Subgame Perfect Consistent Stability[*]

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Abstract

Farsighted game theoretic solution concepts have been shown to provide insightful results in many applications. However, we provide simple examples demonstrating that the existing solution concepts are not sufficiently farsighted, and in this paper we propose a new farsighted solution concept, the Subgame Perfect Consistent Set (SPCS). Based on von Neumann Morgenstern type stability and subgame perfect equilibrium, this solution is shown to lead to more satisfactory predictions in many situations as compared to existing myopic or farsighted solution concepts. Strikingly, the SPCS is shown to always achieve Pareto efficiency in farsighted normal form games. This result is demonstrated in various competitive settings, and is shown to imply, for example, that in contrast with existing farsighted solution concepts, players who follow the SPCS reasoning are always able to share the monopolistic profit in Bertrand and Cournot competition settings, and are always able to achieve coordination and Pareto efficiency in decentralized supply chain settings, even when they are unable to form coalitions.

1 Introduction

Recently, farsighted solution concepts (Chwe 1994, Mauleon and Vannetelbosch (2004, henceforth M&V), Konishi and Ray 2003) were employed to investigate various models of decentralized operations management and supply chains. For example, Sošić (2006) used Chwe's largest consistent set (LCS) and Konishi and Ray's equilibrium process of coalition formation (EPCF) to analyze the single-period two-stage model introduced by Anupindi, Bassok and Zemel (2001, henceforth ABZ), concerning competition and cooperation among geographically separated and independent retailers; Granot and Yin (2008) used both the LCS and M&V's largest cautious consistent set (LCCS) to study a two-stage supply chain consisting of independent suppliers selling complementary products to a downstream retailer. Other recent papers using farsighted solution concepts include Granot and Sošić (2005), who introduced the LCS to the Operations Management (OM) literature and used this concept to study alliance formation in Internet-based supply exchanges, Nagarajan and

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Sošić (2007, 2009), who used the LCS and the EPCF approaches to predict the structure of a supply chain consisting of suppliers selling either substitutable or complementary products to a downstream retailer/assembler, Nagarajan and Bassok (2008), who used the LCS to characterize the structure of stable coalitions of suppliers, and Yin (2010), who extended the analysis and model in Granot and Yin (2008) to situations where the retail price is an endogenous decision variable.

Let us examine in a bit more details the first two settings mentioned above, in order to appreciate the different and insightful results derived using farsighted concepts. Specifically, consider the single-period two-stage model introduced by ABZ wherein, at the beginning of the period (Stage 1), independent retailers facing a stochastic demand for an identical product non-cooperatively place orders for this product. At the end of the period, in Stage 2, the retailers may have an opportunity to increase their overall revenues by transshipping the good from retailers having residual supplies to retailers with residual demands. Granot and Sošić (2003) have shown that if, as essentially suggested by ABZ, a core allocation of the transshipment game associated with Stage 2 is used to allocate the additional revenues at this stage, retailers may have an incentive not to reveal the true values of their residual supplies and demands. If, by contrast, e.g., the Shapely value is used, retailers do have the incentive to reveal their true values of excess supply or demand. However, in this case, subcoalitions may be formed in the second stage, leading to fragmentation and inefficiency. Thus, the conclusion from Granot and Sošić (2003) appears to have been that while truthful revelation of residuals can be achieved using the Shapely value, and stability of the grand coalition of retailers in the second stage can be achieved by using a core allocation, both cannot be achieved. That is, full system efficiency, or first best, cannot be achieved.

However, the core of a cooperative game is a myopic concept, as it follows the paradigm of Nash equilibrium in normal form games. It assumes that if a coalition believes that it can gain by a one-step deviation, it would do so without fully analyzing the implications of this deviation. Indeed, its deviation could lead to deviations by other coalitions, resulting possibly with an outcome in which the original deviating coalition is actually worse off. The LCS concept endows players with the foresight to understand the implications of their actions, and Sošić (2006) has shown that if the revenues from the second stage are allocated using the Shapely value, then the grand coalition of retailers is 'stable', i.e., it is contained in the LCS. Thus, even though according to the (myopic) core concept there may exist a coalition that would benefit from its one-step deviation, according to the LCS, it would not do so, fearing of ending up with a possibly inferior outcome.

Next, let us briefly consider the single period model analyzed by Granot and Yin (2008), which consists of two stages. In the first stage, coalitions of suppliers of complementary products are formed. In the second stage, these coalitions request wholesale prices for their components from a downstream retailer/assembler who orders these components and sells the assembled product in the market at a fixed retail price. The main insight from the paper by Granot and Yin (see also Wang and Gerchak 2003, Gerchak and Wang 2004 and Wang 2006) is that if suppliers adopt the myopic paradigm in the coalition formation stage, then the equilibrium coalition structure consists of stand alone individual suppliers, resulting with the worst possible outcome for all supply chain members. By contrast, if suppliers are farsighted in the sense of the LCCS, the unique coalition structure arising in equilibrium is the grand coalition, in which all supply chain members are best off.

Evidently, as demonstrated above, farsighted solution concepts can lead to insightful results for a variety of supply chain models, and in this paper we make contributions both to the game theory literature concerned with farsighted solution concepts, as well as to the application of these concepts to OM models. Specifically, we introduce in this paper a farsighted solution concept which significantly improves upon Chwe's farsighted reasoning, as embodied by the LCS. To intuitively illustrate one type of improvement, consider the case where a system is at a state z, and that at this stage coalition S_1 can move the system from state z to state z_1. According to the myopic paradigm, if not all members of S_1 are strictly better off at z_1 than at z, S_1 would not make the move. However, according to Chwe's LCS farsighted logic, S_1 may elect to move from z to z_1 even if not all members of S_1 are strictly better off at z_1. Indeed, S_1 would prefer to move if that would trigger subsequent moves by other coalitions, leading, for example, to a unique final state of the game, z^*, at which all members of S_1 are strictly better off as compared to z. This reasoning of farsightedness is quite compelling, but far from being complete. Indeed, consider, for example, another compelling farsighted logic which, as recognized by Chwe (1994, page 323, penultimate paragraph), is completely missing in the LCS. Suppose that aside for S_1, coalition S_2 could also move from state z, leading to another state of the game, z_2, and suppose that subsequent moves by other coalitions lead uniquely to another final state of the game, z'. Of course, if S_2 is strictly better off at z' than at z, then Chwe's LCS reasoning justifies the move by S_2 from z to z_2. However, a move by S_2 to z_2 is justified even if not all members of S_2 are strictly better off at z' as compared to z. Indeed, it is justified as long as S_2 is strictly better off at z' as compared to z^*. That is, given an opportunity to move at z, S_2 would move to z_2 to preempt S_1 from moving to z_1 and resulting in the final outcome z^*.

In this paper we introduce a solution concept that incorporates both types of farsighted reasoning described above and more. Specifically, we introduce an approach to farsightedness that provides a non-cooperative foundation for this concept based on von Neumann Morgenstern type stability and subgame perfect equilibrium. We refer to the set of outcomes derived by this approach as Subgame Perfect Consistent Set (SPCS), and investigate its structure for the class of farsighted normal form games. In particular, we show that even when only single players can deviate, quite remarkably, the SPCS solution leads to (weak) Pareto efficiency in any normal form game having a pure Nash equilibrium. Thus, the new farsightedness concept leads to efficiency in, e.g., the Prisoner's Dilemma, Bertrand games, Cournot games and Decentralized Supply Chains. For all these settings a SPCS is a singleton set containing a Pareto efficient outcome weakly dominating the equilibrium outcome. This is achieved using strategies that are similar to 'grim-trigger' strategies commonly used in folk theorems within the repeated games literature. However, contrary to repeated games in which efficiency is achieved as a possibility, our SPCS achieves efficiency as a necessary implication. We also show that strong Pareto efficiency is obtained when coalitions are allowed in normal form games. Moreover, we show that the SPCS solution leads to efficiency also for normal form games having no pure Nash equilibrium, given that one is willing to accept as a possible outcome of the game a ‘swinging’ behavior in which some of the players keep on moving between states. An important contribution of our approach is that, contrary to many cooperative game models which assume Pareto efficiency from the start, we justify such an assumption in a non-cooperative framework. This is undertaken using strategies that could be thought of as modelling farsighted tacit negotiation.

Finally, we note that other approaches to study farsightedness, some preceding Chwe (e.g., Greenberg 1990), and others following him (e.g., Mariotti 1997 and Xue 1998), were discussed in the literature. Some of these approaches, their interrelationships and their relationship to the SPCS will be discussed in Section 6.

The plan of this paper is as follows. In Section 2 we provide several motivating examples, mostly to demonstrate the advantage of farsighted solutions over myopic solutions and to illustrate some shortcomings in the currently existing farsighted solutions. In Section 3 we introduce the farsighted game and formally define our new farsighted solution concept – the Subgame Perfect Consistent Set (SPCS). The SPCS is similar to the LCS and LCCS in that it is based on a von Neumann Morgenstern (vN-M) type of stability, but differently from them it uses the reasoning of subgame perfection instead of indirect dominance. It is further demonstrated in Section 3 that the SPCS delivers the proper predictions in all the motivating examples previously discussed. In Section 4 we introduce farsighted normal form games, derived from standard one-shot normal form games by allowing players, or coalitions, to publicly and repeatedly, in their turn, change their previous actions. For this class of games we prove that any SPCS consists of essentially a single Pareto efficient outcome, and conversely, any Pareto efficient outcome weakly dominating a pure Nash equilibrium is the single member of a SPCS. Extended results concerning farsighted normal form games without pure Nash equilibrium are further pursued. In Section 5 we illustrate the significance of our findings by demonstrating, for example, that by contrast with the LCS and the LCCS, farsighted players who follow the SPCS reasoning are always able to share the monopolistic profit in Bertrand and Cournot competition settings, and are always able to achieve full coordination and Pareto efficiency in decentralized supply chain settings even when they cannot form coalitions. In Section 6 we provide the formal definitions of the LCS, LCCS and other related solution concepts, including Greenberg’s (1990) stable Standard of Behavior (SB) and its further examination by Xue (1998), and critically compare them to our SPCS while uncovering similarities and differences. We further demonstrate in this section that the various shortcomings of the LCS, as detailed by Chwe in the last section of his seminal contribution, are not valid for the SPCS. Section 7 concludes with a summary and suggestions for future research.

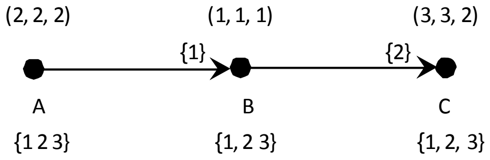

Figure 2.1: Transition tree in Example 2.1

2 Motivating Examples

To illustrate the advantage of farsighted solution concepts over myopic solution concepts, and in order to motivate our new farsighted solution concept, let us first consider the following two simple examples in which we compare the myopic set with the largest consistent set (LCS), introduced by Chwe (1994) (for a formal definition of the LCS and related discussion see Section 6).

Example 2.1 *The coalition formation game depicted in Figure 2.1 has only three states with transitions between states as illustrated in the figure. At state A, the coalition structure is the grand coalition,* $\{1\ 2\ 3\}$*, and the utility to each of the three players is* 2*. Player* 1 *can exit the grand coalition, resulting with the coalition structure* $\{1, 2\ 3\}$*, in which the utility to each of the players is* 1*. Then, from state B, player* 2 *can exit the coalition to state C, in which the utilities to the three players are* $(3, 3, 2)$*. No other moves are possible.*
Now, a move by player 1 *from state A to state B will reduce his utility from* 2 *to* 1*. Hence state A is in the* myopic set*. Furthermore, since state C is terminal, as no move away from it is possible, it is also in the* myopic set*. Thus, for Example 2.1, the* myopic set $= \{A, C\}$*.*
Next, let us consider the farsighted logic, as manifested by the LCS. State B cannot be a final state in the coalition formation game, since at this state player 2 *can move to a final state C and strictly improve his utility. Therefore, when player* 1 *moves to state B, he is certain that player* 2 *will subsequently move to state C. Since player* 1*'s utility at state C strictly exceeds his utility at state A, he will move from this state to state B. Thus, state A is not (farsighted) stable and the LCS of Example 2.1 consists of the singleton set* $\{C\}$*.*

In Example 2.1, player 1's myopia, inherited from the paradigm of Nash equilibrium in normal form games, prevented him from moving from state A, which would have led, via player 2's subsequent move, to a strictly higher utility. In Example 2.2 below, we demonstrate that myopia, which motivates a player to move to a state in which his utility is strictly higher, subsequently leads to a strictly inferior outcome.

Example 2.2 *Consider the coalition formation game displayed in Figure 2.2: Myopically, player* 1 *will move from state A to state B, since at B his utility,* 4*, is strictly higher than at state A, which is* 2*. Clearly though, player* 2 *will move from state B to state C, to increase his utility from* 0 *to* 6*. Thus, a move by player* 1 *from state A, dictated by the myopic paradigm, will prove costly to player* 1*. In Example 2.2, we have* myopic set $= \{C\}$*, while LCS* $= \{A, C\}$*. Thus, according to*

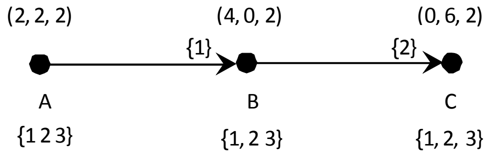

Figure 2.2: Transition tree in Example 2.2

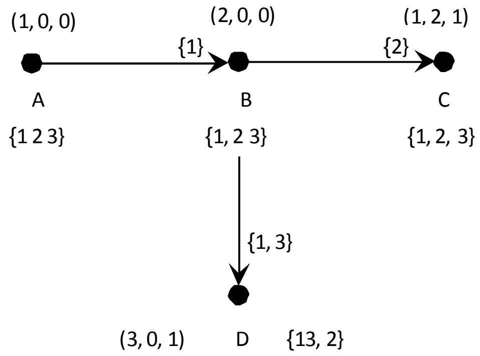

Figure 2.3: Transition tree in Example 2.3

the LCS, player 1 *understands the implication of a move to state A, and, therefore will not move from this state.*

Next, consider the following example which has motivated the introduction of a refinement of the LCS, the largest cautious consistent set (LCCS), by M&V (2004) (for a formal definition of the LCCS and related discussion see Section 6).

Example 2.3 *In the coalition formation game represented by Figure 2.3, states C and D are terminal states and, as such, both are contained in the LCS. State B is not contained in the LCS because there exist strictly profitable moves to final states, state C via a move by player* 2*, or state D via a move by coalition* $\{1, 3\}$*. Let us then consider state A. Since B is not a final state, a move by player* 1 *could subsequently lead to either state C, at which player* 1 *is equal off compared to state A, or to state D, at which player* 1 *is strictly better off than at state A. Since player* 1 *may end up at state C, by definition, state A is contained in the LCS. Thus, in Example 2.3, LCS* $= \{A, C, D\}$.
Intuitively, the nature of the LCCS refinement of the LCS is that a subset of players will move from a state if such a move will not lead to a final outcome at which they are strictly worse off, and could possibly lead to a final outcome at which they are strictly better off. In Example 2.3, a move by player 1 *from state A could either lead to state C, at which he is equal off, or to state D, at which he is strictly better off. Therefore, state A is not contained in the LCCS. That is, LCCS* $= \{C, D\}$.

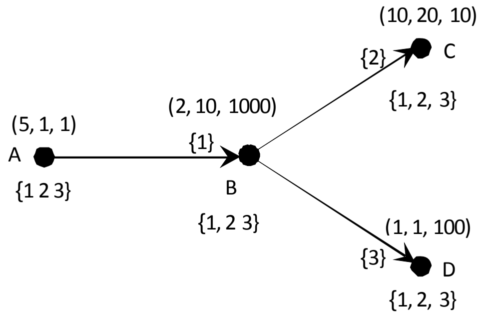

Figure 2.4: Transition tree in Example 2.4

Chwe was aware of a possible criticism of the LCS for being too inclusive and that it may contain possibly non-intuitive outcomes. Explicitly, “If Y is consistent and $a \in Y$, the interpretation is not that a *will be* stable, but that it is *possible* for a to be stable. If an outcome b is not contained in any consistent Y, the interpretation is that b cannot possibly be stable: there is no consistent story in which b is stable” (Chwe 1994; p. 303). Nevertheless, in Example 2.4 below we show that there are outcomes not contained in the LCS and for which there are very consistent stories.

Example 2.4 *Consider the coalition formation game associated with Figure 2.4. Again, since states C and D are terminal states, they both belong to the LCS and the LCCS. State B cannot be a final state, since player* 2 *can move from B to C and strictly increase his utility. Thus, state B is neither in the LCS nor in the LCCS. We further note that according to the farsighted reasoning of both the LCS and the LCCS, player* 3 *will not move from state B to state D, since at state D his utility is* 100*, which is strictly lower than his utility at B.*
Next, consider state A. Since, as clarified above, at state B player 2 *will move to state C and player* 3 *will not move to state D, and since the utility to player* 1 *at state C,* 10*, is strictly larger than his utility at state A, player* 1 *will move to state B. Thus, LCS = LCCS = $\{C, D\}$.*

However, let us consider again state B and player 3’s possible move from state B to state D. Indeed, player 3 is strictly worse off at state D compared to state B. However, being “farsighted”, player 3 also realizes that state B is not a final state, and given an opportunity, player 2 will move from state B to state C. Since player 3 is worse off at state C as compared to state D, it is quite compelling to conclude that given an opportunity to move, player 3 will move from state B to state D, to preempt a move of player 2 from state B to state C. Therefore, even though state $A \notin$ LCS, there is evidently a consistent and compelling story for state A to be stable: player 1 will prefer not to move from state A since he may end up at state D at which he is strictly worse off.

The farsighted consistent set we introduce in this paper (Section 3), referred to as Subgame Perfect Consistent Set (SPCS), contains the states A, C, and D when a pessimistic criterion is used (as in the LCS or LCCS). Our SPCS endows players with a more complete farsightedness than the limited farsightedness associated with the LCS or the LCCS. Indeed, players according

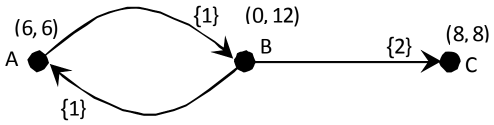

Figure 2.5: Transition tree in Example 2.5

to the SPCS solution gain their farsightedness by analyzing a sequential move tree associated with the game using the subgame perfect equilibrium concept.

Finally consider the following example that illustrates that one may need to analyze an infinite game tree in order to determine a SPCS.

Example 2.5 *Consider the game depicted in Figure 2.5. Clearly, since state C is final, it belongs to the LCS, LCCS, and SPCS. State B cannot be a final state, since player* 1 *can move from it to state A, at which he is strictly better off. Next, consider state A. Since a move by player* 1 *to state B can subsequently lead to a move by player* 1 *back to state A, state A is contained in the LCS. Furthermore, since player* 2 *is strictly better off at state B than at state C, according to the farsighted logic of the LCCS (and the LCS), player* 2 *will not move from state B to state C. Thus, a move by player* 1 *from state A to state B can only be followed by a return move by player* 1 *from state B to state A. Therefore, state A is also in the LCCS, and, we have, LCS = LCCS = $\{A, C\}$.*

But consider now the SPCS solution, and the corresponding game tree associated with Example 2.5. Since the LCS, LCCS as well as our SPCS do not specify for sure which coalition/player is moving at a state in which more than one coalition/player can move, in principle it could happen that player 1 gets a turn to move at state B for any finite number of times before player 2 gets a turn to move at this state. Thus, the game tree associated with Example 2.5 is infinite, wherein, player 1 will move from state A to state B and then for any finite number of times could stay at state B or move from state A to B and back to A.

Notwithstanding the difficulty that may arise from an infinite tree game, the analysis of the game is straightforward. Indeed, it is not a subgame perfect equilibrium strategy for player 1 to remain forever at state B. Thus, state B can never be a final outcome of the game, which implies that it is a subgame perfect equilibrium strategy for player 2 to move from state B to state C once such an opportunity arises. Player 1 is aware of player 2's optimal strategy, and therefore, he will prefer to repeatedly move from state A until player 2 is provided with an opportunity to be the first to move at state B. Thus, state A is not a stable or final outcome of the game according to the farsighted reasoning of the SPCS solution, and in Example 2.5, SPCS = $\{C\}$, and LCS = LCCS = $\{A, C\}$.

3 Model and Solution: The Subgame Perfect Consistent Set

We first introduce in this section our new farsighted solution concept and subsequently illustrate that it delivers the proper predictions for all the motivating examples.

3.1 The Farsighted Game and the SPCS

Consider a dynamic game in which players can act repeatedly and publicly by moving between states they care about. For example, consider a market consisting of firms engaged in price competition, in which the state is the vector of prices set by the firms. Players only care about the final state reached, irrespective of the sequence of actions that lead to it. Thus actions have no significant costs, and the time frame in which they are taken is short with no relevant discounting of the final state utility. Despite the dynamic nature of the game, it can be thought of as a one period interaction between the players.

This setting can be described as a game in effectiveness form (Chwe 1994, see also Greenberg 1990), denoted by $\langle N, Z, (u_i)_{i\in N}, (\rightarrow_S)_{S\subseteq N}\rangle$, where N is a finite set of players; Z is the set of possible states; the utility function $u_i : Z \to \mathbb{R}$ for each $i \in N$ determines player i's utility from a final state $z \in Z$; and the binary relation $\rightarrow_S \subseteq Z \times Z$ for each $S \subseteq N$ describes the players ability to change the current state, where $z^1 \rightarrow_S z^2$ for $z^1, z^2 \in Z$ means that coalition S of players can alter the current state z^1 by moving to a new state z^2 (with the empty coalition always having null effectiveness, i.e. $\rightarrow_\emptyset = \emptyset$). For example, in the price competition setting, Z can be the set of all price vectors possibly set by the firms, and the effectiveness of a singleton coalition $S = \{i\}$, consisting of a single player i, can allow the player to alter the current state by changing their own price without changing the prices set by the other players. Furthermore, in this example, the effectiveness of a coalition consisting of two or more players can describe agreements between members of the coalition that allow particular simultaneous changes in the prices set by all members of the coalition (again without changing the prices set by players outside the coalition).

As mentioned above, we envision a perfect information game, in which all actions are public, which we call the Farsighted Game (FG). The complete description of the game is given by the following sequence of events. First, an initial state in Z is publicly chosen by nature. Then, a possibly infinite sequence of stages initiates, each stage consisting of the following two steps: (Step 1) some coalition is publicly selected by nature; and (Step 2) according to its effectiveness, the selected coalition can publicly keep the current state unchanged or move to a new state in Z.

Several remarks are in order concerning the definition of this game. First, the choice of the initial state by nature is made only to reflect the fact that none of the players can affect this choice – one could alternatively define the game with a prespecified initial state instead of random selection. Second, in a setting where players can repeatedly observe the actions made by others and adjust their own actions accordingly, full farsightedness necessitates unbounded repeated play: actions are made taking into account the fact that any move by a coalition may be counteracted with a further move by another coalition, without limit. Such an approach is in the spirit of modelling farsighted tacit negotiation.

Finally, any selection made by nature can be thought of as a random draw from some prespecified full support distribution. It is possible for our solution concept to be insensitive to the particular distribution used, in the spirit of existing farsighted solution concepts. To understand how this can be achieved, we need to discuss first the strategies in this game. A strategy of a coalition S specifies the action taken by this coalition after any history in the game (even off equilibrium path) at a point where this coalition might be selected to make a choice. A strategy profile is a vector specifying the strategy of each coalition S. When restricted to a subgame that follows a given history, a strategy profile can lead to a distribution over final states: either states in Z from which no coalition moves, or what we call 'swinging' final states in the case of non-converging actions. Therefore when players make choices in the game, they choose between actions that each lead to a distribution over utilities from final states. For ease of presentation, we assume for now that utility functions are extended to assign utility $-\infty$ to a swinging final state, so that it will never be reached. In Section 4 we discuss the possibility and implications of assuming instead that swinging is assigned a finite utility, and so could be considered a final state. When players make a choice, they can use various decision rules. Whichever rule they use, we assume that it is consistent with strict preference for strictly first order stochastically dominating distributions. For our analysis, the important implication of this assumption arises when comparing a degenerate distribution, i.e. some utility s for sure, with a non-degenerate distribution. In this case, the sure outcome s is strictly worse than any non-degenerate distribution that gives positive probability only to utilities at least as high as s. If players use expected utility maximization, the solution may depend on the distribution determining the selections by nature. To allow a distribution invariant solution concept, we can assume, similarly to Chwe (1994) and (particularly to) M&V (2004), that players use a pessimistic criterion to make their choices:

Definition 3.1 *Pessimistic criterion for players: for any two distributions d_i^1, d_i^2 over player i's utilities, player i strictly prefers d_i^1 to d_i^2 if the lowest utility in the support of d_i^1 is strictly higher than that of d_i^2, or if d_i^1 strictly first order stochastically dominates d_i^2. Weak preference and indifference are defined from the strict preference in the usual way.*

This pessimistic criterion implies that a sure outcome s is strictly preferred to any non-degenerate distribution that gives positive probability to some utility strictly lower than s.

The extension of the decision criterion to coalitions is as follows.

Definition 3.2 *Decision criterion for coalitions: for any two distributions d_S^1, d_S^2 over vectors of utilities for players in a coalition S, coalition S strictly prefers d_S^1 to d_S^2 if some players $i \in S$ strictly prefer d_i^1 to d_i^2 while the other players in S (if any) are indifferent, where d_i^1, d_i^2 are the restrictions of the distributions d_S^1, d_S^2 to the utilities of player i. Weak preference and indifference are defined from the strict preference in the usual way.*

In analyzing the farsighted game described above, we seek a solution concept that produces a set of states considered farsighted stable. As mentioned in the Introduction, our proposed solution is based on subgame perfection, with recursion under some decision rule, and a consistency notion in the spirit of vN-M stable sets (von Neumann and Morgenstern 1944). Roughly speaking, this consistency notion requires the solution set X of states to satisfy the following two properties: (a) farsighted internal stability: for each state $z \in X$, no coalition prefers to move away from z, anticipating that such a move would eventually end up in X; and (b) farsighted external stability: for each state $z \notin X$, there always exists a coalition that prefers to move away from z, again anticipating that such a move would eventually end up in X. To formalize where a move might potentially lead to, we say that a state z^2 is reachable from a state z^1 if for the subgame in which z^1 is the initial state, there exists a subgame perfect equilibrium strategy profile such that z^2 has positive probability to be a final state. Thus, formally, we define our solution as follows.

Definition 3.3 *A finite set of states* $X \subseteq Z$ *is a* Subgame Perfect Consistent Set (SPCS) of a Farsighted Game (FG) *if there exists a subgame perfect equilibrium strategy profile such that:*
(i) $z \in X$ *if, and only if, no coalition moves from* z *when it is selected as an initial state (thus making it a final state for sure); and*
(ii) after choices by all coalitions to stay at an initial state $z \in Z$ *(on or off equilibrium path), if* z *is reachable from itself then* z *is the final state for sure; after an initial move from an initial state* $z \in Z$ *by some coalition (on or off equilibrium path), all reachable states in* X*, and only them, have positive probability to be final states.*

We say that a strategy profile supports X as a SPCS if it satisfies the requirements in Definition 3.3 when X is a SPCS. It will be also useful to consider the final states that a SPCS predicts after the selection of initial states outside the consistent set:

Definition 3.4 *For a SPCS* X *and any* $z \in Z \backslash X$*, define the* attraction set*,* $T_X(z)$*, as the set of all states in* X *that, according to some subgame perfect equilibrium supporting* X *as a SPCS, have positive probability to be final states following the selection of* z *as an initial state.*

As we will show in the next subsection, a SPCS trivially exists for the motivating examples in Section 2. More generally, whenever Z is finite and the effectiveness relation is acyclic (as in the first four motivating examples), there exists a non-empty SPCS. To see this observe that for a game satisfying these assumptions, the corresponding FG has a subgame perfect equilibrium that spans all indifferent moves at a given state, achieved by allowing the move choice of a coalition to depend on the realization of the number of coalitions (including the empty coalition) selected to play since the arrival at this state; such a strategy profile is referred to as having full support. Moreover, the strategy profile can be chosen to satisfy the first part of property (*ii*) in Definition 3.3, and to be stationary in the sense that the action of any coalition at a given state is chosen in the same way whenever arriving at this state. Let X be the set of states $z \in Z$ for which, on equilibrium path, the strategy profile dictates no move of any coalition when z is selected as an initial state. This set is non-empty because it includes in particular terminal states according to the effectiveness relation, i.e. states which no coalition can alter, the existence of which is ensured because Z is finite and the effectiveness relation is acyclic. By definition, the strategy profile satisfies property (i) and the first part of property (ii) in Definition 3.3. Moreover, consider a state z' having positive probability to be a final state after some initial move from an initial state (on or off equilibrium path), which implies that z' is reachable. Then, by stationarity of the strategy profile, there does not exist a coalition that moves away from z' when it is selected as an initial state, because the existence of such a coalition would contradict the supposition that z' can be a final state. Therefore z' must be in X. Now consider a reachable state $z' \in X$ after some initial move from an initial state (on or off equilibrium path). Since the strategy profile has full support, there is positive probability that z' is a final state. This verifies the second part of property (ii) in Definition 3.3.

When analyzing farsighted normal form games in Section 4, we will show that the assumptions that Z is finite and the effectiveness relation is acyclic are not necessary for the issue of existence. Indeed, a non-empty SPCS can exist also for games with an infinite Z or when the effectiveness relation is cyclic. However, the following example demonstrates that in this very large set of games, a SPCS may fail to exist. In this example the failure is due to the lack of a subgame perfect equilibrium.

Example 3.1 *There is one player,* Z *is the set of natural numbers,* $u(z) = z$ *for each* $z \in Z$*, and the player can only move from* $z = 1$ *to any* $z > 1$ *(with no other possible moves included in the effectiveness relation). In this example there is no SPCS because there is no subgame perfect equilibrium: when* $z = 1$ *is selected as an initial state, for any choice the player can make there exists a strictly better choice.*

Despite the possibility of non-existence, the following property of a SPCS is useful.

Proposition 3.1 *If a SPCS exists, then it is non-empty.*

Proof. An empty SPCS implies the existence of a subgame perfect equilibrium that dictates a move by some coalition at any state $z \in Z$ on the equilibrium path, i.e. swinging. However, starting from any state $z \in Z$, all players prefer to stay at z which is strictly better than swinging, contradicting property (i) in Definition 3.3. ■

3.2 Applying the SPCS to the Motivating Examples

We now demonstrate the usefulness of our model by providing satisfactory predictions for the motivating examples in Section 2. In each of these examples there is a unique SPCS supported by some strategy profile. We describe this strategy profile for each of the examples and explain why it forms a subgame perfect equilibrium (it will be immediate to see that properties (i) and (ii) in definition 3.3 are satisfied).

In Example 2.1, the unique SPCS is $\{C\}$ with attraction sets $T_X(z) = \{C\}$ for each $z \in Z \backslash X$. This is supported by the strategy profile in which player 1 moves from A to B and player 2 moves from B to C when they are selected to make a choice. This forms a subgame perfect equilibrium because player 2 prefers C to B and player 1 prefers C to A. Similarly in Example 2.2, the unique SPCS is $\{A, C\}$ with the attraction set $T_X(B) = C$. This is supported by the strategy profile in which player 1 does not move from A and player 2 moves from B to C. This forms a subgame perfect equilibrium because player 2 prefers C to B but player 1 prefers A to C.

In Example 2.3, the unique SPCS is $\{C, D\}$ with attraction sets $T_X(z) = \{C, D\}$ for each $z \in Z \backslash X$. This is supported by the strategy profile in which player 1 moves from A to B, player 2 moves from B to C and coalition $\{1, 3\}$ moves from B to D. This forms a subgame perfect equilibrium because player 2 prefers C to B, coalition $\{1, 3\}$ prefers D to B and player 1 prefers the first order stochastically dominating distribution of utilities resulting from C and D to the degenerate distribution of utilities resulting from A.

In Example 2.4, if players use the pessimistic criterion, the unique SPCS is $\{A, C, D\}$ with the attraction set $T_X(B) = \{C, D\}$. This is supported by the strategy profile in which player 1 does not move from A, player 2 moves from B to C and player 3 moves from B to D. This forms a subgame perfect equilibrium because player 2 prefers C to B and player 3 prefers D, the final state after a move to D, to C, the final state after not moving from B and anticipating a later move of player 2 from B to C; player 1 uses the pessimistic criterion to prefer the degenerate distribution of utilities resulting from A to the distribution of utilities from C and D that gives positive probability to D, which is a worse outcome for player 1 than A. Note that in this example the SPCS predicts that player 3 will move from B to D despite the fact that D is worse than B, because this player perceives C rather than B as a final state. This example demonstrates the extent to which the SPCS solution is more farsighted than the LCS and the LCCS, which would predict player 3 not moving from B.

Finally, in Example 2.5, the unique SPCS is $\{C\}$ with attraction sets $T_X(z) = \{C\}$ for each $z \in Z \backslash X$. This is supported by the strategy profile in which player 1 moves from A to B and waits there until player 2 is selected to make a choice, in which case player 2 moves from B to C; however, in the case that player 2 chooses to remain at B, player 1 returns to A and does not move anymore. This strategy profile forms a subgame perfect equilibrium because player 2 prefers C, the final state after a move to C, to A, the final state after not moving from B and anticipating a later move of player 1 from B to A with no further moves. Furthermore, player 1 prefers a move from A, the final state after not moving from A, to C, the final state after moving from A to B and anticipating a later move of player 2 from B to C; in the case that player 2 does not move from B, player 1 prefers A, the final state after returning to A and not moving anymore, to B, the final state after remaining at B with the hope that player 2 will move to C. Note further that player 1 is not restricted by the first part of property (ii) of the SPCS to stay at B after both of them did not move from B, because B is not reachable from itself. This example shows the importance of non-stationary strategies in supporting the desirable efficient final state C. Player 1's choice depends not only on the current state but also on the history of play by player 2: if player 2 has not yet been selected to make a choice, player 1 gives player 2 the opportunity to move to the efficient state C; but after player 2 was selected to make a choice and chose not to move, player 1 gives up and returns to A. Note that the fact that this strategy does not allow player 2 more than one opportunity to move is not crucial: one could imagine player 1 giving player 2 a finite number of such opportunities to move until giving up and returning to A; in any such strategy, player 2 will eventually move to C on the equilibrium path. Similarly to Example 2.4, this example also demonstrates the extent to which the SPCS yields more appropriate predictions than the LCS or the LCCS, which would argue that player 1 will not move from A.

4 Farsighted Normal form games

In this section we analyze normal form games when they are viewed as farsighted games.

Definition 4.1 *(i) A normal form game is a game in effectiveness form for which* $Z = \times_{i \in N} A_i$*, where* A_i *is the set of alternatives available to player* i*, and for each* $z^1, z^2 \in Z$ *and each coalition* $S \subseteq N$*,* $S \neq \emptyset$*, the effectiveness relation* $z^1 \rightarrow_S z^2$ *holds if, and only if,* $z_i^1 = z_i^2$ *for each* $i \notin S$*.*
(ii) A normal form game without coalitions is a normal form game with empty effectiveness relation $\rightarrow_S$ *for all coalitions* S *having at least two members.*

We show that the SPCS solution provides a striking conclusion in the farsighted analysis of normal form games: Pareto efficiency is always achieved. Indeed, aside for the approach of cooperative game analysis, wherein Pareto efficiency is assumed rather than derived, we are unfamiliar with any other single, unified paradigm in the literature that always leads to such a conclusion, in particular when coalition deviations are not permitted.

Note that in our framework, pure Nash equilibrium is defined as follows.

Definition 4.2 *In a normal form game without coalitions,* $e \in Z$ *is a pure Nash equilibrium if* $u_i(e) \geq u_i(z)$ *for each* $i \in N$ *and* $z \in Z$ *such that* $e \rightarrow_{\{i\}} z$*.*[1]

To state the main result we need two additional pieces of terminology/notation. First, we say that a state $z \in Z$ is Pareto efficient if there does not exist a state $z' \in Z$ such that $u_i(z') > u_i(z)$ for all $i \in N$. Similarly, we say that a state $z \in Z$ is strongly Pareto efficient if there does not exist a state $z' \in Z$ such that $u_i(z') \geq u_i(z)$ for all $i \in N$ with a strict inequality for at least one player (i.e., for the grand coalition N, there is no strictly preferred state). Second, whenever a subset $X \subseteq Z$ consists of states that are all equivalent for all players, i.e. there exists $z^* \in X$ such that $u_i(z') = u_i(z^*)$ for each player i and $z' \in X$, we write that $X \approx \{z^*\}$.

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[1] Our analysis also applies when considering Stackelberg games (and extensive form games with perfect information in general) in their reduced normal form. This is justified by the view that in the farsighted perspective, the leader and the follower are indistinguishable. Thus, in this case, the results concerning farsighted normal form games apply when replacing pure Nash equilibria with pure Stackelberg equilibria.

We can now state our main result.

Theorem 4.1 *(1) For any normal form game, with or without coalitions, if $X \subseteq Z$ is a SPCS, then $X \approx \{z^*\}$ for some Pareto efficient $z^* \in Z$; and*
(2a) For any normal form game without coalitions, if $X \approx \{z^*\}$ for some Pareto efficient $z^* \in Z$ such that $u_i(z^*) \geq u_i(e)$ for some pure Nash equilibrium $e \in Z$ and all $i \in N$, then X is a SPCS;
(2b) For any normal form game with coalitions, if $X \approx \{z^*\}$ for some strongly Pareto efficient $z^* \in Z$, then X is a SPCS.[2]

Proof. We first prove (1). The proof is identical for the cases of a normal form game with or without coalitions. Let $X \subseteq Z$ be a SPCS, and consider some subgame perfect equilibrium strategy profile P that supports X as a SPCS under a decision rule that is consistent with strict preference for strictly first order stochastically dominating distributions. We first show that any state $z^2 \in X$ is reachable from any state $z^1 \in Z$. For the subgame in which z^1 is selected as an initial state, consider the strategy profile P' according to which each player i, when selected to make a choice whether to keep the current state or to move to a new state, moves by choosing the alternative $a_i \in A_i$ corresponding to z^2, except when at most one player has not done that, in which case the strategy profile P' continues exactly as P does following either the selection of z^2 as an initial state or, possibly, after an initial move by one player from the initial state z^2. Since P dictates no moves when z^2 is selected as an initial state – see property (i) of the SPCS – z^2 is for sure also the final state according to P'. Moreover, under the chosen decision rule, no player strictly prefers to deviate from P' because no player strictly prefers to deviate from P's dictate to stay at z^2. This proves the reachability of z^2 from z^1.

Now suppose that there are at least two distinct states in X and a player i that is not indifferent between them. Since a SPCS is finite, there is a worst state $\check{z} \in X$ for player i. Consider the case where $\check{z}$ is selected as an initial state and player i is selected to make a choice whether to keep $\check{z}$ or to move to a new state. Since $\check{z} \in X$ and X is a SPCS, player i will keep $\check{z}$, anticipating it as a final state – see property (i) of the SPCS. But note that in case of an initial move by player i off equilibrium path from $\check{z}$ to a new state, the final state must be in X, and, since all states in X are reachable following a move away from $\check{z}$, there is a positive probability that the final state will be strictly better than $\check{z}$ for player i, again because X is a SPCS – see property (ii). Therefore, player i can either choose $\check{z}$ for sure, or alternatively, can choose a first order stochastically dominating distribution of utilities, as $\check{z}$ is the worst state in X for i. Since the first order stochastically dominating distribution of utilities is strictly preferred, the choice of player i to keep $\check{z}$ violates the subgame perfect equilibrium. Since this argument holds for any subgame perfect equilibrium strategy profile, we derived a contradiction to the assumption that X is a SPCS. Therefore, since X cannot be empty by Proposition 3.1, it must consist of states that are all equivalent for all players, i.e. $X \approx \{z^*\}$ for some $z^* \in Z$.

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[2] Under the additional assumption that utility functions are continuous (trivially satisfied when Z is finite), the theorem also holds when modifying the requirement of a SPCS to be a compact set instead of a finite set.

Suppose now that z^* is not Pareto efficient. Then there exists $d \in Z \backslash X$ such that $u_i(d) > u_i(z^*)$ for each $i \in N$. Consider the case where d is selected as an initial state. By the second part of property (*ii*) of the SPCS, any player i, selected to make a choice whether to keep d or move to a new state, anticipates that any choice of an initial move on equilibrium path will lead for sure to a final state equivalent to z^*. Since all players strictly prefer d to z^* and the strategies form a subgame perfect equilibrium, any player anticipates that players selected later to make a choice would prefer to stay at d, since d is reachable from itself, thus making d a final state – see the first part of property (*ii*) of the SPCS. Therefore, no player will move from d, contradicting property (*i*) of the SPCS because $d \notin X$. Thus z^* must be Pareto efficient.

Next we prove (2a). Suppose that $X \approx \{z^*\}$ for some Pareto efficient $z^* \in Z$ such that $u_i(z^*) \geq u_i(e)$ for some pure Nash equilibrium $e \in Z$ and all $i \in N$. Consider the strategy profile according to which no player moves if an initial state in X is selected; if an initial move is made away from an initial state in X, or if all previously selected players stayed at an initial state not in X (except if all were selected and none moved from an initial state reachable from itself), then each $z^f \in X$, the choice of which depends on the realization of the number of coalitions (including the empty coalition) previously selected to play, becomes the final state by each player moving by choosing the alternative corresponding to z^f and then not moving anymore; after any deviation from the above (except after choices by all coalitions to stay at an initial state reachable from itself), each player moves to the alternative corresponding to e and then does not move anymore. This strategy profile clearly satisfies both properties (*i*) and (*ii*) of the SPCS. We now verify that it forms a subgame perfect equilibrium of the FG. First note that on equilibrium path the final state is equivalent for all players to z^*. An initial state not in X will not be a final state because some player would prefer to make an initial move so as to reach a preferred final state in X. Any multiple-move deviation from equilibrium path by a single player leads to the final state e. Since $u_i(z^*) \geq u_i(e)$ for all $i \in N$, no player prefers to deviate from equilibrium path, leading indeed to a final state in X. Moreover, when reaching e off equilibrium path, since e is a pure Nash equilibrium, no player prefers to deviate, as this can only lead to a final state at most as good as e. Consequently the strategy profile is a subgame perfect equilibrium.

Now we prove (2b). Suppose that $X \approx \{z^*\}$ for some strongly Pareto efficient $z^* \in Z$. Consider the stationary strategy profile according to which at any state $z \in Z$, each coalition S which is a strict subset of N stays at z, and each $z^f \in X$, the choice of which depends on the realization of the number of coalitions selected to play since the arrival at z, becomes the final state by the grand coalition N moving from z to z^f (except after choices by all coalitions off equilibrium path to stay at an initial state reachable from itself). This strategy profile clearly satisfies properties (*i*) and (*ii)* of the SPCS. To see that it forms a subgame perfect equilibrium note that a coalition S which is a strict subset of N does not gain from deviation, as this leads to a final state which is either equivalent to z^* or swinging (i.e., utility $-\infty$ for all players). Moreover, since z^* is strongly Pareto efficient, there is no state z strictly preferred to z^* for the grand coalition N, thus there is no gain from deviation also for the grand coalition. ■

For a normal form game without coalitions, the strategies supporting a SPCS in Theorem 4.1 are similar to 'grim-trigger' strategies commonly used in folk theorems within the repeated games literature (see, e.g., Osborne and Rubinstein 1994), adapted to our setting where a repeated stage game is not required. The strategies involve a threat of reaching an undesirable outcome off equilibrium path in order to create incentives to reach a good outcome on equilibrium path. These strategies could be thought of as social norms that are publicly known to all players in the game. Since no player has an incentive to deviate from acting according to these social norms, they are indeed implemented, leading to the outcome of the corresponding SPCS. But note that contrary to folk theorems in repeated games, which do not claim to achieve efficiency, we achieve efficiency as a necessary implication. Thus the contribution of Theorem 4.1 is the conclusion that such social norms must lead to Pareto efficiency, which is achieved rather than assumed.

Theorem 4.1 implies that a game with no Pareto efficient states cannot have a SPCS. Existence of Pareto efficient states is a mild condition satisfied in many interesting settings as demonstrated in the following Example.

Example 4.1 *Prisoner's Dilemma. In this normal form game without coalitions there are two players, player R (the row player) and player C (the column player), each having two available alternatives and utilities as in the following matrix.*

$1,1$	$4,0$
$0,4$	$3,3$

For this game there is a unique SPCS $X = \{z^*\}$ *with* $u_i(z^*) = 3$ *for both* i*, and the attraction sets are* $T_X(z) = X$ *for each* $z \in Z \backslash X$*. This SPCS is supported by the following strategy profile, as in the proof of Theorem 4.1: after a selection of any initial state, each player moves to the alternative corresponding to* z^* *and then does not move anymore, unless some player previously did not do so, in which case each player moves to the alternative corresponding to the Nash equilibrium* $(1,1)$ *and then does not move anymore. As a result, irrespective of the initial state, the SPCS predicts the cooperative outcome in the game. Note that in this example both the LCS and the LCCS are equal to the top-left right-bottom diagonal of the matrix.*
Note also that by part (2b) of Theorem 4.1, the cooperative state forms a SPCS also when modifying this example to a normal form game with coalitions.

Theorem 4.1 provides sufficient but not necessary conditions for a Pareto efficient state to form a SPCS. Indeed, as demonstrated in the following example, we may have normal form games where $X = \{z^*\}$ is a SPCS for some Pareto efficient z^* that does not weakly dominate any pure Nash equilibrium.

Example 4.2 *Consider a normal form game without coalitions with two players (R and C), where* *each player has two alternatives and the utilities are described by the following matrix.*

$1,1$	$1,2$
$0,2$	$2,2$

For simplicity, we denote states in Z *by their utility pairs according to the matrix. In this game the unique pure Nash equilibrium is* $(2,2)$*. By Theorem 4.1, the set* $X = \{(2,2)\}$ *is a SPCS. But note that the set* $X = \{(1,2)\}$ *is also a SPCS. This is true due to the following stationary strategy profile: at* $(1,2)$ *both players stay; at* $(2,2)$ *R stays and C moves to* $(0,2)$*; at* $(0,2)$ *R moves to* $(1,1)$ *and C stays; at* $(1,1)$ *R stays and C moves to* $(1,2)$*. To see that indeed this strategy profile forms a subgame perfect equilibrium note that an attempt of R to improve by moving to* $(2,2)$ *fails because C punishes by moving to* $(0,2)$*; C does not mind moving to* $(0,2)$ *because he would either stay there and receive utility* 2*, or, if R moved to* $(1,1)$*, C would move and end up at* $(1,2)$*; if, on the other hand, C stays at* $(2,2)$ *then R stays there also, leading to payoff* 2 *for C, which is not strictly better than the payoff at* $(1,2)$*.*

Another example going beyond Theorem 4.1 concerns normal form games having a non empty SPCS despite the non-existence of a pure Nash equilibrium.

Example 4.3 *Matching Pennies. Consider a normal form game without coalitions with two players (R and C), where each player has two alternatives and the utilities are described by the following matrix.*

$1,-1$	$-1,1$
$-1,1$	$1,-1$

In this game there is no pure Nash equilibrium, so Theorem 4.1 does not apply. Still, any set consisting of a single state in Z *is a SPCS, supported by a strategy profile similar to the one used in Example 4.2. In this case, the player that receives utility* -1 *in the single state of the given SPCS prefers to accept this utility than to insist on moving, which would lead to swinging with utility* $-\infty$*. For comparison, note that the LCS* $= Z$*, but the prediction of the final state according to the two solution concepts is different: the LCS predicts the initial state, whereas the SPCS predicts the unique state in any given SPCS. Further note that in this example the LCCS is empty, but there are two CCSs (see also discussion in Section 6.1), each corresponding to a diagonal in the matrix. Such sets are also SPCSs, supported by a strategy profile in which the players move only from states not on the SPCS diagonal.*

We now provide an extension of the analysis above that applies to any normal form game, including games having no pure Nash equilibrium. As explained in Section 3, a final state in the game can be either a state in Z to which the sequence of actions in the play converges, or a 'swinging' final state in case of non-convergence. Previously we assumed that swinging is assigned utility $-\infty$, so it is never reached as a final state. Assume now that swinging can be assigned a finite utility, normalized to be 0, and so could be considered a final state. Let $\bar{Z} = Z \cup \{w\}$, where

w denotes any swinging final state. Accordingly consider an extension of the Definition 3.3 that allows w to be a member of a SPCS. Indeed, regardless of the other players' actions, any coalition can force a swinging final state in a farsighted normal form game if it so desires, simply by always electing to change the current state whenever it is selected to make a choice. Note that Proposition 3.1 extends here: a SPCS cannot be empty as this implies that it includes a swinging final state. The possibility of a swinging final state allows us to extend Theorem 4.1 to the following result.

Theorem 4.2 *For any normal form game (with or without coalitions), $X \subseteq \bar{Z}$ is a SPCS if, and only if, $X \approx \{z^*\}$ for some Pareto efficient state $z^* \in \bar{Z}$ such that $u_i(z^*) \geq 0$ for all $i \in N$.*

Proof. Suppose first that $X \subseteq Z$ is a SPCS. Using the argument in the proof of Theorem 4.1, $X \approx \{z^*\}$ for some Pareto efficient $z^* \in Z$. Suppose now that $u_i(z^*) < 0$ for some player $i \in N$. Consider the case where z^* is selected as an initial state and player i is selected to make a choice whether to keep z^* or to move to a new state. Since $z^* \in X$ and X is a SPCS, player i will keep z^*, anticipating it as a final state – see property (i) of the SPCS. But this player could instead adopt a strategy in which he always changes the current state whenever such an opportunity arises, thus forcing a swinging final state. Since the utility, 0, of a swinging final state is strictly higher than $u_i(z^*)$, the choice of player i to keep z^* violates the subgame perfect equilibrium. Thus we derived a contradiction to the assumption that X is a SPCS. Therefore $u_i(z^*) \geq 0$ for all $i \in N$.
For the other direction, suppose that $X \approx \{z^*\}$ for some Pareto efficient $z^* \in Z$ such that $u_i(z^*) \geq 0$ for all $i \in N$.
Consider the strategy profile according to which no coalition moves if an initial state in X is selected; if an initial move is made away from an initial state in X, or if all previously selected coalitions stayed at an initial state not in X (except if all were selected and none moved from an initial state reachable from itself), then each $z^f \in X$, the choice of which depends on the realization of the number of coalitions previously selected to play, becomes the final state by each coalition moving by choosing the alternatives corresponding to z^f and then not moving anymore; after any deviation from the above (except after choices by all coalitions to stay at an initial state reachable from itself), each coalition changes the current state whenever it is selected to make a choice. This strategy profile clearly satisfies both properties (i) and (ii) of the SPCS. We now verify that it forms a subgame perfect equilibrium of the FG. First note that on equilibrium path the final state is equivalent for all players to z^*. An initial state not in X will not be a final state because some coalition would prefer to make an initial move so as to reach a preferred final state in X. Any multiple-move deviation from equilibrium path by a single coalition leads to a swinging final state, in which case each player has utility 0. Since $u_i(z^*) \geq 0$ for all $i \in N$, no coalition prefers to deviate from the equilibrium path, leading indeed to a final state in X. Moreover, when reaching a swinging final state off equilibrium path, no coalition can change the swinging final state because it is forced by the other coalitions when they are selected to make a choice, so no coalition prefers to deviate. Consequently the strategy profile is a subgame perfect equilibrium. ■

Theorem 4.2 provides us with necessary and sufficient conditions for the existence of a non-empty SPCS when swinging is allowed and its normalized utility is zero for all players. Reflecting back on Example 4.3 we can appreciate the effect of allowing a swinging behavior.

Example 4.4 *Matching Pennies with swinging. By Theorem 4.2, the singleton set* $\{w\}$ *consisting of a swinging final state is the unique SPCS. This is true because* w *is the unique Pareto efficient state in* $\bar{Z}$ *providing all players with non-negative utilities (note that any other state is also Pareto efficient but fails the non-negativity condition).*

5 Applications to Various Oligopolistic and Supply Chain Settings

In this section we briefly illustrate the application of the SPCS solution concept to analyze various classical competitive settings.

5.1 Bertrand Competition

Consider a market with n sellers of a homogeneous product that are engaged in price competition, in which each seller i sets price p_i. The sellers face a downward sloping demand function $D(p)$, where p is the lowest among the prices they set. Suppose that the sellers have equal sales power, so that when the same price is set by several sellers, each of them sells the same quantity. Assume further that there are no fixed costs and the unit cost of the product for seller i is c (when the unit costs are not all equal there is no real price competition because the seller with the lowest unit cost can always set price slightly below the second lowest unit cost and gain the entire market alone).

It is well known that when this situation is viewed as a normal form game, there is a unique Nash equilibrium in prices: all sellers set the price $p^* = c$ and the market is split between them equally with zero profit to each of them.

For this setting a SPCS is $X = \{z^*\}$ where z^* is a Pareto efficient state in which several (possibly all) sellers each set the monopolistic price, i.e. the price p that solves

$$\max_p (p - c)D(p),$$

which allows them to share the monopolistic profit equally between them, while the remaining sellers (if any) set some price higher than the monopolistic price and receive zero profits. This follows from Theorem 4.1 because such states z^* are the only Pareto efficient states in Z that weakly Pareto dominate the unique Nash equilibrium state.

5.2 Cournot Competition

Consider a market with n sellers of a homogeneous product that are engaged in quantity competition, in which each seller i sets quantity q_i. The sellers face a decreasing inverse demand function determining the market price as, for simplicity, $\max\{a - Q, 0\}$, where $Q = \sum_i q_i$ is the total quantity sold. Suppose that there are no fixed costs and the unit cost of the product for seller i is c_i. Thus the profit to seller i is $\pi_i \equiv q_i(a - Q - c_i)$ when $Q \leq a$ and zero otherwise.

It is well known that this normal form game has a unique Nash equilibrium in quantities, in which player i sells quantity

$$q_i^* = a - c_i - \frac{1}{n+1} \sum\nolimits_j (a - c_j)$$

and makes profit $\pi_i^* = (q_i^*)^2$ (this holds under the assumption that $a \geq (n+1)\max_j c_j - \sum_j c_j$, otherwise the market is not large enough to include all sellers in equilibrium). In the symmetric case in which $c_i = c$ for all i, each player sells in equilibrium $\frac{a-c}{n+1}$ and makes profit $(\frac{a-c}{n+1})^2$.

To find the SPCS solution for this setting, Theorem 4.1 says that we need to compute the Pareto efficient states that weakly Pareto dominate the unique Nash equilibrium state. Since in any state $z \in Z$, the quantity sold $q_j = \frac{\pi_j}{a-c_j-Q}$ for all j, the profit π_i of any seller i can be written as a function f of the total quantity sold and the other sellers' profits, given by

$$f[Q, (\pi_j)_{j \neq i}] \equiv (Q - \sum\nolimits_{j \neq i} \frac{\pi_j}{a - c_j - Q})(a - c_i - Q).$$

Therefore a Pareto efficient vector of profits, for which $\pi_j \geq \pi_j^*$ for all $j \neq i$, must satisfy that π_i maximizes $f[Q, (\pi_j)_{j \neq i}]$ over the variable Q and subject to the constraint $\pi_i \geq \pi_i^*$.

In the symmetric case, $f[Q, (\pi_j)_{j \neq i}]$ simplifies to $Q(a-c-Q) - \sum_{j \neq i} \pi_j$, for which the maximal Q equals $\frac{a-c}{2}$, independently of the other sellers' profits. In this case seller i's profit is $\pi_i = (\frac{a-c}{2})^2 - \sum_{j \neq i} \pi_j$, as long as $\pi_j \geq \pi_j^*$ for all j (including i), and the total profit is $(\frac{a-c}{2})^2$. Such Pareto efficient states always exist because for all $n \geq 2$, the total profit is always higher than the total Nash equilibrium profit, $n(\frac{a-c}{n+1})^2$. This allows the total profit increase to be shared in some way between the sellers. Therefore the SPCS solution predicts that the sellers would share between them a monopolistic total quantity, allowing them to share a monopolistic total profit in a way that improves for each of them on the Nash equilibrium profit.

5.3 Decentralized Supply Chain

The SPCS can be used to analyze decentralized supply chains and can be shown to lead to full coordination and Pareto efficiency. We demonstrate this general point in the classic setting of a vertical decentralized supply chain, based on the newsvendor model, with a single supplier and several competing retailers. As is well known, system coordination and Pareto efficiency are trivially achieved in a bargaining/negotiation modelling of this setting (see, e.g., Nagarajan and Bassok 2008). We demonstrate that they can also be achieved in a non-cooperative modelling of this problem when players are assumed to be farsighted. Moreover, coordination is achieved contrary to the inability of revenue sharing contracts to do so in general (see Cachon and Lariviere 2005).

There is a market consisting of n retailers that face uncertain demand for a differentiated product, where retailer i's demand $D_i(p_1, ..., p_n, \varepsilon)$ is decreasing in their own unit price p_i, increasing in each of the other retailers' unit prices p_j and depends on a common random variable ε. Each retailer decides about the price p_i and the inventory level q_i to be ordered from a single supplier before demand realization. The supplier decides on the following charges from each retailer: the wholesale price w per unit ordered and the share m received out of the retailer's total revenue. The unit cost for the supplier is c with no fixed costs.

When this arrangement is viewed as a revenue sharing contract signed between the supplier and the retailers, the setting becomes a two stage model: first the supplier sets the charges (w, m) under equal[3] contractual terms across the retailers, and then the retailers set the prices p_i and order the quantities q_i. Solving for Nash equilibrium of the second stage, each retailer's profit, $(1-m)p_i E_\varepsilon \min\{q_i, D_i(p_1, ..., p_n, \varepsilon)\} - wq_i$ (where E_D is the expectation operator over the distribution of the random variable ε), is maximized given the other retailers' choices, leading to equilibrium prices and quantities as a function of the contract parameters (w, m). In the first stage one seeks a contract that coordinates the supply chain by achieving the system optimal quantities and prices. Note that such revenue sharing contracts, when the retailers must be offered equal contractual terms by the supplier, are unable to coordinate the system when the retailers are asymmetric (see Cachon and Lariviere 2005).

The prediction of the SPCS solution for this setting is very different. A state z in the game consists of a vector $[w, m, (p_1, q_1), ...(p_n, q_n)]$, so the wholesale price, the revenue share m, the inventory levels q_i and the prices p_i are viewed as alternatives than can be altered in the game tree without limit. By Theorem 4.1, we look for the Pareto efficient states that weakly dominate the Stackelberg equilibrium. The wholesale price w and the revenue share m do not affect the system profit, but only determine how it is shared between the players. Therefore a SPCS is a single state z^* consisting of optimal prices and quantities for the system, and wholesale price and revenue share values that split the optimal system total profit such that each player is weakly better off than in the equilibrium. Such wholesale price and revenue share values exist whenever the equilibrium system profit is strictly lower than the optimal one. Therefore, in contrast to the traditional equilibrium analysis, the SPCS solution predicts system coordination.

6 Further Examination of the Related Literature

6.1 Existing Farsighted Solution Concepts

We discuss more formally in this section the relevant literature on farsighted solution concepts and critically relate them to the newly introduced SPCS. We start with Chwe's largest consistent set which is based on an indirect dominance relation.

Definition 6.1 *[Chwe 1994, Harsanyi 1974]. We say that a state* $a \in Z$ *is indirectly dominated by a state* $b \in Z$*, or* $a \ll b$*, if there exist states* $a_0, a_1, ..., a_m \in Z$*, where* $a_0 = a$ *and* $a_m = b$*, and coalitions* $S_0, S_1, \ldots, S_{m-1}$*, such that* $a_l \rightarrow_{S_l} a_{l+1}$ *and* $u_i(b) > u_i(a_l)$ *for all* $l = 0, 1, \ldots, m-1$ *and* $i \in S_l$*.*

The interpretation for indirect dominance provided by Chwe is that if $a \ll b$ and b is presumed to be stable, then it is possible, not certain, that the coalitions $S_0, S_l, \ldots, S_{m-1}$ will move from a

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[3] Equal contractual terms are legally required in the USA by the Robinson–Patman Act of 1936.

to b.

Definition 6.2 *[Chwe 1994] We say that a set $Y \subset Z$ is consistent if $a \in Y$ if, and only if, for every coalition S and state $d \in Z$ such that $a \rightarrow_S d$, there exists a state $e \in Y$, such that $e = d$ or $d \ll e$, for which it is not the case that $u_i(e) > u_i(a)$ for all $i \in S$. The largest consistent set (LCS) is the union of all consistent sets.*

A useful way of understanding Chwe's consistent set (and the LCS, in particular) is by defining the function $f : 2^Z \rightarrow 2^Z$ such that $f(X)$ equals the set of all $a \in Z$ such that for every coalition S and state $d \in Z$ such that $a \rightarrow_S d$, there exists a state $e \in X$, such that $e = d$ or $d \ll e$, for which it is not the case that $u_i(e) > u_i(a)$ for all $i \in S$. A set Y is consistent if, and only if, Y is a fixed point of f, i.e., $Y = f(Y)$. Indeed, $f(Y) \supseteq Y$ says that all states in Y are stable when Y is perceived to be the set of stable outcomes, and $f(Y) \subseteq Y$ says that all states not in Y are unstable, as there are deviations away from them that will be carried out. Thus Y is consistent if (i) every state in Y is stable (vN-M internal stability) and (ii) every state in $Z \backslash Y$ is unstable (vN-M external stability), when Y is perceived to be the set of final states.

The apparent similarity between Chwe's consistent set and the SPCS definitions is clear. In both cases, if Y is the stable set then a deviation from any $y \in Y$ will not occur, a deviation from $y \notin Y$ will occur and only states in Y can be final states. The difference between the two concepts is the reasoning that governs the sequence of deviations the players follow after a deviation from a stable or a non-stable state. In the LCS, the reasoning is exclusively indirect dominance, while in the SPCS solution, the sequence of moves leads to a final outcome in the SPCS which is supported by a subgame perfect equilibrium in the Farsighted Game.

An additional similarity between the two solution concepts concerns the belief players are assumed to have on other players' future actions. In the LCS, all players believe in the same final state of a possibly given indirect dominance relation, which provides a reasonable justification for their collaboration to carry out the corresponding sequence of moves that leads to the final state. Similarly, all players believe in the same strategy profile supporting a SPCS, leading to the subgame perfect reasoning that justifies their part in moving to the final state. Both kinds of reasoning can be viewed as resulting from social norms that players possess when entering the game (as further explained in the discussion following Theorem 4.1).

Another difference between the LCS and the SPCS solutions concerns their existence, uniqueness and non-emptiness. In view of the isotonicity of f, the LCS always exists uniquely, simply being the set of all outcomes which can possibly be stable (Proposition 1 in Chwe 1994, proved by employing Tarski's fixed point argument on the function f). However, the LCS may be empty. Sufficient conditions for the non-emptiness of the LCS were provided by Chwe for games with a finite or countably infinite set of states Z. These sufficient conditions include non-existence of certain infinite sequences of indirect dominance. Xue (1997) provided weaker sufficient conditions that also do not require a countably infinite Z. By contrast, a SPCS may not exist (Example 3.1 – note that in this example the LCS exists and consists of all states $z > 1$), but a SPCS must be non-empty when it exists (Proposition 3.1), and there may be multiple non-empty SPCSs, each of which is a largest set in the sense of set inclusion (Theorem 4.1). Theorems 4.1 and 4.2 provide necessary conditions and sufficient conditions (that are also identical in Theorem 4.2) for the existence of a non-empty SPCS for normal form games. The following is another example (adapted from an example by Chwe 1994) that highlights the difference between the LCS and SPCS regarding existence.

Example 6.1 *There is one player, Z is the set of natural numbers, $u(z) = z$ for each $z \in Z$, and the player can move from any z to $z + 1$ (with no other possible moves included in the effectiveness relation). In this example, LCS $= \emptyset$, but there is no SPCS because there is no subgame perfect equilibrium: the player strictly prefers to keep moving to higher and higher states indefinitely, but this means swinging that is assigned utility $-\infty$, thus the player strictly prefer not moving from any state. This example highlights a difference between the two kinds of farsighted reasoning: in the LCS, the player continues to move to higher states although this does not lead to a good outcome, whereas such moves cannot be considered as part of subgame perfect equilibrium strategies supporting a SPCS.*

Finally, the SPCS solution allows more general decision rules than the LCS, as it can, but does not necessarily, assume pessimism. Note also that theorems 4.1 and 4.2 do not depend on an assumption of pessimism.

Two main alternatives to Chwe's seminal contribution were discussed in the literature. In the first approach, prior to Chwe, Greenberg (1990) developed the theory of social situations, and therein introduced the concept of a stable standard of behavior (SB), which is claimed to capture perfect foresight by individuals or coalitions. Some precise relationships between a stable SB and the LCS were investigated by Chwe (1993). Xue (1998) argued that the LCS captures only partial foresight, and by employing Greenberg's (1990) framework, characterized a set of paths which constitutes a stable SB. (See also Mariotti 1997, wherein a solution concept similar to a stable SB is shown to achieve partial efficiency in normal form games with coalitions). The second approach, M&V's (2004) extension to the LCS, is exclusively concerned with a weakening of the pessimistic criterion Chwe employs, according to which coalitions determine whether defections are worthwhile (this will be further explained below).

Let us first consider Xue's (1998) stable paths concept, which is based on Greenberg's (1990) approach and uses his stable SB solution. This concept is defined with respect to a directed graph G whose vertices are the states in Z and whose directed arcs correspond to the effectiveness function, so that if $a \rightarrow_S b$ for some states $a, b \in Z$ and coalition S, then arc ab is in G. Let Π denote the set of all finite (directed) paths in G, and for each $a \in Z$, let Π_a denote all paths that originate from a, including the degenerate path (a) consisting of a alone. The preferences over paths in Π are the preferences over their terminal nodes, and two criteria, optimistic and conservative, were introduced to compare between a path and a set of alternative paths.

Now, for every $a \in Z$, Π_a is interpreted as the set of feasible outcomes when a is the status quo. Xue characterizes the set of paths in Π_a for each $a \in Z$ that might be followed by rational and farsighted players.

Definition 6.3 *[Greenberg 1990, Xue 1998] A standard of behavior (SB) σ for the situation with perfect foresight is a mapping from Z to Π such that for every $a \in Z$, $\sigma(a) \subseteq \Pi_a$. A SB σ is stable if it is both internally and externally stable, i.e. for all $a \in Z$, a path $\alpha \in \Pi_a \backslash \sigma(a) \Longleftrightarrow$ there exists a coalition S and states b, c such that $b \in \alpha$, $b \rightarrow_S c$ and S prefers $\sigma(c)$ to α.*

Xue provided some examples which demonstrate the success of the stable SB to provide reasonable and farsighted predictions. Yet, the two examples below illustrate some drawbacks of this concept of solution.

Specifically, consider our motivating Example 3, wherein we compare the stable SB to the LCCS and the SPCS. In this example the stable SB σ predicts the same states as the LCS, i.e. $\{A, C, D\}$, whereas the LCCS and the unique SPCS predict $\{C, D\}$. The stable SB σ has $\sigma(A) = \{(A)\}$, $\sigma(B) = \{(B, C)\}$, $\sigma(C) = \{(C)\}$ and $\sigma(D) = \{(D)\}$, where each element in these sets represents a stable path. To see this first note that strict preference according to the stable SB is defined for coalitions when each player in the coalition has a strict preference. Therefore coalition $\{1, 3\}$ does not strictly prefer a move from B to D (according to optimism or conservatism) as player 3 is indifferent between C, the final state of the path (B, C), and D, the final state of the unique path (D) in $\sigma(D)$. Therefore the path (B, C) is in $\sigma(B)$. The path (B, D) is not in $\sigma(B)$ because coalition $\{2\}$ strictly prefers a move from B to C (according to optimism and conservatism) as player 2 strictly prefers C, the final state of the unique path (C) in $\sigma(C)$, to D, the final state of the path (B, D). For the same reason, the path (B) is not in $\sigma(B)$. Therefore $\sigma(B) = \{(B, C)\}$. Now $\sigma(A)$ includes only the path (A) because player 1 knows that moving to B will end him up at C, which is indifferent for him to A, so according to both optimism and conservatism he does not move to B. Note also that even if we change the stable SB's coalition strict preference definition and take instead a definition similar to our pessimistic criterion for coalitions (i.e., coalition S strictly prefers a move if all players are weakly better off while at least one player is strictly better off), the stable SB would not change in this example. This is because the modified criterion would make coalition $\{1, 3\}$ strictly prefer a move from B to D, which will eliminate the path (B, C) from $\sigma(B)$, leaving it an empty set (see Figure 4 in Xue 1998). Therefore path (A) still cannot be removed from $\sigma(A)$ because no path from $\sigma(B)$ can lead to a strictly better final state than A, as D is not considered a possible final state from B.

Example 6.2 below illustrates another drawback of the stable SB.

Example 6.2 *Let $N = \{1, 2, 3\}$, $Z = \{A, B, C, D\}$, and consider Figure 6.1. Clearly, player 2 strictly prefers C to D, player 3 strictly prefers D to C, and both players 2 and 3 strictly prefer C and D to B, which can be shown to imply that $\sigma(B) = \emptyset$. That is, even though state B is rejected by both players 2 and 3, a stable SB is silent as to which of the alternatives C or D will replace B (see again Figure 4 in Xue 1998). Therefore, $\sigma(A) = \{(A)\}$, implying that player 1 will not deviate from state A. Consequently, even though player 1 knows that a deviation from A will not end up at state B but rather will subsequently lead either to C or to D, and even though he strictly (and strongly) prefers both states C and D to A, the stable SB predicts that he will stay at state A.*

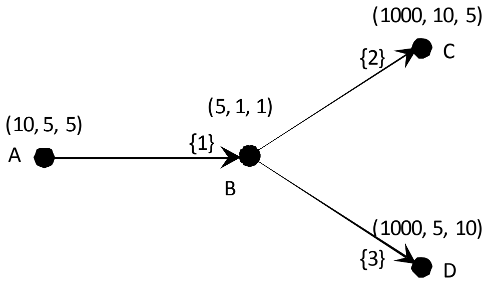

Figure 6.1: Transition tree in Example 6.2

Note that, in contrast to the stable SB, SPCS = LCS = LCCS = $\{C, D\}$*, i.e., all three stable set concepts predict that player* 1 *will indeed deviate from state A, and will end up either at state C or D.*

Next, let us consider the refinement of the LCS, the largest cautious consistent set (LCCS) introduced by M&V (2004). As previously mentioned, the LCCS results from a modification of the LCS pessimistic criterion, as evident in the following algorithmic definition of the LCCS provided by M&V.

Definition 6.4 *[M&V 2004]. Let* $Z^0 \equiv Z$*. Then,* Z^k $(k = 1, 2, \ldots)$ *is inductively defined as follows:* $a \in Z^{k-1}$ *belongs to* Z^k *if, and only if, for every coalition* S *and state* $d \in Z$ *such that* $a \rightarrow_S d$*, there exists a finite distribution* $\alpha = (\alpha(e_1), \ldots, \alpha(e_m))$ *over states such that* $e_j \in Z^{k-1}$*,* ($e_j = d$ *or* $d \ll e_j$) *and* $\alpha(e_j) \in (0, 1)$ *for each* $j = 1, ..., m$ *with* $\sum_{j=1}^{m} \alpha(e_j) = 1$*, for which it is not the case that* $\sum_{j=1}^{m} \alpha(e_j) u_i(e_j) > u_i(a)$ *for all* $i \in S$*. The largest cautious consistent set (LCCS) is* $\cap_{k \geq 1} Z^k$*.*

As M&V explain, the idea behind the LCCS is that once a coalition S deviates from z to z', this coalition should contemplate the possibility of ending up with positive probability at any state z'' not ruled out such that $z' = z''$ or $z' \ll z''$. Hence, a state z is never stable if a coalition S can engage a deviation from z to z', and by doing so there is no risk that some coalition members will end up worse or equal off. The LCCS does provide improved farsighted predictions in some instances, as was shown in motivating Example 3, and was used successfully by Granot and Yin (2008) to derive insightful results for a single-period two-stage supply chain problem.

Although not discussed by M&V, it is important to note that one can easily derive a definition of a cautious consistent set (CCS) based on Definition 6.4. Unfortunately, one cannot prove unique existence of a largest CCS based on a fixed point argument analogue to the one used by Chwe to prove unique existence of the LCS (because in contrast to the function f used by Chwe, a corresponding function associated with the LCCS is not isotonic). Furthermore, the LCCS could be empty in settings where the LCS is ensured to be non-empty. Let us consider the example used by M&V to demonstrate this point.

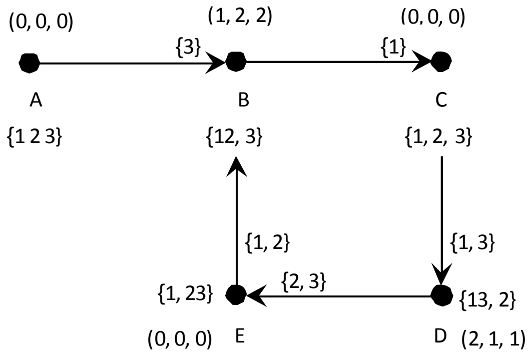

Figure 6.2: Transition tree in Example 6.3

Example 6.3 *Consider the game corresponding to Figure 6.2. It can be easily shown that the LCCS of Example 6.3 is empty. As M&V wrote: "Indeed, it is intuitively reasonable that no outcome can be possibly cautiously stable in this example. Player* 1 *or the coalition formed by players* 2 *and* 3 *cannot end worse off by engaging a move from* $\{12, 3\}$ *and* $\{13, 2\}$*, respectively."*
Nevertheless, even though the LCCS of Example 6.3 is empty, we note that both singleton sets $\{B\}$ *and* $\{D\}$ *are maximal (with respect to set inclusion) CCSs, and, as such, provide insightful predictions of the final states in the game associated with Example 6.3. Thus, while the LCCS could be empty, there may exist a non-empty maximal CCS.*

We note that the SPCS solution can adopt the modified pessimistic criterion of M&V, as is evident, in some sense, from Definitions 3.1 and 3.2. Still, the logic of the LCCS, like that of the LCS, is exclusively based on indirect dominance which provides the players with only limited foresight, and, indeed, as in motivating examples 4 and 5, fails to deliver the proper farsighted predictions.

Yet another approach to farsightedness, which was employed in the analysis of supply chain models, is due to Konishi and Ray (2003). Therein, the authors have introduced a dynamic process of coalition formation (PCF), given by a transition probability $p : Z \times Z \to [0, 1]$ such that $\sum_{y \in Z} p(x, y) = 1$ for each $x \in Z$. A PCF p induces a value function for each player, which captures the infinite horizon discounted payoff to each player starting from any initial point under the Markov process p. A PCF is an equilibrium PCF (EPCF) if a coalitional move to some other state can be supported by the expectation of a higher future value.

Evidently, the EPCF approach to farsightedness and coalition formation is quite different from the reasoning leading to the LCS, LCCS, and SPCS. By contrast with the last three concepts, in which payoffs to players are realized only at the final state and wherein players may entertain different conjectures about other players behavior, in the EPCF players get discounted state-dependent payoffs at each stage, and all players have common beliefs about future moves. Further, as compared to the SPCS solution, in which strategies could be history-dependent, in the EPCF they are Markovian stationary, i.e. they depend only on the current state and not on the mode of arrival to that state.

One of the attractions of the EPCF is that the set of all absorbing states under all deterministic (i.e., $p(x,y) \in \{0,1\}, \forall x,y \in Z$) absorbing EPCFs is a refinement, typically strict, of the LCS when the discount factor is large enough. This result was used by e.g., Sošić (2006) and Nagarajan and Sošić (2007, 2009) in their analysis of several OM models. We note though that in contrast to the SPCS solution, the EPCF approach to coalition formation may fail to achieve efficiency in normal form games, as is evident from Observation 2 and the ensuing discussion in Konishi and Ray (2003). Finally, another approach to characterize farsightedness in coalition formation in the spirit of the LCS using subgame perfection in stochastic games was considered by Flesch et al. (2008).

Brams (1994) theory of moves (TOM) (see also Brams and Wittman 1981) offers a dynamic theory approach to normal form games that bears some resemblance to our farsighted game approach to normal form games. Indeed, TOM attempts to characterize non-myopic equilibria, or stable outcomes, when players in a normal form game think ahead, and the predictions TOM provides depend, as in our case, on the starting point or status quo state. However, TOM is only developed for two-person normal form games in which each player has only two strategies, the players according to TOM play in a strict alternating order, and, more importantly, TOM makes simplifying assumptions to ensure a finite game tree representation of the dynamic normal form game. For example, TOM assumes that if the play returns to the initial state or to any other state that is payoff equivalent to a state chosen earlier, then the play will terminate at this state, making it a final state. Related extensions to TOM's non-myopic equilibria include Kilgour (1984).

6.2 Further Comparison of the SPCS and the LCS

In the last section of his paper, Chwe (1994) noted that many details are left out of the definition of the LCS, leaving important issues blurred and resulting in a limited 'farsightedness'. In this section we revisit the examples that Chwe introduced to demonstrate the shortcomings of the LCS, and show that, in contrast, the SPCS solution provides satisfactory predictions for all these examples.

Example 6.4 *Optimal rather than better choice.* $N = \{1\}$ *(i.e., there is a single player) with* $Z = \{a,b,c\}$, $a \rightarrow_{\{1\}} b$, $a \rightarrow_{\{1\}} c$ *and* $u_1(a) < u_1(b) < u_1(c)$. *LCS* $= \{b,c\}$, *predicting that when* a *is the initial state, player* 1 *will move to either* b *or* c, *even though* c *is strictly preferred to* b. *As Chwe has noted: "This example illustrates how the largest consistent set does not incorporate any idea of "best response": coalitions will move to any, not just the best, of the outcomes which are better than the status quo" (Chwe, 1994: page 322). In this example, the unique SPCS = LCS, since* b *and* c *are terminal states according to the effectiveness relation. However, the attraction set* $T_{\{b,c\}}(a) = \{c\}$, *since* c *is the unique outcome that can be supported as a final state according to a subgame perfect equilibrium in the game tree corresponding to this example. Thus the unique SPCS predicts the proper move from* a *to* c. *Indeed, this demonstrates that in contrast to the LCS, the SPCS* is *predicated on the notion of best response.*

Example 6.5 *Not joining a coalition if it is better otherwise.* $N = \{1, 2\}$ *with* $Z = \{a, b, c\}$, $a \rightarrow_{\{1,2\}} b$, $a \rightarrow_{\{2\}} c$, $u_1(c) < u_1(a) < u_1(b)$ *and* $u_2(a) < u_2(b) < u_2(c)$. *Again, since* b *and* c *are terminal states, and both are preferred by both players to state* a*, the unique SPCS = LCS =* $\{b, c\}$. *Thus from an initial state* a*, the LCS predicts that either coalition* $\{1, 2\}$ *will move to state* b *or player* 2 *will move to state* c. *The difficulty with this prediction, as Chwe noted, is that player* 2 *prefers state* b *to state* c*, thus should prefer not to join player* 1 *to move to* b*, as he could do better by himself moving to* c. *Indeed, there exists a subgame perfect equilibrium in the game tree corresponding to this example, in which the coalition* $\{1, 2\}$ *never moves following the selection of state* a *as an initial state. This choice can be interpreted as resulting from player* 2*'s preference not to join player* 1 *in such a move, and instead to wait until an opportunity arises to move alone from state* a *to state* c. *Consequently the prediction of the SPCS in this case is that player* 2 *will not join player* 1 *in a move to state* b. *Note, however, that the attraction set* $T_{\{b,c\}}(a) = \{b, c\}$*, as there also exists a subgame perfect equilibrium in which the coalition* $\{1, 2\}$ *sometimes moves from* a *to* b*, in which case there is positive probability for* b *to be a final state.*

Example 6.6 *Waiting and not moving until a better opportunity arises.* $N = \{1, 2\}$ *with* $Z = \{a, b, c\}$, $a \rightarrow_{\{1\}} b$, $a \rightarrow_{\{2\}} c$, $u_1(a) < u_1(b) < u_1(c)$, *and* $u_2(a) < u_2(b) < u_2(c)$. *We have LCS =* $\{b, c\}$*, and the prediction of the LCS is that from an initial state* a*, either player* 1 *will move to state* b *or player* 2 *will move to state* c. *The difficulty with this prediction, as pointed out by Chwe, is that both players prefer state* c *to state* b*, so that if player* 1 *is provided with the opportunity to be the first to move from the initial state* a*, he should rather forfeit this opportunity and wait until player* 2 *is able to move. Although for this example the unique SPCS = LCS, the prediction according to the SPCS is that Player* 1*, if provided with the opportunity to move, would not move to state* b. *Indeed, the attraction set* $T_{\{b,c\}}(a) = \{c\}$ *because state* c *is the unique outcome that can be supported by a subgame perfect equilibrium in the game tree corresponding to this example following the selection of state* a *as an initial state.*

Example 6.7 *Moving to preempt other moves that may lead to worse outcomes.* $N = \{1, 2\}$ *with* $Z = \{a, b, c\}$, $a \rightarrow_{\{1\}} b$, $a \rightarrow_{\{2\}} c$, $u_1(a) < u_1(c) < u_1(b)$, *and* $u_2(b) < u_2(c) < u_2(a)$. *Here, LCS =* $\{b, c\}$*, and the prediction is that from an initial state* a*, player* 1 *will move to state* b. *However, as noted by Chwe, even though player* 2 *prefers state* a *to state* c*, he should deviate from* a *to* c *given an opportunity to do so; otherwise player* 1 *will deviate to state* b*, which for player* 2 *is even less preferred than state* c. *Although the unique SPCS = LCS, the attraction set* $T_{\{b,c\}}(a) = \{b, c\}$*, as both these states can be supported as final states according to a subgame perfect equilibrium in the game tree corresponding to this example following the selection of state* a *as an initial state. Thus the SPCS predicts a move from state* a *to either* b *or* c.

In all of the above examples, even though the LCS and the SPCS coincide, they provide different predictions if the game starts from an initial state outside these sets. The SPCS solution is shown to provide the satisfactory prediction, while the LCS, as noted by Chwe, fails to do so.

7 Summary and Future Work

Farsighted players, by contrast with myopic players, recognize the possible reactions by other players to their actions, and deviate from a current state only if they believe that such a deviation would not lead to an inferior outcome. Thus, not surprisingly, an analysis of OM models could yield strikingly different results depending on whether players are assumed to be farsighted or myopic. Indeed, recent contributions to the literature reveal that in competitive situations, farsighted players are more likely to cooperate than myopic players. For example, consider again ABZ (2001) singled-period two-stage model. As shown by Sošić (2006), farsighted players will tend to cooperate and retain the grand coalition structure in the second stage when additional revenues are being shared, while myopic players are more likely to form sub-coalitions in this stage, leading to suboptimal results. Similarly, as shown by Granot and Yin (2008), farsighted suppliers of complementary products will cooperate and form an alliance of all suppliers for the purpose of contracting with the downstream assembler, which will make all supply chain members best off. By contrast, myopic players will not cooperate at all, and each supplier will independently contract with the assembler, resulting with the worst performance for all supply chain members.

In general, the OM literature has adopted Chwe's (1994) approach to farsightedness, wherein it is assumed that players follow the indirect dominance logic. Players will deviate from a status quo state if subsequent deviations by motivated coalitions would eventually result with a better outcome. However, as demonstrated in our paper and, indeed, recognized by Chwe, indirect dominance endows players with limited foresight. Accordingly, we have introduced in this paper a new approach to farsightedness, leading to the Subgame Perfect Consistent Set (SPCS). Our SPCS retains a main feature of the LCS, namely, a vN-M type of stability. However, it replaces Chwe's indirect dominance with subgame perfection. The SPCS is shown to significantly improve upon Chwe's largest consistent set (LCS) and to provide better predictions than the LCS and the LCCS. Furthermore, by contrast with these concepts of stability, (i) it leads to an optimal rather than a better choice, (ii) it allows players not to join coalitions if they so desire, (iii) it lets players wait for a better opportunity rather than move, and (iv) it allows preemptions of other moves that may lead to worse outcomes.

Moreover, we have shown that farsighted players who adopt the SPCS reasoning will achieve full cooperation and overcome misaligned incentives in a variety of settings in which Chwe's type players may fail to do so. Specifically, our farsighted players will always share the monopolistic profit in Bertrand and Cournot competition models and will always achieve supply chain coordination and Pareto efficiency in the decentralized setting of the classical newsvendor model and its variants. Indeed, they will be able to achieve coordination and Pareto efficiency in more complex settings such as a supply chain with a single supplier and competing non symmetric retailers, in which the supplier is obligated to offer identical contractual terms to the retailers. We note that in the latter case, other approaches such as, e.g., revenue sharing contracts, may fail to coordinate the channel (see, e.g., Cachon and Lariviere 2005 and Krishnan and Winter 2011).

Future work may consider the extension of our analysis beyond normal form games to include, for example, coalition formation games. In particular, it would be interesting to investigate whether new insights could be gained by employing our SPCS approach to study coalition formation in the supply chain models investigated by Nagarajan and Sošić (2007, 2009) and Nagarajan and Bassok (2008), who have assumed Chwe-type farsighted players, or Granot and Yin (2008), who have used the LCCS approach.

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