced from slowest to fastest. Hence, the results in this section also refer to the throughput rate of each worker in a BBO line, and the resulting BBOE.

3.1 Throughput rate under dominance

Assume, without loss of generality, that worker 2 dominates worker 1, i.e., \( v_1 \leq v_2 \) for sure. The two assignments, \( 1 \rightarrow 2 \) and \( 2 \rightarrow 1 \), correspond to the slowest-to-fastest and fastest-to-slowest rules, respectively, which are well-defined only in the case of dominance. The slowest-to-fastest personnel assignment rule is known to be optimal in terms of TR when workers have deterministic speeds (Bartholdi & Eisenstein 1996). The rule allows avoiding blockage and attains the maximum TR,

\[
TR_{1 \rightarrow 2}^1 = v_1 + v_2. \tag{3.1}
\]

The opposite, fastest-to-slowest assignment leads to full blockage in the deterministic environment, with the TR, \( TR_{2 \rightarrow 1}^2 = 2v_1 \), slower than that in (3.1).

Under stochastic speeds, the question of setting an appropriate personnel assignment rule is significantly more complicated, and depends on the BBE discussed in Section 2.3. However, in the case of dominance, the slowest-to-fastest assignment is better than the opposite one, as proved in the next proposition. The proposition also explicitly calculates the TR of the workers and of the line.

**Proposition 3.1.** If worker 2 dominates worker 1, then

- the slowest-to-fastest assignment yields

\[
TR_{1 \rightarrow 2}^1 = E[v_1] \leq TR_{2 \rightarrow 1}^2, \quad TR_{1 \rightarrow 2}^1 = E[v_1] + E\left[ \frac{1}{v_2} \right]^{-1}, \tag{3.2}
\]

- the fastest-to-slowest assignment yields

\[
TR_{2 \rightarrow 1}^2 = TR_{1 \rightarrow 1}^2 = E\left[ \frac{1}{v_1} \right]^{-1}, \quad TR_{2 \rightarrow 1}^1 = 2E\left[ \frac{1}{v_1} \right]^{-1}, \tag{3.3}
\]

- the slowest-to-fastest assignment is better: \( TR_{1 \rightarrow 2}^1 \geq TR_{2 \rightarrow 1}^2 \).

11
Proof. When no blockage can occur, \( v_{1,n}^{in} = v_{1,n} \) for any cycle \( n \) (see proof of Proposition 2.1). Consequently, (2.12) becomes an equality,

\[
TR_{1}^{1\rightarrow 2} = E[v_{1}].
\]

By dominance, this TR is not greater than the TR of worker 2. The TR of the BB line is

\[
TR^{1\rightarrow 2} = TR_{1}^{1\rightarrow 2} + TR_{2}^{1\rightarrow 2} = E[v_{1}] + E \left[ \frac{1}{v_{2}} \right]^{-1}.
\]

Under full blockage, the hand-off positions are alternating between 1 and 0. The average position of \( 1/2 \), substituted into (2.14), leads to \( TR_{2}^{2\rightarrow 1} = TR_{1}^{2\rightarrow 1} \) and \( TR^{2\rightarrow 1} = TR_{1}^{2\rightarrow 1} + TR_{2}^{2\rightarrow 1} = 2E \left[ \frac{1}{v_{1}} \right]^{-1}. \) The slowest-to-fastest \( TR_{1}^{1\rightarrow 2} \) is greater than the opposite one, \( TR_{2}^{1\rightarrow 2} \), since each of the two terms in (3.2), \( E[v_{1}] \) and \( E \left[ \frac{1}{v_{2}} \right]^{-1} \), is greater than \( E \left[ \frac{1}{v_{1}} \right]^{-1}. \) This follows, respectively, from Jensen’s inequality and the dominance assumption.

Proposition 3.1 shows that under dominance, the TR of the first worker can never exceed that of the second worker. Additionally, while the TR in BB in \( 1 \rightarrow 2 \) ordering depends on the specific distribution of the speed of worker 2 (even when holding the expected speed fixed), it depends on the speed of worker 1 only through the expected value.

### 3.2 Bucket Brigade effect under dominance

Having determined the TR, we now develop the closed form expressions for the BBE. Then we show that the BBE of a team with dominance is positive and bounded from above by 1, where this bound is tight.

The BBE in a no blockage case is determined in the following.

**Proposition 3.2.** The BBE of worker 1 and the BBE of the line are

\[
BBE_{1}^{1\rightarrow 2} = E[v_{1}] E \left[ \frac{1}{v_{1}} \right] - 1, \quad BBE^{1\rightarrow 2} = \frac{E[v_{1}] E \left[ \frac{1}{v_{1}} \right] - 1}{1 + E \left[ \frac{1}{v_{1}} \right] E \left[ \frac{1}{v_{2}} \right]^{-1}}.
\] (3.4)

Moreover, both expressions are zero when the speed of worker 1 is deterministic, and are increasing with a mean preserving spread of the speed of worker 1. In particular, both expressions are positive if and only if the speed of worker 1 is not deterministic.

**Proof.** By combining (2.15) and (3.2),

\[
BBE_{1}^{1\rightarrow 2} = \frac{E[v_{1}] - E \left[ \frac{1}{v_{1}} \right]^{-1}}{E \left[ \frac{1}{v_{1}} \right]^{-1}} = E[v_{1}] E \left[ \frac{1}{v_{1}} \right] - 1.
\]
The $BBE^{1\rightarrow 2}$ is now obtained from (2.18). Since $E[v_1] E\left[\frac{1}{v_1}\right] = 1$ when $v_1$ is deterministic, both expressions are zero in this case. A mean preserving spread of $v_1$ adds a non-deterministic difference $\delta$ with expectation $E_{\delta|v_1} \delta = 0$ conditional on each realization of $v_1$. Conditioning on $v_1$, we can write a double expectation expression, where the outer expectation is taken with respect to $v_1$ and the inner expectation is taken with respect to $\delta$ conditional on $v_1$. Then applying Jensen’s inequality,

$$E\left[\frac{1}{v_1 + \delta}\right] = E\left[E\left[\frac{1}{v_1 + \delta | v_1}\right]\right] > E\left[\frac{1}{v_1 + E[\delta | v_1]}\right] = E\left[\frac{1}{v_1}\right],$$

thus $E\left[\frac{1}{v_1}\right]$ increase with a mean preserving spread of $v_1$. Since both $BBE^{1\rightarrow 2}$ and $BBE^{1\rightarrow 2}$ are increasing functions of $E\left[\frac{1}{v_1}\right]$, both also increase with a mean preserving spread of $v_1$. That both expressions are positive when the speed of worker 1 is not deterministic follows from applying a mean preserving spread to a deterministic $v_1$. □

Proposition 3.2 implies that in a no blockage case, the BB line necessarily increases the TR of the two workers. That is, when the speed of worker 1 is not deterministic,

$$TR^{1\rightarrow 2} = TR_1 (1 + BBE^{1\rightarrow 2}_1) + TR_2 > TR_1 + TR_2$$

This phenomenon is explained as follows. Contrary to parallel workers, wherein each worker completes a whole item, the BB cycle workload is divided between the workers based on their relative speeds. Figure 3.1 demonstrates the hand-off points of three cycles, $n$, $n+1$ and $n+2$. When $\frac{v_1}{v_2}$ is relatively high (cycle $n+1$), worker 1 completes a relatively large portion of the workload, whereas relatively low $\frac{v_1}{v_2}$ (cycle $n+2$) leads to an earlier hand-off point and to worker 1 being assigned a smaller workload portion. Hence on average, a larger portion of the workload is performed by worker 1 when assigned a higher speed. Note however that worker 2’s workload portion in any cycle is determined based on the previous cycle’s speeds, independently from the current cycle, thus the TR contribution of worker 2 is the same as when working alone. To conclude, $BBE^{1\rightarrow 2}_2 = 0$, but $BBE^{1\rightarrow 2}_1 > 0$, leading to a positive BBE.

Figure 3.1: Larger workload performed by relatively faster worker
The example below demonstrates Proposition 3.2 by showing the growth of the $BBE_{1 \rightarrow 2}^{1}$ and $BBE_{1 \rightarrow 2}^{2}$ when the spread of speed $v_1$ increases.

**Example 3.1.** Consider two workers processing identical items, where worker 1 is slower than worker 2. Specifically suppose that speed $v_1$ is distributed uniformly on the interval $[e_1 - w_1, e_1 + w_1]$ and speed $v_2$ is distributed uniformly on the interval $[e_2 - w_2, e_2 + w_2]$, where $e_1 + w_1 \leq e_2 - w_2$.

The growth of $BBE_{1 \rightarrow 2}^{1}$ and $BBE_{1 \rightarrow 2}^{2}$ as functions of $w_1$ is calculated from (3.4), and equals 

$BBE_{1 \rightarrow 2}^{1} = e_1 \left( \frac{\ln(e_1 + w_1) - \ln(e_1 - w_1)}{2w_1} \right) - 1$ and $BBE_{1 \rightarrow 2}^{2} = \frac{1}{2} \frac{e_1 \left[ \ln(e_1 + w_1) - \ln(e_1 - w_1) \right] - 2w_1}{w_1 + w_2 \left[ \ln(e_2 + w_2) - \ln(e_2 - w_2) \right]}$. These functions are depicted in Figure 3.2.

![Figure 3.2: The growth of the Bucket Brigade effect with the spread of the v_1 distribution for e_1 = 1, e_2 = 2 and w_2 = 0.1](image)

Proposition 3.2 and Example 3.1 have shown that the BB line improves throughput, and its effect, $BBE_{1 \rightarrow 2}^{1}$, can have significant impact. However, since worker 2 does not contribute to the effect and worker 1 is constrained by worker 2, $BBE_{1 \rightarrow 2}^{2}$ is bounded as well. The next proposition determines this bound.

**Proposition 3.3.** Let $v_1$ and $v_2$ be arbitrarily distributed over the supports $[a, b] \text{ and } [b, \infty]$, respectively, where $0 < a \leq b$. Then,

- given $v_2$, the maximum value of $BBE_{1 \rightarrow 2}^{1}$ is

$$\max_{v_1} BBE_{1 \rightarrow 2}^{1} = a \left( 1 + bE \left[ \frac{1}{v_2} \right] \right) E \left[ \frac{1}{v_2} \right] \left( \sqrt{\frac{b - a}{a \left( 1 + bE \left[ \frac{1}{v_2} \right] \right)}} - 1 \right)^2,$$  \hspace{1cm} (3.5)

  where the probability distribution of $v_1$ attaining this maximum is

$$Pr(v_1 = a) = \frac{a \left( 1 + bE \left[ \frac{1}{v_2} \right] \right)}{b - a} \left( \sqrt{\frac{b - a}{a \left( 1 + bE \left[ \frac{1}{v_2} \right] \right)}} - 1 \right),$$  \hspace{1cm} (3.6)

$$Pr(v_1 = b) = 1 - Pr(v_1 = a), \quad Pr(v_1 = t) = 0 \text{ for } a < t < b,$$

14
• given \( v_1 \), the maximum value of \( BBE^{1 \rightarrow 2} \) is

\[
\max_{v_2} BBE^{1 \rightarrow 2} = \frac{E[v_1] E\left[\frac{1}{v_1}\right] - 1}{1 + b E\left[\frac{1}{v_1}\right]},
\]

where the \( v_2 \) attaining this maximum is \( v_2 = b \) deterministically.

The proof is in the appendix.

When maximizing \( BBE^{1 \rightarrow 2} \) jointly on \( v_1 \) and \( v_2 \), we obtain the following.

**Proposition 3.4.** In a BB line with no blockage,

\[
\sup_{v_1,v_2} BBE^{1 \rightarrow 2} = 1. \tag{3.7}
\]

**Proof.** Let \( v_1 \) and \( v_2 \) be arbitrarily distributed over the supports \([a, b]\) and \([b, \infty]\), respectively, where \(0 < a \leq b\). By substituting \( v_2 = b \) deterministically into (3.5), we get the maximum \( BBE^{1 \rightarrow 2} \) as,

\[
\max_{v_1,v_2} BBE^{1 \rightarrow 2} = \frac{1}{b} \left( \sqrt{a + b} - \sqrt{2a} \right)^2. \tag{3.8}
\]

Now, the supremum is obtained by maximizing (3.8) over \( a \) and \( b \), leading to \( BBE^{1 \rightarrow 2} \) tending to 1 as \( a \) tends to 0 for any \( b \).

The next example illustrates Propositions 3.3 and 3.4.

**Example 3.2.** Consider a team of two workers, processing items of a single type. For a standard item, the speed of the two workers is identical, \( v_1 = v_2 = b \). However, the response of the workers to a faulty item is different. Worker 2, when facing a faulty item, is able to repair it and keeps the processing speed unchanged, \( v_2 = b \). For worker 1 to repair a faulty item takes much more time, so that the speed reduces to \( a \), \( v_1 = a < b \). Denote by \( p \) the probability of facing a faulty item.

In case of parallel workers, the TR of worker 1, \( TR_1 = E \left[ \frac{1}{v_1} \right]^{-1} = \left( \frac{p}{a} + \frac{1-p}{b} \right)^{-1} = \frac{ab}{bp + a(1-p)} \). It drops down when \( a \) decreases, and can even go to zero when \( a \) goes to zero. The TR of worker 2 is constant, \( TR_2 = b \), and the total TR of the team is,

\[
TR_1 + TR_2 = \frac{ab}{bp + a(1-p)} + b. \tag{3.9}
\]

In case the workers are assigned 1 \( \rightarrow \) 2 in a BB line, an equivalent situation is where items are identical but the speed of worker 1 is stochastic and the speed of worker 2 is deterministic. Worker 2 assists worker 1 by taking the items at hand-offs and completing the process (standard and faulty) with the regular constant speed, \( b \). As a result, the TR of worker 1 becomes (see Proposition 3.1), \( TR_1^{1 \rightarrow 2} = E[v_1] = pa + (1-p)b \), and the TR of the team

\[
TR^{1 \rightarrow 2} = pa + (1-p)b + b. \tag{3.10}
\]
The $BBE^{1\rightarrow 2}$ is obtained by substituting (3.9) and (3.10) into (2.16). For each $p > 0$, $BBE^{1\rightarrow 2}$ decreases in $a$ (as its partial derivative with respect $a$ is negative). Figure 3.3 presents the $BBE^{1\rightarrow 2}$ as a function of $p$ for various $a$. For a given $a$, the value of $p$ for which $BBE^{1\rightarrow 2}$ is maximal is presented in (3.6). At $a = 0$, it decreases in $p$, tending to its upper limit of 1 (see Proposition 3.4) as $p$ tends to zero. In this case, the TR of the BB line doubles the TR of two parallel workers.

Consider now the fastest-to-slowest assignment, $2 \rightarrow 1$. Worker 2, having higher capabilities, is fully blocked by the slower worker 1. As a result, worker 2’s TR in the BB line is smaller than the single worker TR, and the BBE of the line becomes negative. This statement is proved in the next proposition.

**Proposition 3.5.** The BBE of fastest-to-slowest is negative,

$$BBE^{2\rightarrow 1} < 0.$$  

**Proof.** The proposition is obtained from (2.16) and (3.3), as

$$BBE^{2\rightarrow 1} = \frac{TR_1 - TR_2}{TR_1 + TR_2} < 0.$$  

Note that by exchanging the workers in the BB line, a full blockage case turns into no blockage with positive BBE, as shown above. From Propositions 3.4 and 3.5 it follows that under dominance, the best BBE of a team of two workers is positive and bounded by 1.

### 3.3 Bucket Brigade effect in a K-worker line

The BBE becomes even more significant in a multiple-worker line. When $K$ workers are assigned slowest-to-fastest along a BB line in a scenario where worker $i+1$ dominates worker $i$, $i = 1, 2, ..., K-1$, the TR of the line is the sum of the individual worker TRs, accelerated by the BBEs of all workers except for the last one. This result generalizes the one obtained for 2 workers in Proposition 3.1. We now show this by calculating the individual $TR_i^{1\rightarrow \cdots \rightarrow K}$ for each worker $i = 1, 2, ..., K$, and the total TR of the line, $TR^{1\rightarrow \cdots \rightarrow K}$.

![Figure 3.3: BBE values for Example 3.2 ($b = 10$)](image)
Proposition 3.6. The individual TR of $K$ workers under dominance is

$$TR^1 \rightarrow \cdots \rightarrow K = E[v_i], \ i = 1, 2, ..., K - 1, \ TR^K = E \left[ \frac{1}{v_K} \right]^{-1},$$

the TR of the BB line is

$$TR^1 \rightarrow \cdots \rightarrow K = \sum_{i=1}^{K-1} E[v_i] + E \left[ \frac{1}{v_K} \right]^{-1}.$$

Proof. The $n^{th}$ cycle time, $CT_n = \frac{1-x_{K-1,n}}{v_{K,n}}$, is independent of the speed, $v_{i,n}$, during this cycle of all workers except for the last one. Thus the TR associated with worker $i = 1, 2, ..., K - 1$ is

$$TR^i = \lim_{T \to \infty} \sum_{n=1}^{N(T)} \frac{CT_n v_{i,n}}{\sum_{n=1}^{N(T)} CT_n} = \lim_{T \to \infty} \frac{1}{N(T)} \sum_{n=1}^{N(T)} CT_n v_{i,n} = \frac{E[CT v_i]}{E[CT]} = E[v_i],$$

and the TR of worker $K$ is

$$TR^K = \lim_{T \to \infty} \sum_{n=1}^{N(T)} \frac{CT_n v_{K,n}}{\sum_{n=1}^{N(T)} CT_n} = \lim_{T \to \infty} \frac{1}{N(T)} \sum_{n=1}^{N(T)} CT_n v_{K,n} = \frac{E[1-x_{K-1}]}{E \left[ \frac{1-x_{K-1}}{v_K} \right]} = E \left[ \frac{1}{v_K} \right]^{-1}.$$

Therefore, the TR of the BB line is

$$TR^1 \rightarrow \cdots \rightarrow K = \sum_{i=1}^{K-1} E[v_i] + E \left[ \frac{1}{v_K} \right]^{-1}. \quad (3.11)$$

Similarly to a two-worker line, the BBE of worker $i = 1, 2, ..., K - 1$ is $BBE^i = E[v_i]E \left[ \frac{1}{v_i} \right] - 1$, which is positive, and the BBE of the $K^{th}$ worker is zero. Now, (3.11) is rewritten as

$$TR^1 \rightarrow \cdots \rightarrow K = \sum_{i=1}^{K-1} TR_i (1 + BBE^i) + TR_K.$$

This increased TR may be compared with that of $K$ parallel workers,

$$TR = \sum_{i=1}^{K} TR_i.$$

4 General worker speed parameters

In this section we analyze general two worker BB and BBO lines, i.e. when the worker speeds cannot be ordered by dominance. To obtain further insights into the general case, the TR must be evaluated more carefully. As in the previous section, the main component to be evaluated in a BB
line is $E[x]$, the expected location of the steady-state hand-off position. To this end we re-examine the equation (2.2) describing the dynamics of the system as a discrete Markov process, assuming that $\frac{v_1}{v_2}$ has cdf $F_{v_1/v_2}(\xi)$ and probability density function (pdf) $f_{v_1/v_2}(\xi)$ for $\xi \in (0, \infty)$. Recall that the state $x_n=1$ means blockage, since the two workers are positioned at 1, and this blockage state is immediately and with certainty followed by the state $x_{n+1}=0$. Denoting the cdf of $x_n$ by $F_{x_n}(z)$ for $z \in [0, 1]$ and its pdf by $f_{x_n}(z)$, the cdf corresponding to the next hand-off follows from (2.2) by the Markov transition probability equation

$$F_{x_n}(z) = Pr(x_n \leq z) = \int_0^1 Pr(x_n \leq z | x_{n-1} = y) f_{x_{n-1}}(y) dy + Pr(x_{n-1} = 1) + Pr(x_{n-1} = 0) F_{v_1/v_2}(z),$$

(4.1)

where $Pr(x_n \leq z | x_{n-1} = y) = F_{v_1/v_2}(\frac{z}{1-y})$ for $0 < z < 1$. Equation (4.1) determines the evolution of the distribution of the first worker’s position at the hand-offs. Note that when the speed ratio is a continuous random variable, probability mass is given only to states $x_{n-1} = 0$ and $x_{n-1} = 1$, thus they are separated out in this equation. In case there exists a limit distribution, $F_{x}(z) = \lim_{n \to \infty} F_{x_n}(z)$,

we have

$$E[x] = 1 - \int_0^1 F_{x}(z) dz.$$

Applying integration by parts to (4.1), $F_{x}(z)$ is determined by the integral equation

$$F_{x}(z) + \int_0^1 F_{x}(y) f_{v_1/v_2}(\frac{z}{1-y}) \frac{z}{(1-y)^2} dy = 1, \ 0 < z < 1,$$

(4.2)

or equivalently

$$F_{x}(z) + \int_z^{\infty} F_{x}(1 - \frac{z}{t}) f_{v_1/v_2}(t) dt = 1, \ 0 < z < 1,$$

(4.3)

with typical jumps at $x = 0$ and $x = 1$. Since the transition from the state $x = 1$ to $x = 0$ is certain, the steady-state probability is equal for these two states, i.e., the values of the two discontinuity jumps in the function $F_{x}(z)$ at 0 and 1 are equal. Thus the solution has the qualitative form depicted in Figure 4.1 (in fact, we show below that the graph in the figure is the exact solution for a particular instance of this equation).

In a BBO line, denoting the cdf of $x_n$, as a function of the upstream worker $i = 1, 2$ in cycle $n$, by $F_{x_n,i}(z) = Pr(x_n \leq z, r(n) = i)$ for $z \in [0, 1]$ and its pdf by $f_{x_n,i}(z)$, the cdf corresponding to
the next hand-off follows from (2.3) by the Markov transition probability equation

\[
F_{x_n,i}(z) = \int_0^1 Pr(x_n \leq z, r(n) = i|x_{n-1} = y, r(n-1) = i)f_{x_{n-1},i}(y)dy
\]

\[
+ \int_0^1 Pr(x_n \leq z, r(n) = i|x_{n-1} = y, r(n-1) = 3-i)f_{x_{n-1},3-i}(y)dy
\]

\[
= \int_0^1 F_{v_i/v_{3-i}} \left( \frac{z}{1-y} \right) f_{x_{n-1},i}(y)dy + \int_0^z F_{v_i/v_{3-i}} (z-y)f_{x_{n-1},3-i}(y)dy
\]  \hspace{1cm} (4.4)

In case there exists a limit distribution, \( F_{x,i}(z) \), applying integration by parts to (4.4), \( F_{x_n,i}(z) \) is determined by the integral equation

\[
F_{x,i}(z) + \int_0^1 F_{x,i}(y) f_{v_i/v_{3-i}} \left( \frac{z}{1-y} \right) \frac{z}{(1-y)^2}dy - \int_0^z F_{x,3-i}(y) f_{v_i/v_{3-i}} (z-y)dy = F_{x,i}(1), \quad 0 < z < 1,
\]

or equivalently

\[
F_{x,i}(z) + \int_0^\infty F_{x,i}(1 - \frac{z}{t}) f_{v_i/v_{3-i}} (t)dt - \int_0^z F_{x,3-i}(y) f_{v_i/v_{3-i}} (z-y)dy = F_{x,i}(1), \quad 0 < z < 1, \hspace{1cm} (4.5)
\]

where \( F_{x,i}(0) = 0 \) for each \( i \) and \( F_{x,1}(1) + F_{x,2}(1) = 1 \). The solution for a particular instance of this equation (the specification is provided at the beginning of Section 4.2) is illustrated in Figure 4.2, where the two workers have different expected speeds and almost the same standard deviations. The two graphs (blue and red) correspond to the identity of the upstream worker, which, differently from the BB system, may be either \( i = 1 \) or \( i = 2 \). As seen in the figure, there is a higher probability for the slower (blue) worker to be placed upstream. Note that the lack of blockage in the BBO system implies that there are no jumps in the cdfs.

To derive explicit solutions we specialize to a family of continuous distributions which is useful in describing worker speeds in a variety of general, realistic settings. It is natural to assume

Figure 4.1: Cumulative distribution function for first worker’s position in BB

![Cumulative distribution function](image)
a distribution with a finite, non-negative support, which has a unimodal shape, but may not necessarily be symmetric. Specifically, we assume that the speed of worker \( i = 1, 2 \) has a generalized continuous beta distribution \( \text{beta}(\alpha_i, \beta_i) \) on the interval \([0, M]\) for \( \alpha_i, \beta_i \geq 1 \) and \( M > 0 \), so that the pdf for worker \( i \) is

\[
f_{v_i}(z) = \frac{\left( \frac{z}{M} \right)^{\alpha_i-1} \left( 1 - \frac{z}{M} \right)^{\beta_i-1}}{M \cdot B(\alpha_i, \beta_i)}
\]

for \( z \in [0, M] \), where \( B(a, b) \) is the Euler beta function defined by

\[
B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)},
\]

where \( \Gamma(a) \) is the Euler gamma function defined by

\[
\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} dt.
\]

For integer \( a > 0 \). Note also that

\[
e_i \equiv E[v_i] = M \frac{\alpha_i}{\alpha_i + \beta_i}, \quad s_i \equiv \text{std}[v_i] = e_i \sqrt{\frac{\beta_i}{\alpha_i + \beta_i + 1}} = e_i \sqrt{\frac{M - e_i}{M \alpha_i + e_i}}
\]

and

\[
E\left[ \frac{1}{v_i} \right] = \frac{M - \alpha_i - 1}{\alpha_i + \beta_i - 1}.
\]

For this distribution, the pdf of the speed ratio \( v \), defined by

\[
f_{v_1/v_2}(z) \equiv \int_0^{\min\{M, M/z\}} t f_{v_1}(zt) f_{v_2}(t) dt,
\]

Figure 4.3: Beta pdf examples (\( M = 51 \))
is evaluated as
\[
f_{v_1/v_2}(z) = \begin{cases} 
\frac{B(\alpha_1+\alpha_2,\beta_2)}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)} \sum_{n=0}^{\infty} \frac{z^{\alpha_1+n}}{n!} \prod_{m=0}^{n-1} \frac{(\alpha_1+\alpha_2+m)(1-\beta_1+m)}{(\alpha_1+\alpha_2+m+1)(1-\beta_1+m)} & \text{if } 0 < z < 1 \\
\frac{B(\alpha_1+\alpha_2,\beta_1)}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)} \sum_{n=0}^{\infty} \frac{z^{-\alpha_2+n+1}}{n!} \prod_{m=0}^{n-1} \frac{(\alpha_1+\alpha_2+m)(1-\beta_2+m)}{(\alpha_1+\alpha_2+m+1)(1-\beta_2+m)} & \text{if } z \geq 1,
\end{cases}
\]
where we note that the summations in this expression are finite when \(\beta_1, \beta_2\) are positive integers.

To determine the expected hand-off position, \(E[x]\), we look for solutions to Equation (4.3) and to Equation (4.5) of the form
\[
F_x(z) = \sum_{j=0}^{d} a_j (1 - z)^j, \quad 0 < z < 1,
\]
for some integer \(d > 0\) and some coefficients \(a_j, j = 0, 1, \ldots, d\). Under this solution, Equation (4.3) can be rewritten as
\[
\sum_{j=0}^{d} \left( (1 - z)^j + z^j \int_{z}^{\infty} t^{-j} f_{v_1/v_2}(t) dt \right) a_j - 1 = 0, \quad 0 < z < 1,
\]
where
\[
f_z^\infty t^{-j} f_{v_1/v_2}(t) dt = \frac{B(\alpha_1+\alpha_2,\beta_2)}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)} \sum_{n=0}^{\infty} \frac{1-z^{\alpha_1+n-j}}{n(\alpha_1+n-j)} \prod_{m=0}^{n-1} \frac{(\alpha_1+\alpha_2+m)(1-\beta_1+m)}{(\alpha_1+\alpha_2+m+1)(1-\beta_1+m)}
\]
+ \(\frac{B(\alpha_1+\alpha_2,\beta_1)}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)} \sum_{n=0}^{\infty} \frac{z^{\alpha_2+n+1-j}}{n(\alpha_2+n+j)} \prod_{m=0}^{n-1} \frac{(\alpha_1+\alpha_2+m)(1-\beta_2+m)}{(\alpha_1+\alpha_2+m+1)(1-\beta_2+m)}\),

with a similar derivation for Equation (4.5).

### 4.1 Exact solutions for special cases of a BB line

An exact solution can be derived for a BB system in special cases. When \(\beta_1 = 1\), the pdf \(f_{v_1/v_2}(z)\) simplifies to
\[
f_{v_1/v_2}(z) = \frac{\alpha_1}{B(\alpha_2,\beta_2)} z^{\alpha_1-1} \int_{0}^{\min\{1/z\}} t^{\alpha_1+\beta_2-1} (1-t)^{\beta_2-1} dt
\]
for \(z \in (0, \infty)\). Equation (4.7) simplifies to
\[
\sum_{j=0}^{d} \left( (1 - z)^j + \frac{\alpha_1 \Gamma(\alpha_2+\beta_2)}{(\alpha_1-j)\Gamma(\alpha_2)} \left( \frac{\Gamma(\alpha_2+j)}{\Gamma(\alpha_2+\beta_2+j)} - \frac{\Gamma(\alpha_1+\alpha_2)z^{\alpha_1}}{\Gamma(\alpha_1+\alpha_2+\beta_2)} \right) \right) a_j - 1 = 0, \quad 0 < z < 1.
\]

The coefficient of \(z^d\) being zero implies the condition
\[
\alpha_1 = \frac{d}{1 + (-1)^d \frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_2)} \frac{\Gamma(\alpha_2+d)}{\Gamma(\alpha_2+d+\beta_2)}}.
\]

Under this condition, we require the coefficients of \(z^j\) for \(j = 0, \ldots, d-1\) and the coefficient of \(z^{\alpha_1}\) in Equation (4.8) to equal zero. This yields a system of \(d + 1\) linear equations with variables \(a_j\) for
and \( j = 0, ..., d \), thus an exact solution can be obtained, including \( E[x] \).

**Example 4.1.** Consider \( \beta_1 = \beta_2 = 1 \). In this case

\[
f_{v_1/v_2}(z) = \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2}z^{\alpha_1-1}\min\{1, z^{\alpha_1-\alpha_2}\}
\]

and Equation (4.7) takes the simpler form

\[
\sum_{j=0}^{d} \left( (1 - z)^j + \frac{\alpha_1\alpha_2}{\alpha_1 - j} \left[ \frac{z^j}{\alpha_2 + j} - \frac{z^{\alpha_1}}{\alpha_1 + \alpha_2} \right] \right) a_j - 1 = 0, \quad 0 < z < 1.
\] (4.10)

Furthermore, (4.9) becomes \( \alpha_1 = \frac{d}{1 + (-1)^{d+\frac{d}{2}}}, \) i.e. \( \alpha_1 = \alpha_2 + d \) when \( d \) is odd, and \( \alpha_1 = \frac{d(\alpha_2 + d)}{2\alpha_2 + d} \) when \( d \) is even. When \( d = 1 \), equating the coefficients of (4.10) to zero implies the system of equations \( 2a_0 + a_1 = 1 \) and \( \alpha_2 a_0 + (\alpha_2 + 1)a_1 = 0 \), for which the unique solution is \( a_0 = \frac{1 + \alpha_2}{\alpha_2 + 2} \) and \( a_1 = -\frac{\alpha_2}{\alpha_2 + 2}. \) Thus \( F_x(z) = \frac{\alpha_2 + 1}{\alpha_2 + 2} = \frac{M - \varepsilon(1-z)}{2M - \varepsilon z} \) for \( z \in (0, 1) \) and \( E[x] = \frac{1}{2}. \) By Equation (2.10), since \( E[\frac{1}{v_2}]^{-1} = M \frac{\alpha_2 + 1}{\alpha_2 + \beta_2 - 1} = M \left( 1 - \frac{1}{\alpha_2} \right), \) we have \( TR^{1\to 2} = 2M \left( 1 - \frac{1}{\alpha_2} \right) = 2M \left( 2 - \frac{M}{v_2} \right). \)

When \( d = 2 \), similar derivations show that \( F_x(z) = \frac{-\alpha_2^2(z-2)+\alpha_2^2(z-4)+\alpha_2(2^2+2)+2}{\alpha_2+3\alpha_2^2+5\alpha_2+4} \) for \( z \in (0, 1) \) (see Figure 4.1 for \( \alpha_1 = \frac{4}{3}, \alpha_2 = 2 \)) and \( TR^{1\to 2} = \frac{3M(\alpha_2-1)(\alpha_2+3\alpha_2^2+5\alpha_2+4)}{\alpha_2(2\alpha_2+3)(\alpha_3^2+\alpha_2+2)}. \)

When the speed of the second worker is deterministic with no dominance relation between the workers, the speed ratio is distributed beta(\( \alpha_1, \beta_1 \)) on the interval \( [0, \frac{M}{v_2}] \) for \( M > v_2 > 0. \) Assuming again \( \beta_1 = 1 \), Equation (4.3) becomes

\[
\sum_{j=0}^{d} \left( (1 - z)^j + \frac{\alpha_1}{\alpha_1 - j} \left[ \frac{(v_2z)^j}{\alpha_2 + j} - \frac{(v_2z)^{\alpha_1}}{\alpha_1 + \alpha_2} \right] \right) a_j - 1 = 0, \quad 0 < z < 1.
\]

A closed form solution can be found by equating again the coefficients to zero, implying the condition

\[
\alpha_1 = \frac{d}{1 + (-\frac{v_2}{M})^d}
\]

and a system of linear equations determining the solution in terms of \( a_j \) for \( j = 0, ..., d. \)

**Example 4.2.** When \( d = 1 \), \( v_1 \) is distributed beta(\( \frac{M}{M-v_2}, 1 \)) on the interval \( [0, M] \), \( F_x(z) = \frac{M + v_2(z-1)}{2M - v_2} \) for \( z \in (0, 1) \) and \( TR^{1\to 2} = 2v_2. \) When \( d = 2 \), \( v_1 \) is distributed beta(\( \frac{2M^2}{M^2+v_2^2}, 1 \)) on the interval \( [0, M] \), \( F_x(z) = \frac{(M-v_2)^2(M^2+v_2^2)/2M^2} {2M^4-4M^3v_2+5M^2v_2^2-2Mv_2^3+v_2^4} \) for \( z \in (0, 1) \) and \( TR^{1\to 2} = \frac{3v_2(2M^4-4M^3v_2+5M^2v_2^2-2Mv_2^3+v_2^4)}{(3M^2+v_2^2)(M^2-2Mv_2+2v_2^2)}. \)

### 4.2 Numeric approximations

When an exact solution is not available, we analyze numerically (using Wolfram Mathematica) approximate polynomial solutions to Equation (4.3) and to Equation (4.5). To this end, a system
of \( d + 1 \) linear equations with variables \( a_j \) for \( j = 0, ..., d \) is obtained by evaluating (4.7), or evaluating (4.5) with (4.6) substituted, at \( z = \frac{k}{d} \) for \( k = 0, ..., d \). Higher approximation precision is achieved with higher values of \( d \). We found that \( d = 2 \) provides a very good approximation for most parameter values (see, e.g., Figure 4.2 for \( \alpha_1 = \alpha_2 = 6, \beta_1 = 52 \) and \( \beta_2 = 50 \)). As an illustration, Figures 4.4 and 4.5 depict the TR and the BBE as a function of the expected worker speeds \( e_i \) and their standard deviations \( s_i \), where \( M = 51 \). These figures demonstrate how the conclusions of Proposition 3.1 extend to the case of partial blockage. Figure 4.4 shows that the TR increases with both workers’ expected speed and decreases with their standard deviation. The impact of both expectation and standard deviation of the worker speed on the TR is significantly more substantial for the second worker as compared to the first worker. In particular, the TR is close to zero when the coefficient of variation of the second worker’s speed is sufficiently high. The impact of the parameters on the system TR can be explained by the fact that the second worker contributes to the TR exactly as when working in parallel, and the contribution of the first worker is less sensitive to the distribution parameters since it is bounded by that of the second worker.

![Figure 4.4: Throughput rate in a BB line (\( e_2 = 10, s_2 = 3.5 \)).](image)

To understand the effect of the workers’ parameters on the BBE, as shown in Figure 4.5, recall that the BBE is expected to increase with the standard deviation of the first worker, and decrease with the frequency of blockage in the line. The BBE monotonically increases with the expected speed of the second worker and decreases with its standard deviation (Figure 4.5 (c) and (d)). This holds because such changes not only increase the TR of the second worker (at the same amount in BB as when working in parallel), but also have a similar effect on the TR of the first worker as blockage situations become more rare. When the first worker is concerned (Figure 4.5 a and b),
the BBE decreases with the expected speed, and increases with the standard deviation. The effect when increasing the expected speed holds because blockage becomes more frequent, consequently the BB advantage over parallel workers diminishes.

Possibly the most important observation of this numerical study can be seen in the Figure 4.5 (e) and (f), which show that in most situations the BBE value is positive and reaches significant values when the order of the workers can be freely determined. In stochastic environments, the BB line generally provides a higher TR as compared to two parallel workers. Note that lower BBE values occur when one worker has lower expected speed and standard deviation than the other, and the order of workers does not affect the TR.

![Figure 4.5: Bucket Brigade Effect in a BB line (e2 = 10 , s2 = 3.5)](image)

Additional insights can be gained by considering the map of parameters showing the relative advantage of the worker orders, 1 → 2 and 2 → 1, as depicted in Figure 4.6. In this figure, each curve is the locus of $s_1$ and $s_2$ for which the TR is identical for the two possible orders, where below the curve the 1 → 2 order is preferred and the opposite holds above the curve. When both workers have identical expected speed, it is preferred to locate the one with the higher standard
deviation first. Note that the slowest-to-fastest assignment in terms of expected speeds is not always the preferred order, as this decision depends also on the standard deviations. In particular, when the standard deviation of one worker is significantly higher than the other, a fastest-to-slowest assignment is preferred.

Figure 4.6: Curves of indifference to workers order (upper left $e_1 = 3$ to lower right $e_1 = 7$; all $e_2 = 5$)

For a BBO line, the TR and BBOE are depicted in Figures 4.7 and 4.8. In general, just as in BB, we observe that the TR of BBO increases with the expected speed and decreases with the standard deviation of each worker. The BBOE, to the contrary, decreases with the expected speed and increases with the standard deviation of each worker. This follows since the effect of the parameters is stronger with respect to parallel workers than with BBO, and hence the numerator of BBOE is less sensitive than the denominator. Finally, the TR of BBO can be significantly larger than that of BB and parallel workers, and BBOE can be larger than 1 and significantly larger than BBE, which is bounded from above by 1.

Figure 4.7: Throughput rate in a BBO line ($e_2 = 10$, $s_2 = 3.5$)
5 Conclusions and managerial implications

In this section we summarize the main results of this paper according to three categories: (1) analytical results for the general case, (2) analytical results in the case of dominance, and (3) general numeric results.

The main result in the first category is that the TR of the last worker in a BB line is always the same as the TR of this worker when in parallel, i.e. $E\left[\frac{1}{v_2}\right]^{-1}$. This result implies that the difference between the TR of a two-worker BB line and two parallel workers (namely, the BBE) stems solely from the difference between the TR of the first worker in the BB system and the TR of this worker when working in parallel. When BBO is concerned, the TR of each worker is at least the TR achieved when in parallel, and bounded from above by its expected speed.

In the case of dominance, when the workers are ordered from slowest to fastest, the BB and BBO are equivalent, so all results refer to both systems. In this case, the TR of the first worker in BB line is equal to $E[v_1]$, which is larger than the TR of this worker when working in parallel in a stochastic setting. Together with the equal TR of the second worker, as stated above, we can conclude that the BBE is always positive in the case of dominance, namely, BB works better than a system of two parallel workers. In addition, the BBE increases with the mean preserving spread of the first worker. The positive value of the BBE is tightly bounded from above by the value of 1, namely, the TR of a two-worker BB can be at most twice the TR of the parallel worker system.

The numerical experiments, which were conducted on general worker speeds, enable us to gain some more insights. When considering the TR of the BB and BBO, we can see that (1) the TR increases with the expected speed of both workers, and decreases with their standard deviations; and (2) in BB, the TR is mostly affected by the expected speed and standard deviation of the second worker. When the BBE is concerned, we identify an opposite effect of the two workers’ parameters: the BBE decreases (increases) with the expected speed of the first (second) worker, and increases (decreases) with the standard deviation of the first (second) worker. This result provides insights regarding the preferred order of the workers given their parameters. For example, when both workers have the same expected speed, it is better to locate first the worker with the higher standard deviation. Clearly, a worker with lower expected speed and higher standard
deviation will be located first. However, without such domination, namely, when one worker has both lower expected speed and lower standard deviation than the other worker, the preferred order will depend on the parameter values. This implies that in some cases, the preferred order will be fastest to slowest with respect to the expected speeds. This happens when the standard deviation of the worker with the higher expected speed is large enough compared to that of the other worker. Finally, when workers can overtake each other, namely in a system without blockage (BBO), we see that the advantage of the BBO over parallel workers decreases with the expected speed and increases with the standard deviation of both workers. Still, in general, the TR of BBO can be significantly higher than the TR of both the BB and parallel workers.

A Appendix

A.1 Proof of Proposition 3.3

Given \( v_2 \), denote by \( A \) the value of \( E \left[ \frac{1}{v_2} \right] \). Denote by \( u(t) \) the probability density of \( v_1 \) and suppose that \( u(t) \) is bounded from above by a constant \( U \), such that \( U > \frac{1}{b-a} \),

\[
0 \leq u(t) \leq U \quad \text{for} \quad a \leq t \leq b.
\]

Consider the following optimal control problem with state variables \( z_1(t), z_2(t) \) and \( z_3(t) \), and the control variable \( u(t) \). The state variables represent the cdf of \( v_1 \), partial expectation of \( v_1 \), and partial expectation of \( \frac{1}{v_1} \), as follows

\[
\frac{d}{dt}z_1(t) = u(t), \quad z_1(a) = 0, \quad z_1(b) = 1, \quad (A.1)
\]

\[
\frac{d}{dt}z_2(t) = u(t)t, \quad z_2(a) = 0,
\]

\[
\frac{d}{dt}z_3(t) = \frac{u(t)}{t^2}, \quad z_3(a) = 0.
\]

The objective is to maximize \( BBE^{1\rightarrow2} \), which in terms of the state variables is

\[
BBE^{1\rightarrow2} = A \frac{z_2(b)z_3(b) - 1}{A + z_3(b)}.
\]

Its maximization is equivalent to the maximization of

\[
J = \frac{z_2(b)z_3(b) - 1}{A + z_3(b)}.
\]

The necessary optimality conditions for the above problem take form of the maximum principle, which states that there exist costate variables \( \psi_1(t), \psi_2(t) \) and \( \psi_3(t) \), satisfying the following conditions:
costate dynamic equations

\[
\frac{d}{dt} \psi_1(t) = -\frac{\partial H}{\partial z_1}, \quad \frac{d}{dt} \psi_2(t) = -\frac{\partial H}{\partial z_2}, \quad \frac{d}{dt} \psi_3(t) = -\frac{\partial H}{\partial z_3},
\]

where the Hamiltonian function, \( H \), is

\[
H = \psi_1(t) u(t) + \psi_2(t) u(t) t + \psi_3(t) \frac{u(t)}{t};
\]

costate boundary conditions

\[
\psi_2(b) = \frac{\partial J}{\partial z_2(b)} = \frac{z_3(b)}{A + z_3(b)}, \quad \psi_3(b) = \frac{\partial J}{\partial z_3(b)} = \frac{A z_2(b) + 1}{(A + z_3(b))^2};
\]

the optimal control, \( u^*(t) \), maximizes the Hamiltonian at each \( t \),

\[
u^*(t) = \arg\max_{0 \leq u(t) \leq U} H. \quad \text{(A.2)}
\]

Since the right-hand sides of the costate dynamic equations equal zero, the three costate variables are constants. We denote them by \( \psi_1, \psi_2 \) and \( \psi_3 \), and notice that \( \psi_2 \) and \( \psi_3 \) are strictly positive, which follows from the costate boundary conditions. Now, the Hamiltonian is rewritten as

\[
H = \frac{u(t)}{t} \left( \psi_2 t^2 + \psi_1 t + \psi_3 \right).
\]

Denote by \( t_1 \) and \( t_2 \) the two roots of the quadratic equation,

\[
\psi_2 t^2 + \psi_1 t + \psi_3 = 0,
\]

and consider the following cases:

1. \( t_1 \) and \( t_2 \) are complex. In this case the multiplier of \( u(t) \) in the Hamiltonian is positive over \([a, b]\), and from (A.2) \( u^*(t) = U \). Then, \( z_1(b) = U(b-a) > 1 \), which contradicts the boundary condition \( z_1(b) = 1 \).

2. \( t_1 = t_2 \). This case is disregarded similarly to the previous one.

3. \( t_1 < a < b < t_2 \). The multiplier of \( u(t) \) in the Hamiltonian is negative over \([a, b]\), and from (A.2) \( u^*(t) = 0 \). As a result, \( z_1(b) = 0 \), which contradicts the boundary condition \( z_1(b) = 1 \).

4. \( a < t_1 < b < t_2 \). From (A.2), \( u^*(t) = U \) for \( a \leq t \leq t_1 \), and \( u^*(t) = 0 \) for \( t_1 \leq t \leq b \). Then from (A.1), \( U(t_1 - a) = 1 \). When \( U \) tends to infinity, \( t_1 \) tends to \( a \). As a result, \( v_1 \) tends to be equal to \( a \) deterministically with \( BBE^1 \rightarrow 2 = 0 \).

5. \( t_1 < a < t_2 < b \). From (A.2), \( u^*(t) = U \) for \( t_2 \leq t \leq b \), and \( u^*(t) = 0 \) for \( a \leq t \leq t_2 \). Then
from (A.1), \( U(b - t_2) = 1 \). When \( U \) tends to infinity, \( t_2 \) tends to \( b \). As a result, \( v_1 \) tends to be equal to \( b \) deterministically with \( \text{BBE}^{1\rightarrow 2} = 0 \).

6. \( a < t_1 < t_2 < b \). From (A.2), \( u^*(t) = U \) for \( a \leq t \leq t_1 \) and for \( t_2 \leq t \leq b \), and \( u^*(t) = 0 \) for \( t_1 \leq t \leq t_2 \). In this case we have seven unknowns \( t_1, t_2, \psi_1, \psi_2, \psi_3, z_2(b) \) and \( z_3(b) \), which satisfy the seven conditions,

\[
U(t_1 - a + b - t_2) = 1, \quad \frac{1}{2} U \left( t_1^2 - a^2 + b^2 - t_2^2 \right) = z_2(b), \quad U \left( \ln \frac{t_1}{a} + \ln \frac{b}{t_2} \right) = z_3(b),
\]

\[
\psi_2 = \frac{z_3(b)}{A + z_3(b)}, \quad \psi_3 = \frac{A z_2(b) + 1}{(A + z_3(b))^2} t_{1,2} = \frac{-\psi_1 + \sqrt{\psi_1^2 - 4 \psi_2 \psi_3}}{2 \psi_2}.
\]

When solving the conditions for \( U \) tending to infinity, \( t_1 \) tends to \( a \) and \( t_2 \) tends to \( b \). As a result, \( v_1 \) tends to be distributed as stated in the proposition with a positive \( \text{BBE}^{1\rightarrow 2} \).

The last 6-th case presents the optimal solution.

Given \( v_1 \), the \( \text{BBE}^{1\rightarrow 2} \) increases with \( E \left[ \frac{1}{v_2} \right] \), and attains the maximum for the maximal \( E \left[ \frac{1}{v_2} \right] \), which equals \( \frac{1}{b} \), when \( v_2 = b \) deterministically.

References


