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The ordinal Nash social welfare function

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Abstract

A social welfare function entitled 'ordinal Nash' is proposed. It is based on risk preferences and assumes a common, worst social state (origin) for all individuals. The crucial axiom in the characterization of the function is a weak version of independence of irrelevant alternatives. This axiom considers relative risk positions with respect to the origin. Thus, the resulting social preference takes into account non-expected utility risk preference intensity by directly comparing certainty equivalent probabilities. The function provides an interpretation of the Nash-utility-product preference aggregation rule. Necessary and sufficient conditions for the function to produce complete and transitive binary relations are characterized.

JEL classifications: C7; D7

Keywords: Axiomatic approach; Nash program; Non expected utility

1. Introduction

In his classic work, Arrow (1951) introduced the concept of a social welfare function. Arrow's axiomatic analysis led to the well known impossibility result, according to which no social welfare function exists that satisfies the axioms of universal domain, Pareto principle, independence of irrelevant alternatives (IIA) and non-dictatorship. Arrow's path breaking conclusion was extensively analyzed thereafter, resulting in a variety of modified impossibility and possibility theorems.

An interesting question concerns the axiomatization of the Nash social welfare function – the preference aggregation rule which assumes the existence of a common worst social state (or origin) for all individuals in society, and ranks social states according to the product of individual utility differences with respect to the origin. This paper examines the axiomatic justification of an aggregation rule in the spirit of the Nash social welfare function, aggregating non-expected utility individual risk preferences over social states to social ranking of the deterministic social states. The axiomatization of the Nash social welfare function, due to Kaneko and Nakamura (1979), relies heavily on the expected utility (EU) assumption. Thus the motivation for this paper may be viewed as studying the combined implications of the normative approach suggested by the Nash preference aggregation rule together with the descriptive approach offered by the non-EU literature.

The first aim of this paper is the axiomatic justification of the proposed preference aggregation rule. Apart from assuming the existence of an origin, the crucial axiom in the characterization is a weak version of IIA, in which only

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relative risk positions with respect to the origin are considered. The second aim is to provide an ordinal interpretation for the aggregation rule. This is achieved by defining the social preference without the use of utility multiplication, but instead by comparing certainty equivalent probabilities. This comparison is undertaken directly when the social states being compared are different for precisely two individuals. When a greater number of individuals show interest in the comparison, it is undertaken indirectly following a sequence of intermediate steps, in which precisely two individuals are involved at each step. A third aim of the paper is the extension of the 'ordinal Nash' bargaining solution that was introduced by Rubinstein et al. (1992), a reinterpretation of the classic Nash bargaining solution. This solution was defined directly in terms of non-EU risk preferences and physical alternatives and was characterized by an outcome that is immune to all possible appeals.[1]

The domain of preference profiles being investigated consists of profiles of preferences having a representation entitled 'origin biseparable' (OB). This is a multiplicatively separable representation for *elementary* lotteries, i.e. lotteries over two social states, one of which is the origin. These preferences include the EU preferences as a special case, as well as families that accommodate many known violations like the 'Allais paradox' and the 'common ratio effect'. Over the domain of all OB preference profiles we show an impossibility result. This result is then recovered by restricting the domain to the maximal sub-domain which is consistent with the axioms that characterize the social welfare function. Our axiomatic characterization relies on the existence of one comprehensive profile in the domain and some of its permutations. This profile can be viewed as representing preferences over private commodity bundles that exhibit free disposal of individual welfare with no consumption externalities. Other profiles in the domain are not restricted up-front in any way. The analysis utilizes the notion of induced utilities, as introduced in Hanany and Safra (2000).

The results include conditions on the domain of preference profiles, which ensure that the ordinal Nash social welfare function yields a complete and transitive binary relation over the set of social states. For a society of size two, we define necessary and sufficient conditions that characterize the domain of the function, specifically as a result of the transitivity requirement. These conditions state that the pair of preferences are representable with equal probability distortion functions up to positive power transformations.[2] For a larger society, we define similar necessary and sufficient conditions for the transitivity requirement and sufficient conditions for the completeness property. The latter conditions include smoothness and mild convexity of the utility image of the set of social states.

It is interesting to note the connection of the results in this paper to the general conclusion reached within the social choice literature, discussing the aggregation of individual preferences under subjective EU (SEU) theory. This line of inquiry aims at aggregating both outcome utility functions and subjective probabilities. When SEU axioms are imposed on both social and individual preferences, it is shown that the strict Pareto principle does not generally hold. Mongin (1995) shows within the framework of Anscombe and Aumann (1963) SEU theory that under sufficient conditions of individual preference diversity (when the outcome utility functions are affinely independent), the axioms of SEU and strict Pareto are consistent only when all individual subjective probability beliefs are identical. This result and other similar work may be compared with our conclusion that the preference aggregation requirements are consistent only when all individual preferences are compatible with the same objective probability distortion function, up to positive power transformations, over the set of elementary lotteries.

The paper is also related to the social choice literature on cardinal utilities. Our analysis emphasizes the role of the origin in the preference aggregation rule, as opposed to its other two main properties. The first property is the individual cardinal utility assumption, meaning that individual utilities represent preferences uniquely up to a positive affine transformation. The second property is the non-comparability of cardinal utilities, which means that social preference is equal for cardinally equivalent individual utility profiles. Defining cardinally equivalent profiles such that the affine transformations are not necessarily interdependent across individuals, such a property is referred to as an

[1] A bargaining model stems from an entirely different motivation than that which guides social choice. Apart from this crucial distinction between this paper and RST's, the main additional difference is the requirement in this paper that a social welfare function aggregates preferences to a complete and transitive ordering of all social states, whereas a bargaining solution in general only picks (at least) one of these social states as a bargaining outcome. Thus a bargaining solution may not be rationalized by a social welfare function, i.e. there may not exist a preference relation over all social states, the maximization of which always yields bargaining outcomes.

[2] In Hanany (2005) we derive the same conditions when axiomatizing a two-player bargaining solution similar to the ordinal Nash, in a framework where the agreements reached by players in different bargaining situations are considered as a way to reveal the bargainers' (not necessarily transitive) preferences as a group.

assumption of no interpersonal comparability of cardinal individual welfare. Sen (1970) showed that without the origin assumption there is an impossibility result for cardinality and non-comparability. This impossibility result is analogue to Arrow's impossibility, which results from ordinality and non-comparability. It was pointed out by Sen that an escape route from cardinality with non-comparability could be found within a framework similar to Nash (1950) bargaining theory, through the restriction of the domain to profiles with an origin.[3] This method of achieving a possibility result shows the extent to which the origin assumption is crucial for the aggregation rule investigated in this paper. One can easily check that non-comparability of cardinal utility holds for the Nash social welfare function. This is the case since applying any affine transformations to the individual utility functions does not change the social preference. It would appear that the possibility result draws not on cardinal but "ordinal comparability" that utilizes the origin. Moreover, it is a modest type of ordinal comparability, since it doesn't imply egalitarianism as does full ordinal comparability (d'Aspremont and Gevers, 1977). A natural question arises regarding the implications of assuming this kind of ordinal origin comparability without the cardinality assumption. The answer to this question can be found in this paper, as it confronts the necessity of the acceptable interpretation of cardinal utilities as vNM utilities that represent EU risk preferences.

The paper is organized as follows: Section 2 states the axioms, Section 3 characterizes the social welfare function for the case of two individuals, Section 4 specifies the conditions for a general, finite size of society and Section 5 develops an axiomatization of the function over the domains being investigated.

2. Axioms and social welfare function

Consider a finite society of individuals, denoted by $\mathbf{N} = \{1, \ldots, n\}$. The society is associated with a fixed, non-empty and compact (in some topological space) set $\mathbf{X}$ of possible social states and a social state $\mathbf{x}^0$ (origin). State $\mathbf{x}^0$ is considered the worst for all individuals in the society. Each member k of the society has a preference relation $\succsim_k$ (complete and transitive binary relation) over the set of simple (finite) lotteries over $\mathbf{X} \cup \{\mathbf{x}^0\}$. A lottery of the form $p\mathbf{x} + (1-p)\mathbf{x}^0$, where p is the probability of $\mathbf{x} \in \mathbf{X}$ and $1-p$ is the probability of $\mathbf{x}^0$, is called an *elementary* lottery and is denoted $p\mathbf{x}$. Abusing notation, a degenerate lottery with prize $\mathbf{x}$ is denoted by $\mathbf{x}$. As usual, $\sim_k$ and $\succ_k$ denote the symmetric and asymmetric components of $\succsim_k$, respectively. A profile of preferences $\langle \succsim_k \rangle_{k=1}^n$ is denoted by $\succsim$.

Consider a preference relation $\succsim_k$ for which there exists an onto and strictly increasing function $g_k : [0,1] \to [0,1]$ and a function $v_k : \mathbf{X} \to \mathbb{R}$ normalized with $v_k(\mathbf{x}^0) = 0$, such that for every pair $p\mathbf{x}, q\mathbf{y}$ of elementary lotteries, $p\mathbf{x} \succsim_k q\mathbf{y} \Leftrightarrow g_k(p)v_k(\mathbf{x}) \geq g_k(q)v_k(\mathbf{y})$. This representation is not necessarily applicable for general lotteries, thus we call it an 'origin biseparable' (OB) representation. Let $\mathcal{P}$ be the set of all such preference relations. The set $\mathcal{P}$ contains (Ghirardato and Marinacci, 2001) *biseparable preferences*, when the latter is restricted to preferences over lotteries. This in turn includes the entire family of 'rank-dependent utility' (RDU) preferences (Quiggin, 1982; Weymark, 1981) and 'disappointment aversion' (DA) family (Gul, 1991). The set $\mathcal{P}$ also contains the family of 'disagreement linear' (DL) preferences introduced by Grant and Kajii (1995) (when disagreement is replaced by $\mathbf{x}^0$), for which $g_k(p) = p$, $\forall p \in [0,1]$. DL preferences contain EU preferences.

We consider social welfare functions $W : \mathcal{D} \to \mathcal{B}$, where $\mathcal{D} \subseteq \mathcal{P}^n$ and $\mathcal{B}$ is the set of all binary relations R over $\mathbf{X}$. Thus unlike Kaneko and Nakamura (1979), we are not concerned with social risk preferences, but instead use individual risk preferences as an information basis for social choice. This modelling is justified by viewing only elements of $\mathbf{X}$ as possible social outcomes. A further justification relates to the departure from EU preferences. Aggregation of non-EU preferences to possibly non-EU social preferences over lotteries on $\mathbf{X}$ leads to strong restrictions on the domain $\mathcal{D}$. This conflicts with our aim to characterize the widest possible domain consistent with the axioms. The symmetric and asymmetric components of a social binary relation R are denoted by I and P, respectively.

[3] A different way to resolve the individual cardinal utilities impossibility, that was discussed extensively in the social choice literature, is to weaken the non-comparability assumption to unit comparability or full comparability. These assumptions strengthen the cardinally equivalent utility profiles requirement by imposing a connection between the positive affine transformations of individual utilities. Unit comparability is achieved when all these transformations have the same positive multiplier and full comparability results when it is further required that all possess the same added constant. Adding an anonymity axiom, these weaker assumptions were shown to characterize the utilitarian (d'Aspremont and Gevers, 1977) and leximin (Deschamps and Gevers, 1978) aggregation rules.

For each profile in the domain $\mathcal{D}$, we assume for simplicity of presentation that (1) for every $\mathbf{x} \in \mathbf{X}$, $\mathbf{x} \gtrapprox \mathbf{x}^0$,[4] (2) there exists $\mathbf{y} \in \mathbf{X}$ such that $\mathbf{y} \gg \mathbf{x}^0$, and (3) for some origin biseparable representations $g_k v_k$, the set $\{\langle v_k(\mathbf{x})\rangle_{k=1}^n \in \mathbb{R}^n | \mathbf{x} \in \mathbf{X}\}$ is compact.[5] We also assume that the domain contains one $\mathbf{x}^0$-comprehensive profile and some of this profile's permutations. A profile $\succsim$ is $\mathbf{x}^0$-comprehensive if for any $i \in \mathbf{N}$, $\mathbf{x} \in \mathbf{X}$ and $p \in [0, 1]$, there exists a unique state $\mathbf{y} \in \mathbf{X}$ such that $\mathbf{y} \sim_i p\mathbf{x}$ and $\mathbf{y} \sim_k \mathbf{x}$ for $k \neq i$. Such profiles can be viewed as exhibiting free disposal of individual welfare, where $\mathbf{X}$ may be viewed as a set of perfectly divisible private commodity bundles, with no consumption externalities. In particular this implies that $\mathbf{X}$ must be sufficiently rich, e.g. a real convex set. Another example in which $\mathbf{x}^0$-comprehensive profiles exist is when $\mathbf{X}$ is the set of lotteries over allocations of an indivisible good, including not allocating it to any member of the society, where the latter is equivalent to $\mathbf{x}^0$ for each i. Denote by $\mathcal{D}^{\text{CM}}$ the set of all $\mathbf{x}^0$-comprehensive profiles in $\mathcal{P}^n$. We therefore state the following domain restriction, which allows the consideration of sub-domains that include one $\mathbf{x}^0$-comprehensive profile and its two-individual permutations. Other profiles in the domain are not restricted up-front in any way. Denote by π a permutation of the individuals $(1, \ldots, n)$. For any $i, j \in \mathbf{N}$, let $\pi^{ij} : \mathbf{N} \to \mathbf{N}$ such that $\pi_i = j$, $\pi_j = i$ and $\pi_k = k$ for all $k \in \mathbf{N} \setminus \{i, j\}$.

Definition 1. **CMP** (contain a comprehensive profile with two-individual permutations): *W* is defined over a non-empty domain $\mathcal{D} \subseteq \mathcal{P}^n$ having as member a profile $\succsim \in \mathcal{D}^{\text{CM}}$ and for all $i, j \in \mathbf{N}$, $\langle \succsim_{\pi_k^{ij}} \rangle_{k=1}^n \in \mathcal{D}$.

The social preference characterized by our axioms takes into account the relative position of states with respect to the origin $\mathbf{x}^0$. The relative position is defined by the probability, according to which a mixture of one state with $\mathbf{x}^0$ is equivalent to a second state. Consider a society consisting of two individuals. Suppose for the pair of states $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, that $p\mathbf{x} \sim_1 \mathbf{y}$, $q\mathbf{y} \sim_2 \mathbf{x}$ and $p \leq q$. In this case, we understand that individual 1 prefers state $\mathbf{x}$ over state $\mathbf{y}$ and individual 2 has opposing preferences. However, in terms of the certainty equivalent probabilities p and q, state $\mathbf{x}$ is located in a higher position for 2 than $\mathbf{y}$ is for 1. This may be interpreted as saying that individual 1 exhibits welfare improvement when moving from state $\mathbf{y}$ to state $\mathbf{x}$, improvement that is higher than the welfare improvement that individual 2 exhibits when moving from state $\mathbf{x}$ to state $\mathbf{y}$. Thus we could argue that society should prefer the state $\mathbf{x}$ over the state $\mathbf{y}$. In the case of Pareto dominance, where such a comparison is not possible, the preference is defined to agree with the Pareto rule, i.e. the dominating state is preferred.

We refer to the social preference defined in this way as the 'ordinal Nash' social preference. Our results show the equivalence of this definition to the Nash social preference (Kaneko and Nakamura, 1979). The choice of name for the social preference is also justified by its connection to the 'ordinal Nash' bargaining solution of Rubinstein et al. (1992), in particular to their definition of an 'appeal' in bargaining. The function that assigns, to every preference profile of the society, the corresponding ordinal Nash social preference is called the 'ordinal Nash social welfare function'. The definition of the function for society of an arbitrary, finite size is postponed to Section 4.

Definition 2. The ordinal Nash social welfare function, denoted ON, assigns for a preference profile $\succsim \in \mathcal{P}^2$, a binary relation $R \in \mathcal{B}$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x}R\mathbf{y}$ if either $\mathbf{x} \gtrapprox \mathbf{y}$, or there exist $i, j \in \mathbf{N}$ and $p, q \in [0, 1]$ such that $p\mathbf{x} \sim_i \mathbf{y}$, $q\mathbf{y} \sim_j \mathbf{x}$ and $p \leq q$.

We now state the axioms imposed on a social welfare function *W*. The first and second axioms are the standard weak order assumption and strict Pareto optimality condition.

Definition 3. **WO** (weak order): For each $\succsim \in \mathcal{D}$, $R = W(\succsim)$ is a complete and transitive binary relation over $\mathbf{X}$.

Definition 4. **PAR** (Pareto optimality): Let $\succsim \in \mathcal{D}$ and $R = W(\succsim)$. For $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if $\mathbf{x} \gtrapprox \mathbf{y}$ and $\mathbf{x} \succ_j \mathbf{y}$ for some j, then $\mathbf{x}P\mathbf{y}$.

[4] We use the following notation: $\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \mathbf{x} \gtrapprox_k \mathbf{y}$, $\forall k \in \mathbf{N}$ and $\mathbf{x} \gg \mathbf{y} \Leftrightarrow \mathbf{x} \succ_k \mathbf{y}$, $\forall k \in \mathbf{N}$. The notation $\geq$ and $>$ is used for corresponding comparisons of vectors in $\mathbb{R}_+^n$.

[5] An equivalent statement of this property may be written by replacing the quantifier 'for some' by 'for all'. This holds because any origin biseparable representation of $\succsim_k$ is of the form $(g_k)^{\alpha_k}(v_k)^{\alpha_k}$ for some origin biseparable representation $g_k v_k$ of $\succsim_k$ and some $\alpha_k > 0$.

The third axiom is the standard anonymity condition, in which the social preference is independent of the individuals' name or order. Thus permuting individual preferences should not change the social preference.

Definition 5. **ANM** (anonymity): Let π be any permutation of the individuals $(1, \ldots, n)$ such that $\succsim, \langle \succsim_{\pi_k} \rangle_{k=1}^n \in \mathcal{D}$. Let $R = W(\succsim)$ and $R^\pi = W(\langle \succsim_{\pi_k} \rangle_{k=1}^n)$. For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x}R\mathbf{y}$ if and only if $\mathbf{x}R^\pi\mathbf{y}$.

The fourth axiom is a weak version of IIA combined with neutrality. It requires that preference profiles of identical structure with respect to corresponding pairs of social states, ought to lead to identical social preference between these pairs. This is an adaptation to non-EU preferences of an axiom used by Kaneko and Nakamura (1979). Below we discuss how under an extra domain restriction, this axiom can be broken down to two separate axioms, i.e. IIA and neutrality. Note that only preferences over elementary lotteries are considered in the profiles associated in the axiom.

Definition 6. **IIA-NEU** (independence of irrelevant alternatives with neutral property): Let $\succsim, \succsim' \in \mathcal{D}$, $R = W(\succsim)$ and $R' = W(\succsim')$. Let $\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}$ such that for each k and any $p \in [0, 1]$, $\mathbf{x} \sim_k p\mathbf{y} \Leftrightarrow \tilde{\mathbf{x}} \sim'_k p\tilde{\mathbf{y}}$ and $p\mathbf{x} \sim_k \mathbf{y} \Leftrightarrow p\tilde{\mathbf{x}} \sim'_k \tilde{\mathbf{y}}$. Then $\mathbf{x}R\mathbf{y}$ if and only if $\tilde{\mathbf{x}}R'\tilde{\mathbf{y}}$.

The axiom **IIA-NEU** is crucial to the characterization of the social welfare function, since it is the only axiom that involves the origin. The axiom involves the probability for which one social state is the certainty equivalent of an elementary lottery involving a better social state. This certainty equivalent probability is viewed as a measure for the position of one social state on the scale from the origin to the better state. The axiom says that this kind of measure should matter for the social preference. In other words, if in one profile states $\mathbf{x}$ and $\mathbf{y}$ are related by the same certainty equivalent probability as do states $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ in another profile, then the social ranking for the first profile between $\mathbf{x}$ and $\mathbf{y}$ should be the same as the social ranking for the second profile between $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$.[6] The axiom implies non-comparability for cardinal utilities, since for EU preferences the social ranking will depend on utility representations up to independently chosen positive affine transformations. Instead, this axiom suggests a kind of ordinal origin comparability. Finally, the axiom involves a neutrality property, since it relates different pairs of states in different profiles. In all of the results in the paper, under an additional domain restriction the axiom **IIA-NEU** can be broken down to the following two separate axioms. The first, **IIA**, is the same as **IIA-NEU**, except that only one pair of states is considered.

Definition 7. **IIA** (independence of irrelevant alternatives): Let $\succsim, \succsim' \in \mathcal{D}$, $R = W(\succsim)$ and $R' = W(\succsim')$. Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that for each k and any $p \in [0, 1]$, $\mathbf{x} \sim_k p\mathbf{y} \Leftrightarrow \mathbf{x} \sim'_k p\mathbf{y}$ and $p\mathbf{x} \sim_k \mathbf{y} \Leftrightarrow p\mathbf{x} \sim'_k \mathbf{y}$. Then $\mathbf{x}R\mathbf{y}$ if and only if $\mathbf{x}R'\mathbf{y}$.

The second axiom, **NEU**, says that the social preference is independent of the actual identity of social states.

Definition 8. **NEU** (neutrality): Let $\succsim, \succsim' \in \mathcal{D}$ for which there exists a bijection $\psi : \mathbf{X} \to \mathbf{X}$ such that for every $k \in \mathbf{N}$, $p, q \in [0, 1]$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $p\mathbf{x} \succsim_k q\mathbf{y} \Leftrightarrow p\psi(\mathbf{x}) \succsim'_k q\psi(\mathbf{y})$. Let $R = W(\succsim)$ and $R' = W(\succsim')$. Then $\mathbf{x}R\mathbf{y}$ if and only if $\psi(\mathbf{x})R'\psi(\mathbf{y})$.

This separation of **IIA-NEU** to two axioms is possible under the following domain restriction **CN**, which assumes a sufficiently rich domain. In order to simplify the presentation, our results (except for the proof of Proposition 1 below) are stated using the combined axiom **IIA-NEU** and without the extra domain restriction **CN**.

Definition 9. **CN** (closure under neutrality): W is defined over a non-empty domain $\mathcal{D} \subseteq \mathcal{P}^n$ such that for any $\succsim \in \mathcal{D}$ and any bijection $\psi : \mathbf{X} \to \mathbf{X}$, there exists $\succsim' \in \mathcal{D}$ such that for all $k \in \mathbf{N}$ and $p, q \in [0, 1]$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $p\mathbf{x} \succsim_k q\mathbf{y} \Leftrightarrow p\psi(\mathbf{x}) \succsim'_k q\psi(\mathbf{y})$.

[6] **IIA–NEU** conditions on information given by the certainty equivalent probabilities, whereas Arrow's **IIA** only requires that individuals' ordinal ranking of states be the same in both profiles. Thus the condition of **IIA –NEU** implies the condition of Arrow's **IIA**. Consequently the former axiom is implied by the latter under the restriction $\mathbf{x} = \tilde{\mathbf{x}}$ and $\mathbf{y} = \tilde{\mathbf{y}}$.

The first implication of our axioms is a result showing the particular utility information relevant for the social ranking. Since we are not necessarily in the EU framework, the relevant information is a utility definition that generalizes vNM utility to non-EU preferences and relies on the origin. This approach requires the notion of induced utilities, as was first introduced in Hanany and Safra (2000), referring to the Nash bargaining solution. Given $k \in \mathbf{N}$ and a preference $\succsim_k \in \mathcal{P}$, consider functions $u_k : \mathbf{X} \times \{\mathbf{x} \in \mathbf{X} | \mathbf{x} \succ_k \mathbf{x}^0\} \to \mathbb{R}_+$ that are increasing with respect to $\succsim_k$ in their first argument, decreasing in the second, and satisfy $u_k(\mathbf{x};\mathbf{x}) = 1$, $\mathbf{x} \sim_k \mathbf{x}^0 \Rightarrow u_k(\mathbf{x};\mathbf{y}) = 0$ and $u_k(\mathbf{x};\mathbf{y})u_k(\mathbf{y};\mathbf{x}) = 1$. The set of all such functions for any $\succsim_k \in \mathcal{P}$ is denoted by $\mathcal{U}$.

Definition 10. The induced utility mapping is the function $IU : \mathcal{P} \to \mathcal{U}$, that is defined by

$$IU(\succsim_k)(\mathbf{x};\mathbf{y}) = \begin{cases} p & \text{if} \quad \mathbf{x} \sim_k p\mathbf{y} \\ \dfrac{1}{p} & \text{if} \quad \mathbf{y} \sim_k p\mathbf{x} \end{cases}.$$

The function $u_k = IU(\succsim_k)$ is the *induced utility* of $\succsim_k$. It is well defined since the function g_k in the origin biseparable representation $g_k v_k$ of $\succsim_k \in \mathcal{P}$ is strictly increasing. Moreover,

$$u_k(\mathbf{x};\mathbf{y}) = \begin{cases} g_k^{-1}\left(\dfrac{v_k(\mathbf{x})}{v_k(\mathbf{y})}\right) & \text{if} \quad \mathbf{y} \succsim_k \mathbf{x} \\ \dfrac{1}{g_k^{-1}\left(\dfrac{v_k(\mathbf{y})}{v_k(\mathbf{x})}\right)} & \text{if} \quad \mathbf{x} \succ_k \mathbf{y} \end{cases}.$$

For an EU preference with utility function v_k, $u_k(\mathbf{x};\mathbf{y}) = v_k(\mathbf{x})/v_k(\mathbf{y})$.

Let $\mathbf{u}(\mathbf{x};\mathbf{y})$ denote the vector $\langle u_k(\mathbf{x};\mathbf{y})\rangle_{k=1}^n$, where u_k are the induced utility functions of $\succsim_k$. For any W satisfying **IIA-NEU** on $\mathcal{D} \subseteq P^n$, define the set A_W, characterizing W in terms of induced utility vectors, as follows.

$$A_W = \{\mathbf{s} \in \mathbb{R}_+^n \,|\, \exists \succsim \in \mathcal{D}, \mathbf{x}, \mathbf{y} \in \mathbf{X} \text{ such that } \mathbf{u}(\mathbf{x};\mathbf{y}) = \mathbf{s},\ R = W(\succsim) \text{ and } \mathbf{x}R\mathbf{y}\}$$

Proposition 1. *Let W satisfy* **IIA-NEU** *on a non-empty* $\mathcal{D} \subseteq P^n$. *For any* $\succsim \in \mathcal{D}$, $R = W(\succsim)$ *and* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ *such that* $\mathbf{y} \gg \mathbf{x}^0$, $\mathbf{x}R\mathbf{y} \Leftrightarrow \mathbf{u}(\mathbf{x};\mathbf{y}) \in A_W$.

Proof. Let $\succsim \in \mathcal{D}$ and $R = W(\succsim)$. Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $\mathbf{y} \gg \mathbf{x}^0$. If $\mathbf{u}(\mathbf{x};\mathbf{y}) \in A_W$, then there exist $\succsim' \in \mathcal{D}$ with associated induced utility vector $\mathbf{u}'$ and $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}$, such that $\mathbf{u}'(\tilde{\mathbf{x}};\tilde{\mathbf{y}}) = \mathbf{u}(\mathbf{x};\mathbf{y})$, $R' = W(\succsim')$ and $\tilde{\mathbf{x}}R'\tilde{\mathbf{y}}$. Thus $\mathbf{x}R\mathbf{y}$ by **IIA-NEU**. The other direction is immediate from the definition of A_W.

When **IIA-NEU** is replaced by the separate axioms **IIA** and **NEU** under the domain restriction **CN**, the proof involves an additional step after establishing $\tilde{\mathbf{x}}R'\tilde{\mathbf{y}}$. Let the bijection in the definition of **CN**, $\psi : \mathbf{X} \to \mathbf{X}$, satisfy $\psi(\tilde{\mathbf{x}}) = \mathbf{x}$ and $\psi(\tilde{\mathbf{y}}) = \mathbf{y}$. Let $\succsim'' \in \mathcal{D}$ such that for every $k \in \mathbf{N}$, $p, q \in [0, 1]$ and $\mathbf{w}, \mathbf{z} \in \mathbf{X}$, $p\mathbf{w} \succsim'_k q\mathbf{z} \Leftrightarrow p\psi(\mathbf{w}) \succsim''_k q\psi(\mathbf{z})$. Let $R'' = W(\succsim'')$. Then $\mathbf{x}R''\mathbf{y}$ by **NEU**. Applying **IIA** with $\succsim$ and $\succsim''$, it follows that $\mathbf{x}R\mathbf{y}$. □

3. Society of size two

In this section we axiomatically characterize the social welfare function when only two individuals make up the society. We start by the following proposition, which characterizes the social preference using the set A_W that appears in Proposition 1.[7] Then we present an impossibility result, showing that the axioms are inconsistent over the domain $\mathcal{P}^2$. The impossibility is then recovered to a possibility result by restricting the domain.

Proposition 2. *Let* $n = 2$. Let *W satisfy* **WO**, **PAR**, **ANM** *and* **IIA-NEU** *under the domain restriction* **CMP**.

[7] Proposition 2 is not restricted to OB preferences. It holds in general for preferences that are continuous and monotone with respect to the relation of first order stochastic dominance for elementary lotteries.

(1) $W = ON$.
(2) *Let* $\succsim \in \mathcal{D}$ *and* $R = ON(\succsim)$. *For any* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x}R\mathbf{y}$ *if, and only if,* $\mathbf{y} \gg \mathbf{x}^0$ *implies* $u_1(\mathbf{x};\mathbf{y})u_2(\mathbf{x};\mathbf{y}) \geq 1$. *Furthermore, not* $\mathbf{x} \gg \mathbf{x}^0$ and $\mathbf{x}R\mathbf{y}$ *imply not* $\mathbf{y} \gg \mathbf{x}^0$.

Proof. In order to prove (1), we first show that $A_W = \{\mathbf{s} \in \mathbb{R}^2_+ | s_1 s_2 \geq 1\}$. Let $\mathbf{s}, \mathbf{t} \in \mathbb{R}^n_+$ such that $\mathbf{s} < \mathbf{t}$. For $\mathbf{s} > \mathbf{0}$, denote $(1/s_1, 1/s_2)$ by $\mathbf{s}^{-1}$. Denote $\mathbf{s}' = (s_2, s_1)$. Let $\succsim \in \mathcal{D} \cap \mathcal{D}^{\text{CM}}$. Let $R = W(\succsim)$ and $R' = W(\succsim_2, \succsim_1)$. By monotonicity of $\mathbf{u}$ and since $\succsim \in \mathcal{D}^{\text{CM}}$, there exist $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$ such that $\mathbf{x} < \mathbf{z}$, $\mathbf{y} \gg \mathbf{x}^0$, $\mathbf{s} = \mathbf{u}(\mathbf{x};\mathbf{y})$ and $\mathbf{t} = \mathbf{u}(\mathbf{z};\mathbf{y})$. Therefore $\mathbf{s}' = \mathbf{u}'(\mathbf{x};\mathbf{y})$, where $\mathbf{u}'$ is the induced utility vector of $\langle \succsim_2, \succsim_1 \rangle$. Moreover, $\mathbf{s} > \mathbf{0}$ implies $\mathbf{x} \gg \mathbf{x}^0$ and $\mathbf{z} \gg \mathbf{x}^0$. Thus $\mathbf{s} > \mathbf{0}$ implies $\mathbf{s}^{-1} = \mathbf{u}(\mathbf{y};\mathbf{x})$ and $\mathbf{t}^{-1} = \mathbf{u}(\mathbf{y};\mathbf{z})$. By **ANM**, $\mathbf{x}R\mathbf{y}$ if, and only if, $\mathbf{x}R'\mathbf{y}$. Using Proposition 1, the following three properties hold. (1) $\mathbf{s} \in A_W$ if, and only if, $\mathbf{s}' \in A_W$, (2) $\mathbf{0} < \mathbf{s} \notin A_W$ implies $\mathbf{s}^{-1} \in A_W$ (by completeness of R), and (3) $\mathbf{s} \in A_W$ implies $\mathbf{t} \in A_W$ and $\mathbf{t}^{-1} \notin A_W$ (this is because $\mathbf{s} \in A_W$ implies $\mathbf{z}P\mathbf{x}R\mathbf{y}$ by **PAR**, consequently $\mathbf{z}P\mathbf{y}$ by transitivity).

We now use the above three properties to show that $\mathbf{s} \in A_W$ if, and only if, $s_1 s_2 \geq 1$. Suppose that $\mathbf{s}, \mathbf{s}' \in A_W$. If $0 < s_1 s_2 < 1$, then $\mathbf{s}^{-1} > \mathbf{s}'$, since $(\mathbf{s}^{-1})_1 = 1/s_1 > s_2 = (\mathbf{s}')_1$ and $(\mathbf{s}^{-1})_2 = 1/s_2 > s_1 = (\mathbf{s}')_2$. Therefore $0 < s_1 s_2 < 1$ implies $\mathbf{s}^{-1} \in A_W$ and $\mathbf{s} \notin A_W$, a contradiction. If $s_1 s_2 = 0$, then for any $\tilde{\mathbf{s}} > \mathbf{s}$ such that $0 < \tilde{s}_1 \tilde{s}_2 < 1$, $\tilde{\mathbf{s}} \in A_W$, again a contradiction. Hence, $\mathbf{s} \in A_W$ implies $s_1 s_2 \geq 1$. To prove the converse, suppose that $s_1 s_2 \geq 1$. If $s_1 s_2 = 1$, then $\mathbf{s}^{-1} = \mathbf{s}'$. Thus $\mathbf{s}, \mathbf{s}' \in A_W$. If $s_1 s_2 > 1$ but $\mathbf{s} \notin A_W$, then $\mathbf{s}^{-1} \in A_W$. Thus $s_1 s_2 \leq 1$, a contradiction.

Assuming that (2) holds, we are ready to show that $W = ON$. Let $\succsim \in \mathcal{D}$, $R = W(\succsim)$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. By Proposition 1, it follows that $\mathbf{y} \gg \mathbf{x}^0$ implies $\mathbf{x}R\mathbf{y} \Leftrightarrow u_1(\mathbf{x};\mathbf{y})u_2(\mathbf{x};\mathbf{y}) \geq 1$. Furthermore, we show that not $\mathbf{y} \gg \mathbf{x}^0$ implies $\mathbf{x}R\mathbf{y}$. Suppose that not $\mathbf{y} \gg \mathbf{x}^0$ but $\mathbf{y}P\mathbf{x}$. **PAR** implies not $\mathbf{x} \gtrapprox \mathbf{y}$. Thus there exist $i \neq j$ and $p \in [0, 1]$ satisfying $\mathbf{y} \sim_i 0\mathbf{x}$ and $p\mathbf{y} \sim_j \mathbf{x}$. Therefore for any $\mathbf{z} \in \mathbf{X}$ such that $\mathbf{z} \succ_i \mathbf{x}^0$ and $\mathbf{x}^0 \sim_i 0\mathbf{z}$ and $p\mathbf{x}^0 \sim_j \mathbf{z}$, it follows that $\mathbf{x}^0 P \mathbf{z}$ by **IIA-NEU**. This contradicts **PAR**. Hence, $\mathbf{x}R\mathbf{y}$ if, and only if, $\mathbf{y} \gg \mathbf{x}^0$ implies $u_1(\mathbf{x};\mathbf{y})u_2(\mathbf{x};\mathbf{y}) \geq 1$. Thus assuming (2), $W = ON$.

It is left to prove (2). Let $\succsim \in \mathcal{D}$, $R' = ON(\succsim)$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Either $\mathbf{x} \gtrapprox \mathbf{y}$, or $\mathbf{y} \gtrapprox \mathbf{x}$, or there exist $i, j \in \mathbf{N}$ and $p, q \in [0, 1]$ such that $p\mathbf{x} \sim_i \mathbf{y}$, $q\mathbf{y} \sim_j \mathbf{x}$, where by definition $p = u_i(\mathbf{y};\mathbf{x})$, $q = u_j(\mathbf{x};\mathbf{y})$. Therefore $\mathbf{y} \gg \mathbf{x}^0$ implies $\mathbf{x}R'\mathbf{y} \Leftrightarrow \prod_k u_k(\mathbf{x};\mathbf{y}) \geq 1$. If not $\mathbf{y} \gg \mathbf{x}^0$, then $\mathbf{x}R'\mathbf{y}$, since either $\mathbf{x} \gtrapprox \mathbf{y}$, or $0 = p \leq q$. Therefore, for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x}R'\mathbf{y}$ if, and only if, $\mathbf{y} \gg \mathbf{x}^0$ implies $u_i(\mathbf{x};\mathbf{y})u_j(\mathbf{x};\mathbf{y}) \geq 1$. It remains to show that not $\mathbf{x} \gg \mathbf{x}^0$ and $\mathbf{x}R\mathbf{y}$ imply not $\mathbf{y} \gg \mathbf{x}^0$. This follows since not $\mathbf{x} \gg \mathbf{x}^0$ and $\mathbf{y} \gg \mathbf{x}^0$ imply $\prod_k u_k(\mathbf{x};\mathbf{y}) = 0$ and thus not $\mathbf{x}R\mathbf{y}$, a contradiction. □

Proposition 2 provides one direction of our axiomatic characterization. The rest of the section is devoted to the other direction, dealing with consistency of the axioms. In particular, the only issue is whether the social binary relation defined by ON is complete and transitive. Therefore, Corollary 1 is meaningful. Let $\mathcal{D}^C \subseteq \mathcal{P}^2$ be the set of all $\succsim$ for which $R = ON(\succsim)$ is a complete binary relation.

Corollary 1.

(1) $\mathcal{D}^C = \mathcal{P}^2$.
(2) *If* $\succsim_k$ *are EU, then* $R = ON(\succsim)$ *is also transitive*.

Proof.

(1) According to the induced utilities definition, for any $\mathbf{x}^0 < \mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $k \in \mathbf{N}$, $u_k(\mathbf{x};\mathbf{y})u_k(\mathbf{y};\mathbf{x}) = 1$. Thus either $\prod_k u_k(\mathbf{x};\mathbf{y}) \geq 1$, or $\prod_k u_k(\mathbf{y};\mathbf{x}) \geq 1$. Therefore R is complete by Proposition 2.
(2) Suppose that $\succsim_k$ are EU. For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ where $\mathbf{y} \gg \mathbf{x}^0$ and $k \in \mathbf{N}$, $u_k(\mathbf{x};\mathbf{y}) = v_k(\mathbf{x})/v_k(\mathbf{y})$. Let $\mathbf{x}R\mathbf{y}R\mathbf{z}$. If not $\mathbf{y} \gg \mathbf{x}^0$, or not $\mathbf{z} \gg \mathbf{x}^0$, then $\mathbf{x}R\mathbf{z}$ by Proposition 2. If $\mathbf{y} \gg \mathbf{x}^0$ and $\mathbf{z} \gg \mathbf{x}^0$, then $\prod_k u_k(\mathbf{x};\mathbf{z}) = \prod_k v_k(\mathbf{x})/v_k(\mathbf{z}) = \prod_k \left[v_k(\mathbf{x})/v_k(\mathbf{y})\right]\left[v_k(\mathbf{y})/v_k(\mathbf{z})\right] = \prod_k u_k(\mathbf{x};\mathbf{y})\prod_k u_k(\mathbf{y};\mathbf{z}) \geq 1$. Therefore $\mathbf{x}R\mathbf{z}$. □

The corollary demonstrates a domain, over which the social welfare function ON satisfies **WO**. Unfortunately, there exist examples of preference profiles in $\mathcal{P}^2$, for which the social binary relation is not transitive. This conclusion is stated below in Proposition 3 as an impossibility result.

Proposition 3. *There exists no social welfare function W satisfying* **WO**, **PAR**, **ANM** *and* **IIA-NEU** *over the domain* $\mathcal{D}=\mathcal{P}^2$.

Proof. By Proposition 2, $W = ON$. The impossibility is proved using a preference profile in $\mathcal{P}^2$, for which ON produces a non-transitive social binary relation. Such a profile is demonstrated in the following example. □

Example 1. Let $\mathbf{X}=\{\mathbf{x}\in\mathbb{R}^2_+|\sum_k x_k \le 9\}$ and $\mathbf{x}^0=(0,0)$ Let $\succsim\in\mathcal{P}^2$, where $\succsim_1$ is an EU preference with a vNM utility function $v_1(\mathbf{x})=x_1$ and $\succsim_2$ is a Disappointment Averse (DA) preference, for which $v_2(\mathbf{x})=x_2$, $g_2(p)=p/1+(1-p)\beta_2$ and $\beta_2=1$. The corresponding induced utility functions satisfy $u_1(\mathbf{x};\mathbf{y})=x_1/y_1$ and $u_2(\mathbf{x};\mathbf{y})=2/(1+y_2/x_2)$ for $\mathbf{y}\succsim_2\mathbf{x}$. Let $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{X}$, where $\mathbf{x}=(6,1)$, $\mathbf{y}=(3,3)$ and $\mathbf{z}=(2,6)$. Then $\prod_k u_k(\mathbf{x};\mathbf{y})=6(/3)2/(1+3/1)=1$, $\prod_k u_k(\mathbf{y};\mathbf{z})=(3/2)2/(1+6/3)=1$ and $\prod_k u_k(\mathbf{x};\mathbf{z})=(6/2)2/(1+6/1)<1$. Therefore, by Proposition 2, $\mathbf{x}I\mathbf{y}I\mathbf{z}$, but $\mathbf{z}P\mathbf{x}$.

Example 1 motivates the investigation of domains, over which ON produces a transitive social binary relation. In the example, the individuals have different probability distortion functions, one of them fits the EU assumption, while the other does not. It turns out that this fact is crucial to transitivity, as can be seen by Proposition 4. The result proves a necessary and sufficient condition for transitivity, defined as follows.

Definition 11. **UD** (uniform distortion): A pair of preferences $\succsim\in\mathcal{P}^2$ satisfies **UD** if there exists a bijection $\phi:\mathbf{X}\to\mathbf{X}$ such that for every $p,q\in[0,1]$ and $\mathbf{x},\mathbf{y}\in\mathbf{X}$,

$$p\mathbf{x}\succsim_1 q\mathbf{y}\Leftrightarrow p\phi(\mathbf{x})\succsim_2 q\phi(\mathbf{y}).$$

The set of all preference pairs satisfying **UD** is denoted by $\mathcal{D}^{\mathrm{UD}}$.

Proposition 4. *Let* $\succsim\in\mathcal{D}^{\mathrm{CM}}$ *and* $R=ON(\succsim)$. *The followings are equivalent*:

(1) *R is transitive*
(2) $\succsim\in\mathcal{D}^{\mathrm{UD}}$
(3) *for some* $\delta>0$, $g_1=(g_2)^\delta$
(4) $\succsim_k$ *have origin biseparable representations* $g_k v_k$, *such that* $g_1=g_2$.

Proof. Let $g_k v_k$ be an origin biseparable representation of $\succsim_k$. First we show that (2) implies (4). Suppose that $\succsim\in\mathcal{D}^{\mathrm{UD}}$. Define $\tilde{v}_1$ by $\tilde{v}_1(\mathbf{x})=v_2[\phi(\mathbf{x})]$. Then $\succsim_1$ can be represented for any elementary lottery $p\mathbf{x}$ by $g_2(p)\tilde{v}_1(\mathbf{x})$. This is exactly (4).

Now we show that (4) implies (1). Suppose that $g_1=g_2=g$. Let $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{X}$ such that $\mathbf{x}R\mathbf{y}R\mathbf{z}$. If $\mathbf{x}\gtrapprox\mathbf{z}$, or not $\mathbf{x}\gg\mathbf{x}^0$, or not $\mathbf{y}\gg\mathbf{x}^0$, or not $\mathbf{z}\gg\mathbf{x}^0$, then $\mathbf{x}R\mathbf{z}$ by Proposition 2. Otherwise, by Proposition 2, either $u_k(\mathbf{x};\mathbf{y})\ge 1$ for both $k\in\mathbf{N}$, or there exist $i\in\mathbf{N}$ such that $g^{-1}(v_i(\mathbf{y})/v_i(\mathbf{x}))=u_i(\mathbf{y};\mathbf{x})\le u_j(\mathbf{x};\mathbf{y})=g^{-1}(v_j(\mathbf{x})/v_j(\mathbf{y}))$. Thus $\prod_k v_k(\mathbf{x})/v_k(\mathbf{y})\ge 1$. Similarly, $\prod_k v_k(\mathbf{y})/v_k(\mathbf{z})\ge 1$. Therefore $\prod_k v_k(\mathbf{x})/v_k(\mathbf{z})\ge 1$. Let $i,j\in\mathbf{N}$ such that $\mathbf{z}\succsim_i\mathbf{x}$ and $\mathbf{x}\succsim_j\mathbf{z}$. Then $u_i(\mathbf{x};\mathbf{z})=g^{-1}(v_i(\mathbf{x})/v_i(\mathbf{z}))\ge g^{-1}(v_j(\mathbf{z})/v_j(\mathbf{x}))=u_j(\mathbf{z};\mathbf{x})$. Thus $\mathbf{x}R\mathbf{z}$ by Proposition 2. Hence R is transitive.

Next we show that (1) implies (3). Let $f_2:[0,1]\to[0,1]$ such that $\forall\alpha\in[0,1]$, $f_2(\alpha)=g_1[g_2^{-1}(\alpha)]$. Let $\alpha_1,\beta_1\in[0,1]$, $\alpha_2=f_2(\alpha_1)$, $\beta_2=f_2(\beta_1)$ and $\mathbf{x}^0<\mathbf{w}\in\mathbf{X}$. By monotonicity of both v_i and since $\succsim\in\mathcal{D}^{\mathrm{CM}}$, there exist $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{X}$ such that $\mathbf{x}\sim_2\mathbf{w}$, $v_1(\mathbf{x})=\alpha_2\beta_2 v_1(\mathbf{w})$, $v_1(\mathbf{y})=\alpha_2 v_1(\mathbf{w})$, $v_2(\mathbf{y})=\beta_1 v_2(\mathbf{w})$, $\mathbf{z}\sim_1\mathbf{w}$ and $v_2(\mathbf{z})=\alpha_1\beta_1 v_2(\mathbf{w})$. Therefore $u_2(\mathbf{z};\mathbf{y})=g_2^{-1}(\alpha_1)=g_1^{-1}(\alpha_2)=u_1(\mathbf{y};\mathbf{z})$ and $u_2(\mathbf{y};\mathbf{x})=g_2^{-1}(\beta_1)=g_1^{-1}(\beta_2)=u_1(\mathbf{x};\mathbf{y})$. Thus $\mathbf{x}I\mathbf{y}I\mathbf{z}$. Transitivity implies $\mathbf{x}I\mathbf{z}$. Thus $g_2^{-1}(\alpha_1\beta_1)=u_2(\mathbf{z};\mathbf{x})=u_1(\mathbf{x};\mathbf{z})=g_1^{-1}(\alpha_2\beta_2)$. Therefore we get Cauchy's power functional equation $\forall\alpha_1,\beta_1\in[0,1]$, $f_2(\alpha_1\beta_1)=f_2(\alpha_1)f_2(\beta_1)$. The unique solution f_2, that is continuous and strictly increasing, satisfies for some $\delta_2>0$ and $\forall\alpha\in[0,1]$, $f_2(\alpha)=\alpha^{\delta_2}$. Hence, $\forall p\in[0,1], g_1(p)=f_2[g_2(p)]=[g_2(p)]^{\delta_2}$.

Finally, that (3) implies (2) follows by normalizing each v_k such that its image is [0,1] and defining ϕ such that $v_1(\mathbf{x})=[v_2[\phi(\mathbf{x})]]^\delta$. □

Note that in Proposition 4, $\mathcal{D} \subseteq \mathcal{D}^{\mathrm{CM}}$ is needed only for (2), (3) and (4) to be implied by (1). The equivalence of properties (2)–(4) and each implying (1) hold also under the weaker domain restriction **CMP**. Roughly speaking, the "closer" is a profile $\succsim$ to $\mathcal{D}^{\mathrm{CM}}$, the "closer" it is also to $\mathcal{D}^{\mathrm{UD}}$ as a result of transitivity implications.

Proposition 4 shows that EU is not necessary to derive transitivity of the *ON* social binary relation. We can have profiles of non-EU preferences, as long as all preferences violate the EU assumption in a similar way. Note that the result permits the characterization of transitivity through a condition that depends only on the distortion functions g_k, not on the functions v_k. This separation is permissible due to the origin biseparable property of the preferences in $\mathcal{P}$.

Following are examples of subsets of $\mathcal{D}^{\mathrm{UD}}$ formed by intersection with known families of preferences.

Example 2.

(1) DL preferences. The set $\mathcal{D}^{\mathrm{UD}}$ contains all DL preference (Grant and Kajii (1995)) profiles, including all EU preference profiles. DL preferences can be represented with $g(p) = p, \forall p \in [0, 1]$.
(2) RDU preferences. The set $\mathcal{D}^{\mathrm{UD}}$ contains all profiles of RDU preferences, for which the distortion functions are g^{δ_k} for some g and $\delta_k > 0, \forall k \in \mathbf{N}$. In other words, the individual distortion functions are the same up to a positive power transformation. This allows for a very wide range of risk attitudes, as characterized by both the power parameter δ_k and by the entirely unrestricted curvature of the functions v_k. For example, since g may not be the identity function, this allows behavior which is consistent with the 'Allais paradox' and the 'common ratio effect'. This behavior is excluded by EU or DL preferences. Note that taking a positive power transformation of both g and v_k preserves the property UD, but does not preserve a RDU representation for general lotteries. This demonstrates that the kind of restriction implemented by UD is in a sense weak, since only preferences over elementary lotteries are involved. In other words, the requirement that all individuals have identical RDU distortion functions for general lotteries is much more restrictive than the property UD.[8]
(3) DA preferences. The set $\mathcal{D}^{\mathrm{UD}}$ contains all profiles of DA preferences, for which $g(p) = p/[1 + (1 - p)\beta], \forall k \in \mathbf{N}$. In other words, the individual disappointment aversion parameters β_k are equal in these profiles (see Example 1 for the case of non-transitivity under which this condition is violated).

Proposition 4 states that the *ON* social binary relation is transitive only for all the profiles in the set $\mathcal{D}^{\mathrm{UD}}$. This seems to be a disappointing result, since it limits the applicability of the preference aggregation rule implied by the axioms. But it also has an interesting positive implication concerning the role of the EU assumption in the derivation of the Nash social welfare function. It turns out that the axioms that imply the preference aggregation rule are consistent in the case of EU preferences only because these preferences have a common feature with respect to risk attitude. The important issue is the initial similarity in risk attitudes (equality of distortion functions up to positive power transformations). Whether it is EU or some other assumption on risk attitude, the crucial factor is that all individuals agree on it. The results show that the social welfare function applies to a rich setting, within which EU or DL preferences constitute an extremely small subset. This conclusion suggests that the EU assumption is too restrictive in the analysis of the connection between individual and social choice.

We are now able to provide the main characterization theorem of this section. The theorem states that the function *ON* is characterized uniquely by the axioms on $\mathcal{D}^{\mathrm{UD}}$, where it satisfies **WO** and moreover has a Nash-like utility product representation.

Theorem 1. *Let $n = 2$ and consider a domain $\mathcal{D}$ satisfying the domain restrictions* **CMP** *and $\mathcal{D} \setminus \mathcal{D}^{\mathrm{CM}} \subseteq \mathcal{D}^{\mathrm{UD}}$.*

(1) *ON is the unique social welfare function satisfying* **WO**, **PAR**, **ANM** *and* **IIA-NEU**. *Furthermore, $\mathcal{D} \subseteq \mathcal{D}^{\mathrm{UD}}$, e.g. $\mathcal{D} = \mathcal{D}^{\mathrm{UD}}$.*
(2) *For every $\succsim \in \mathcal{D}$, there exists an origin biseparable representation $\langle gv_1, gv_2\rangle$ such that for every $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x}R\mathbf{y}$ if, and only if, $\prod_k v_k(\mathbf{x}) \geq \prod_k v_k(\mathbf{y})$.*

Proof. The part of (1) which states that *ON* is implied by the axioms follows immediately from Proposition 2. Proposition 4 implies $\mathcal{D} \cap \mathcal{D}^{\mathrm{CM}} \subseteq \mathcal{D}^{\mathrm{UD}}$. We now prove (2). Let $\langle gv_k\rangle_{k=1}^{n}$ be an origin biseparable representation of $\succsim$.

[8] In the case of expected utility this observation has no implication since uniform distortion holds for general lotteries by not being distorted at all.

Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. If not $\mathbf{y} \gg \mathbf{x}^0$, then $\mathbf{x}R\mathbf{y}$ by Proposition 2 and $\prod_i v_i(\mathbf{x}) \geq \prod_i v_i(\mathbf{y}) = 0$. If $\mathbf{x}, \mathbf{y} \gg \mathbf{x}^0$, then $\mathbf{x}R\mathbf{y}$ if, and only if, either $v_k(\mathbf{x}) \geq v_k(\mathbf{y})$ for both $k \in \mathbf{N}$, or there exist $i \in \mathbf{N}$ such that $g^{-1}(v_i(\mathbf{y})/v_i(\mathbf{x})) \leq g^{-1}(v_j(\mathbf{x})/v_j(\mathbf{y}))$. Therefore, $\mathbf{x}R\mathbf{y}$ if, and only if, $\prod_k v_k(\mathbf{x}) \geq \prod_k v_k(\mathbf{y})$.

It is left to prove consistency of the axioms by showing that *ON* satisfies them. Completeness and transitivity follow from Corollary 1 and Proposition 4 (the proof does not rely on being a member of $\mathcal{D}^{\mathrm{CM}}$). If $\mathbf{x} \gtrapprox \mathbf{y}$ and $\mathbf{x} \succ_j \mathbf{y}$ for some j, then $\prod_k v_k(\mathbf{x}) > \prod_k v_k(\mathbf{y})$. Therefore $\mathbf{xPy}$. Hence **PAR** is satisfied. Axiom **ANM** is satisfied, since for any permutation π of the players and any $\mathbf{x} \in \mathbf{X}$, $\prod_k v_{\pi_k}(\mathbf{x}) = \prod_k v_k(\mathbf{x})$. Now we check axiom **IIA-NEU**. $\mathbf{x}R\mathbf{y} \gg \mathbf{x}^0$ implies $\prod_k u_k(\mathbf{x}; \mathbf{y}) \geq 1$ by Proposition 2. $u_k(\mathbf{x}; \mathbf{y}) = u'_k(\tilde{\mathbf{x}}; \tilde{\mathbf{y}})$ for each $k \in \mathbf{N}$ implies $\prod_k u'_k(\tilde{\mathbf{x}}; \tilde{\mathbf{y}}) \geq 1$. Therefore $\tilde{\mathbf{x}}R'\tilde{\mathbf{y}}$. If not $\mathbf{y} \gg \mathbf{x}^0$, then not $\tilde{\mathbf{y}} \gg \tilde{\mathbf{x}}^0$. Thus $\tilde{\mathbf{x}}R'\tilde{\mathbf{y}}$. Hence **IIA-NEU** is satisfied. □

Utilizing the *ON* social welfare function, in the case of EU preferences, Theorem 1 provides an interpretation of Nash's utility product maximization principle for social choice. The theorem also extends this interpretation by considering non-EU preference profiles in $\mathcal{P}$. Furthermore, this extension yields the largest subset of $\mathcal{D}^{\mathrm{CM}}$ for which the interpretation is valid, i.e. for which the *ON* social welfare function satisfies **WO**. For this extended domain, when choosing appropriately the functions v_k to have identical distortion functions g_k, the information provided by v_k is sufficient to derive the social preference. This conclusion holds despite the fact that v_k generally convey only partial information about the preferences in $\mathcal{P}$.

To conclude this section, the next proposition shows that the axioms are independent under the domain restrictions of Theorem 1.

Proposition 5. **WO**, **PAR**, **ANM** and **IIA-NEU** *are independent under the domain restrictions* **CMP** *and* $\mathcal{D} \subseteq \mathcal{D}^{\mathrm{UD}}$.

Proof. The proof establishes independence by providing, for each axiom, a social welfare function that violates this axiom but satisfies the other axioms.

(1) **WO**. Consider the social welfare function defined by unanimity, i.e. $\mathbf{x}R\mathbf{y}$ if, and only if, $\mathbf{x} \gtrapprox \mathbf{y}$. This function clearly satisfies **PAR**, **ANM** and **IIA-NEU**. To see that it violates **WO**, consider a profile in $\mathcal{D} \cap \mathcal{D}^{\mathrm{CM}}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$ such that $\mathbf{x} \gg \mathbf{z}$ but $\mathbf{x} \not\gtrapprox \mathbf{y}$, $\mathbf{y} \not\gtrapprox \mathbf{x}$, $\mathbf{y} \not\gtrapprox \mathbf{z}$ and $\mathbf{z} \not\gtrapprox \mathbf{y}$. Then $\mathbf{x}I\mathbf{y}I\mathbf{z}$, but $\mathbf{x}P\mathbf{z}$.
(2) **PAR**. The proof clearly follows by considering the social welfare function that assigns for each profile in the domain $\mathcal{D}$ the social preference defined by indifference for all states in $\mathbf{X}$.
(3) **ANM**. Consider the social welfare function defined by dictatorship, i.e. $\mathbf{x}R\mathbf{y}$ if, and only if, $\mathbf{x} \succsim_1 \mathbf{y}$. This function clearly violates **ANM**, while satisfying **WO**, **PAR** and **IIA-NEU**.
(4) **IIA-NEU**. Under the domain restriction $\mathcal{D} \subseteq \mathcal{D}^{\mathrm{UD}}$, consider origin biseparable representations gv_k of $\succsim_k$. Consider the social welfare function defined by utilitarianism, i.e. $\mathbf{x}R\mathbf{y}$ if, and only if, $v_1(\mathbf{x}) + v_2(\mathbf{x}) \geq v_1(\mathbf{y}) + v_2(\mathbf{y})$. This function clearly satisfies **WO**, **PAR** and **ANM**. To see that it violates **IIA-NEU**, consider $\succsim, \langle \succsim_2, \succsim_1 \rangle \in \mathcal{D} \cap \mathcal{D}^{\mathrm{CM}}$. Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $v_1(\mathbf{x}) + v_2(\mathbf{x}) = v_1(\mathbf{y}) + v_2(\mathbf{y})$ and $v_k(\mathbf{x}) \neq v_k(\mathbf{y})$ for both k. Let $\alpha_1, \alpha_2 \in (0, 1)$ and $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}$ such that $\alpha_1 \neq \alpha_2$ and for both i and $j \neq i$, $v_i(\tilde{\mathbf{x}}) = \alpha_j v_j(\mathbf{x})$ and $v_i(\tilde{\mathbf{y}}) = \alpha_j v_j(\mathbf{y})$. Since $v_i(\tilde{\mathbf{x}})/v_i(\tilde{\mathbf{y}}) = v_j(\mathbf{x})/v_j(\mathbf{y})$ for both i, the conditions of **IIA-NEU** are satisfied. However, $v_1(\tilde{\mathbf{x}}) + v_2(\tilde{\mathbf{x}}) \neq v_1(\tilde{\mathbf{y}}) + v_2(\tilde{\mathbf{y}})$. Thus $\mathbf{x}I\mathbf{y}$, but not $\tilde{\mathbf{x}}I\tilde{\mathbf{y}}$, violating the axiom. □

4. Society of size greater than two

In the general case, where the society has more than two individuals, the *ON* interpretation can be extended to allow comparisons between social states that are different for more than two individuals. Such a comparison is not as clear as in the case of $n = 2$. For example, in the case where $n = 3$, a social state may be preferable for one individual but worse for the other two, thus a direct comparison as in the case where $n = 2$ is not possible. In this scenario, the use of the majority rule for example, would not express the comparison in welfare improvement as considered in the former definition. In order to incorporate such comparisons, we extend the social preference definition by utilizing its transitivity property when a direct comparison is not possible. This extension is shown to be implied by the axioms in the main characterization Theorem 2 of the next section.

Definition 12. Given a profile $\succsim \in \mathcal{P}^n$, for any $l = 1, 2, \ldots$, define the binary relations $R^l \in \mathcal{B}$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

(1) $\mathbf{x}R^1\mathbf{y}$ if there exist $i, j \in \mathbf{N}$ such that $\mathbf{x} \sim_k \mathbf{y}$ for each $k \in \mathbf{N} \setminus \{i, j\}$ and either $\mathbf{x} \succsim_k \mathbf{y}$ for $k \in \{i, j\}$, or there exist $p, q \in [0, 1]$ such that $p\mathbf{x} \sim_i \mathbf{y}$, $q\mathbf{y} \sim_j \mathbf{x}$ and $p \leq q$.
(2) For any $l \geq 2$, $\mathbf{x}R^l\mathbf{y}$ if there exist $\mathbf{z} \in \mathbf{X}$ such that $\mathbf{x}R^{l-1}\mathbf{z}R^1\mathbf{y}$.

Definition 13. The ordinal Nash social welfare function, denoted ON, assigns for a preference profile $\succsim \in \mathcal{P}^n$, a binary relation $R \in \mathcal{B}$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x}R\mathbf{y}$ if

(1) $\mathbf{x}R^1\mathbf{y}$ or,
(2) $\mathbf{x}R^l\mathbf{y}$ for $l > 1$ and $\#\{k \in \mathbf{N} | x_k \neq y_k\} > 2$.

This definition, despite being long, is a straightforward generalization of the one for $n = 2$. The latter is provided by R^1, except that all individuals other than the two which are involved in the direct comparison must be indifferent between the states being compared. For R^l where $l \geq 2$, we get a preference that takes into account higher transitivity implications of R^1. Definition 13 is related to the extension of the ordinal Nash bargaining outcome to $n > 2$, which was proposed without axiomatic characterization by Burgos et al. (2002a, b)[9]. When $n > 2$ and the social states being compared matter for more than two individuals, the preference is determined indirectly through a sequence of comparisons as in the $n = 2$ scenario. An illustration of an indirect comparison is presented in the following example.

Example 3. Let $n = 3$, $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n_+ | \sum_k x_k \leq 9\}$ and $\mathbf{x}^0 = (0, 0, 0)$. Let $\succsim \in \mathcal{P}^n$, where $\succsim_k$ are DL preferences, i.e. $g_k(p) = p \forall p \in [0, 1]$, and $v_k(\mathbf{x}) = x_k$. Let $\mathbf{x} = (3, 3, 3)$ and $\mathbf{y} = (1, 4, 4)$. Then xR^1y does not hold since $\mathbf{x}, \mathbf{y}$ are different for all $k \in \mathbf{N}$. Nevertheless, $\mathbf{z} = (2, 4, 3) \in \mathbf{X}$ satisfies $\mathbf{x}P^1\mathbf{z}$, since $\mathbf{x}, \mathbf{z}$ matter only for individuals 1 and 2, $2/3\mathbf{x} \sim_1 \mathbf{z}$, $3/4\mathbf{z} \sim_2 \mathbf{x}$ and $2/3 < 3/4$. Similarly, $\mathbf{z}P^1\mathbf{y}$, since $\mathbf{z}, \mathbf{y}$ matter only for individuals 1 and 3, $1/2\mathbf{z} \sim_1 \mathbf{y}$, $3/4\mathbf{y} \sim_3 \mathbf{z}$ and $1/2 < 3/4$. Thus, $\mathbf{x}R^2\mathbf{y}$ and therefore $\mathbf{x}R\mathbf{y}$. Note that in order to deduce $\mathbf{x}P\mathbf{y}$, we must show that $\mathbf{y}R^l\mathbf{x}$ does not hold for any $l \geq 2$.

Although the ON social preference definition is extended based on a sequence of comparisons, thus ensuring transitivity when a direct comparison is not possible, transitivity does not necessarily hold in general. The example of non-transitivity given in Section 3 is also relevant here if we assume that all individuals except two are indifferent between the states in the example. Furthermore, in contrast to the case where $n = 2$, the extended social binary relation is not necessarily complete, since the existence of a sequence which permits indirect comparison is not guaranteed for all pairs of social states. Hence, we are interested in domains over which ON satisfies the axioms and in particular **WO**. As before, the induced utilities are useful in this analysis. The following proposition is an analogue of Proposition 4. It extends the necessary and sufficient conditions for the ON social binary relation to be transitive (proof in the appendix). The definition of $\mathcal{D}^{\mathrm{UD}}$ extends to $n > 2$ in a straightforward way by requiring, for every $k > 1$, the existence of ϕ_k satisfying $p\mathbf{x} \succsim_1 q\mathbf{y} \Leftrightarrow p\phi_k(\mathbf{x}) \succsim_k q\phi_k(\mathbf{y})$ for any $p, q \in [0, 1]$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$.

Proposition 6. *Let $\succsim \in \mathcal{D}^{\mathrm{CM}}$ and $R = ON(\succsim)$. The followings are equivalent*:

(1) *R is transitive*
(2) $\succsim \in \mathcal{D}^{\mathrm{UD}}$
(3) *for each $k \in \mathbf{N}$* there exists some $\delta_k > 0$ *such that* $g_1 = (g_k)^{\delta_k}$
(4) *there exist origin biseparable representations $g_k v_k$ such that for each $k \in \mathbf{N}$, $g_k = g_1$.*

Proposition 7 and its interpretations are analogous to part (2) of Theorem 1. The proposition states conditions under which the function ON satisfies **WO** and has a Nash-like utility product representation (proof in the appendix). As before, let $\mathcal{D}^C \subseteq \mathcal{P}^n$ be the set of all $\succsim$ for which $R = ON(\succsim)$ is a complete binary relation (the analysis of the set $\mathcal{D}^C$ appears at the end of this section).

[9] Burgos et al. define their extension by allowing a bargainer to seek a concession from only one other bargainer at a time. They also show that allowing a bargainer to seek concessions simultaneously from any number of the other bargainers may result in non-existence of a bargaining outcome. This impossibility may be used to justify our extension of the social preference for $n > 2$.

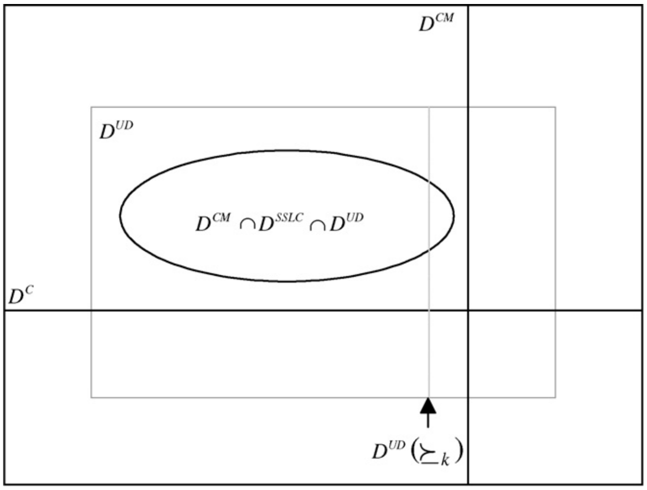

Fig. 1. Venn diagram of domain restrictions.

Proposition 7. *Let* $\succsim \in \mathcal{D}^C \cap \mathcal{D}^{\text{UD}}$, where $\succsim_k$ *have origin biseparable representations* gv_k, *and let* $R = ON(\succsim)$. *For any* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x}R\mathbf{y} \Leftrightarrow \prod_k v_k(\mathbf{x}) \geq \prod_k v_k(\mathbf{y})$.

The axioms **PAR** and **ANM** do not impose any further constraints on the domain investigated. Unlike the case of $n = 2$, a different conclusion holds for the axiom **IIA-NEU**, since there exist domains for which the *ON* social welfare function does not satisfy this axiom. Proposition 8 considers domains as of Propositions 6 and 7, and presents necessary and sufficient conditions under which the axiom **IIA-NEU** is satisfied (see Fig. 1). Note that we are still not confined to the EU case.

Definition 14. Given $k \in \mathbf{N}$ and $\succsim_k \in \mathcal{P}$, $\mathcal{D}^{\text{UD}}(\succsim_k)$ is the set of all preference profiles $\succsim' \in \mathcal{P}^n$ such that for each $i \in \mathbf{N}$ there exists a bijection $\phi_i : \mathbf{X} \to \mathbf{X}$ such that for every $p, q \in [0, 1]$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

$$p\mathbf{x} \succsim_k q\mathbf{y} \Leftrightarrow p\phi_i(\mathbf{x}) \succsim'_i q\phi_i(\mathbf{y}).$$

Proposition 8. *Let* $\mathcal{D} \subseteq \mathcal{D}^{\text{CM}} \cap \mathcal{D}^C \cap \mathcal{D}^{\text{UD}}$. *ON* satisfies **IIA-NEU** *on* $\mathcal{D}$ *if and only if* $\mathcal{D} \subseteq \mathcal{D}^{\text{UD}}(\succsim_k)$ *for some* $k \in \mathbf{N}$ *and* $\succsim_k \in \mathcal{P}$.

Proof. Let $\succsim, \succsim' \in \mathcal{D}$ and let gv_k, $g'v'_k$ be the respective origin biseparable representations. Let $R = ON(\succsim)$, $R' = ON(\succsim')$ and let u_k, u'_k be the induced utility functions of $\succsim_k, \succsim'_k$, respectively.

To prove the 'if' direction, suppose without loss of generality that $\mathcal{D} \subseteq \mathcal{D}^{\text{UD}}(\succsim_1)$. For $\succsim'$, let ϕ_1 be the bijection that appears for $i = 1$ in the definition of $\mathcal{D}^{\text{UD}}(\succsim_1)$. Defining $\tilde{v}_1$ by $\tilde{v}_1(\mathbf{x}) = v_1[\phi_1(\mathbf{x})]$, $\succsim_1$ can be represented for any elementary lottery $p\mathbf{x}$ by $g'(p)\tilde{v}_1(\mathbf{x})$. Thus without loss of generality we can assume $g = g'$. Verifying IIA, suppose $u_k(\mathbf{x}; \mathbf{y}) = u'_k(\tilde{\mathbf{x}}; \tilde{\mathbf{y}})$ for every $k \in \mathbf{N}$. Thus $v_k(\mathbf{x})/v_k(\mathbf{y}) = v'_k(\tilde{\mathbf{x}})/v'_k(\tilde{\mathbf{y}})$ for every $k \in \mathbf{N}$. If $\mathbf{x}R\mathbf{y} \gg \mathbf{x}^0$, then $\prod_k v_k(\mathbf{x}) \geq \prod_k v_k(\mathbf{y})$ by Proposition 7. Therefore $\prod_k v'_k(\tilde{\mathbf{x}}) \geq \prod_k v'_k(\tilde{\mathbf{y}})$. Thus $\tilde{\mathbf{x}}R'\tilde{\mathbf{y}}$. If not $\mathbf{y} \gg \mathbf{x}^0$, then $\prod_k v_k(\mathbf{y}) = \prod_k v'_k(\tilde{\mathbf{y}}) = 0 \leq \prod_k v'_k(\tilde{\mathbf{x}})$. Thus $\tilde{\mathbf{x}}R'\tilde{\mathbf{y}}$, again by Proposition 7. Hence, *ON* satisfies **IIA-NEU** on $\mathcal{D}$.

To prove the 'only if' direction, suppose *ON* satisfies **IIA-NEU** on $\mathcal{D}$. Let $f : [0, 1] \to [0, 1]$ such that $\forall \alpha \in [0, 1]$, $f(\alpha) = g'[g^{-1}(\alpha)]$. Let $\alpha, \beta \in [0, 1]$, $i, j, l \in \mathbf{N}$ and $\mathbf{x}^0 < \mathbf{w}, \tilde{\mathbf{w}} \in \mathbf{X}$ such that $\mathbf{w} \sim_k \tilde{\mathbf{w}}$ for $k \in \mathbf{N} \setminus \{i, j, l\}$. Then by monotonicity of v_k and since $\succsim, \succsim' \in \mathcal{D}^{\text{CM}}$, there exist $\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}$ such that $\mathbf{x} \sim_k \mathbf{w}$ for $k \in \mathbf{N} \setminus \{i, j\}$, $v_i(\mathbf{x}) = \alpha v_i(\mathbf{w})$, $v_j(\mathbf{x}) = \beta v_j(\mathbf{w})$, $\mathbf{y} \sim_k \mathbf{w}$ for $k \in \mathbf{N} \setminus \{l\}$, $v_l(\mathbf{y}) = \alpha\beta v_l(\mathbf{w})$, $\tilde{\mathbf{x}} \sim_k \tilde{\mathbf{w}}$ for $k \in \mathbf{N} \setminus \{i, j\}$, $v'_i(\tilde{\mathbf{x}}) = f(\alpha)v'_i(\tilde{\mathbf{w}})$, $v'_j(\tilde{\mathbf{x}}) = f(\beta)v'_j(\tilde{\mathbf{w}})$, $\tilde{\mathbf{y}} \sim_k \tilde{\mathbf{w}}$ for $k \in \mathbf{N} \setminus \{l\}$ and $v'_l(\tilde{\mathbf{y}}) = f(\alpha\beta)v'_l(\tilde{\mathbf{w}})$. Thus, $\prod_k v_k(\mathbf{x}) = \prod_k v_k(\mathbf{y})$. Therefore $\mathbf{x}I\mathbf{y}$ by Proposition 7. Furthermore, for every $k \in \mathbf{N}$, $u'_k(\tilde{\mathbf{x}}; \tilde{\mathbf{y}}) = u_k(\mathbf{x}; \mathbf{y})$. Thus $\tilde{\mathbf{x}}I'\tilde{\mathbf{y}}$ by **IIA-NEU**. Therefore $\prod_k v'_k(\tilde{\mathbf{x}}) = \prod_k v'_k(\tilde{\mathbf{y}})$. Thus we have Cauchy's power functional equation $\forall \alpha, \beta \in [0, 1]$, $f(\alpha\beta) = f(\alpha)f(\beta)$. The unique solution *f*, that is continuous and strictly increasing, satisfies for some $\delta > 0$ and $\forall \alpha \in [0, 1]$, $f(\alpha) = \alpha^\delta$. Hence, $\forall p \in [0, 1]$, $g'(p) = f[g(p)] = [g(p)]^\delta$. Thus, for every $k \in \mathbf{N}$, $\succsim'_k$ is represented for any elementary lottery $p\mathbf{x}$ by $g(p)[v'_k(\mathbf{x})]^{1/\delta}$. Normalizing all v_k, v'_k such that their image is [0, 1], we can define ϕ_i, ϕ'_i such that $v_1(\mathbf{x}) = v_i[\phi_i(\mathbf{x})]$ and $v_1(\mathbf{x}) = [v'_i[\phi'_i(\mathbf{x})]]^{\delta}$. Hence $\mathcal{D} \subseteq \mathcal{D}^{\mathrm{UD}}(\succsim_1)$. □

Note that in Proposition 8, $\mathcal{D} \subseteq \mathcal{D}^{\mathrm{CM}}$ is needed only for the 'only if' direction. The 'if' direction holds also under the weaker domain restriction **CMP** and $\mathcal{D} \subseteq \mathcal{D}^{C} \cap \mathcal{D}^{\mathrm{UD}}$. Similarly to Proposition 4, roughly speaking, the "closer" are two profiles $\succsim, \succsim'$ to $\mathcal{D}^{\mathrm{CM}}$, the "closer" they are also to $\mathcal{D}^{\mathrm{UD}}(\succsim_k)$ for some $\succsim_k$ in these profiles, as a result of implications of **IIA-NEU**.

In order to ensure completeness, it is possible to add a smoothness and convexity assumption as sufficient conditions. The extra assumption may be applied only to the utility image of the set **X** and not to the distortion functions g_k. The convexity condition we suggest is not very strong since it merely applies to the set of log-utilities of the social states in **X**.

Definition 15. **SSLC** (Smoothness and Strict Log-Convexity): A preference profile $\succsim \in \mathcal{P}^n$ satisfies **SSLC** if there exist origin biseparable representations $g_k v_k$ such that $\{\langle \log v_k(\mathbf{x})\rangle_{k=1}^n \in \mathbb{R}^n | \mathbf{x}^0 < \mathbf{x} \in \mathbf{X}\}$ has a smooth boundary and is strictly convex.[10]

Let $\mathcal{D}^{\mathrm{SSLC}} \subseteq \mathcal{P}^n$ be the set of all profiles $\succsim$ that satisfy **SSLC**.[11] The property **SSLC** implies geometric conditions which are sufficient for the completeness of the social binary relation. These conditions are presented in Lemma A.2 and are used in the following Proposition 9 (the lemma is presented in the appendix). Proposition 9 is analogous to Corollary 1. The difference between the two is the addition of the **SSLC** condition here to ensure completeness of the social binary relation in the case where $n > 2$.

Proposition 9. *The domain* $\mathcal{D}^{\mathrm{CM}} \cap \mathcal{D}^{\mathrm{SSLC}} \cap \mathcal{D}^{\mathrm{UD}} \subseteq \mathcal{D}^{C}$, i.e. *the function ON yields a complete binary relation on* $\mathcal{D}^{\mathrm{CM}} \cap \mathcal{D}^{\mathrm{SSLC}} \cap \mathcal{D}^{\mathrm{UD}}$.

Proof. Let $\succsim \in \mathcal{D}^{\mathrm{CM}} \cap \mathcal{D}^{\mathrm{SSLC}} \cap \mathcal{D}^{\mathrm{UD}}$. By completeness of the relation $\geq$ on the set $\{\prod_k v_k(\mathbf{x}) \in \mathbb{R}_+ | \mathbf{x} \in \mathbf{X}\}$, it suffices to show that for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\prod_k v_k(\mathbf{x}) \geq \prod_k v_k(\mathbf{y})$ implies $\mathbf{x}R\mathbf{y}$. Suppose $\prod_k v_k(\mathbf{x}) \geq \prod_k v_k(\mathbf{y})$ and assume first that $\prod_k v_k(\mathbf{x}), \prod_k v_k(\mathbf{y}) > 0$. Let $C = \{\langle v_k(\mathbf{z})\rangle_{k=1}^n \in \mathbb{R}_+^n | \mathbf{x}^0 < \mathbf{z} \in \mathbf{X}\}$ and $\mathbf{a}, \mathbf{b} \in C$ correspond to $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. The conditions of Lemma A.2 hold, since $\succsim \in \mathcal{D}^{\mathrm{CM}} \cap \mathcal{D}^{\mathrm{SSLC}}$ and $v_k(\mathbf{x}^0) = 0$. Thus there exist sequences $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^m$, $\{\mathbf{c}^l \in C\}_{l=1}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$, $\{j_l \in \mathbf{N}\}_{l=2}^m$ such that $\mathbf{z}^1 = \mathbf{x}$, $\mathbf{z}^m = \mathbf{y}$ and for every $2 \leq l \leq m$, $\mathbf{c}^l = \langle v_k(\mathbf{z}^l)\rangle_{k=1}^n$, $c_k^{l-1} = c_k^l$ for each $k \in \mathbf{N} \setminus \{i_l, j_l\}$ and $c_{i_l}^{l-1} c_{j_l}^{l-1} \geq c_{i_l}^l c_{j_l}^l$. Moreover, if $\#\{k \in \mathbf{N} | x_k \neq y_k\} \leq 2$, then this condition holds for $m = 2$. If $\mathbf{c}^{l-1}, \mathbf{c}^l > \mathbf{0}$, then $\mathbf{z}^{l-1}, \mathbf{z}^l \gg \mathbf{x}^0$ and $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ for each $k \in \mathbf{N} \setminus \{i_l, j_l\}$. Moreover, either $\mathbf{z}^{l-1} \succsim_k \mathbf{z}^l$ for $k \in \{i_l, j_l\}$, or $u_{i_l}(\mathbf{z}^l; \mathbf{z}^{l-1}) = g^{-1}(c_{i_l}^l / c_{i_l}^{l-1}) \leq g^{-1}(c_{j_l}^{l-1} / c_{j_l}^l) = u_{j_l}(\mathbf{z}^{l-1}; \mathbf{z}^l)$. Thus $u_{i_l}(\mathbf{z}^l; \mathbf{z}^{l-1}) u_{j_l}(\mathbf{z}^l; \mathbf{z}^{l-1}) \geq 1$. If not $\mathbf{c}^l > 0$, in particular if not $\mathbf{c}^{l-1} > 0$, then not $\mathbf{z}^l \gg \mathbf{x}^0$. Hence, $\mathbf{x}R\mathbf{y}$ by Lemma A.1(in the Appendix A). Suppose that $\prod_k v_k(\mathbf{y}) = 0$. If $\prod_k v_k(\mathbf{x}) > 0$, then there exist $\hat{\mathbf{y}} \in \mathbf{X}$ for which $\#\{k \in \mathbf{N} | \hat{y}_k \neq y_k\} \leq 2$ and $\prod_k v_k(\mathbf{x}) \geq \prod_k v_k(\hat{\mathbf{y}}) > \prod_k v_k(\mathbf{y})$. Thus $\mathbf{x}R\hat{\mathbf{y}}R\mathbf{y}$ by Lemma A.1. Therefore $\mathbf{x}R\mathbf{y}$ by transitivity of R. If $\prod_k v_k(\mathbf{x}) = \prod_k v_k(\mathbf{y}) = 0$, then there exist sequences $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$, $\{j_l \in \mathbf{N}\}_{l=2}^m$ such that $\mathbf{z}^1 = \mathbf{x}$, $\mathbf{z}^m = \mathbf{y}$ and for every $2 \leq l \leq m$, $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ for each $k \in \mathbf{N} \setminus \{i_l, j_l\}$ and $\prod_k v_k(\mathbf{z}^l) = 0$. Thus $\mathbf{z}^{l-1} R \mathbf{z}^l$ by Lemma A.1. Therefore $\mathbf{x}R\mathbf{y}$. □

Example 4. Considering again Example 3. The conditions of Proposition 9 hold. We can conclude that $\mathbf{x} = (3, 3, 3) P \mathbf{y} = (1, 4, 4)$ since the utility product 27 is strictly larger than the utility product 16. Furthermore, $\mathbf{x}$ is the unique optimal social state in **X**, since it is the only state that maximizes the individual utility product gain over the social state $\mathbf{x}^0$.

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[10] A set $A \subseteq \mathbb{R}^n$ is strictly convex if for any $\mathbf{a}, \mathbf{b} \in A$ and $\lambda \in (0, 1)$, $\lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$ is an interior point of A.

[11] Note that in order to check whether a preference profile satisfies the property **SSLC**, any origin biseparable representations $g_k v_k$ may be chosen for preferences in that profile. This is true because any origin biseparable representation of $\succsim_k$ is of the form $(g_k)^{\alpha_k}(v_k)^{\alpha_k}$, for some $\alpha_k > 0$. Thus, checking whether the set $\{\langle \log [v_k(\mathbf{x})]^{\alpha_k}\rangle_{k=1}^n \in \mathbb{R}^n | \mathbf{x}^0 < \mathbf{x} \in \mathbf{X}\}$ is strictly convex does not depend on the value of α_k chosen.

5. Characterization of the ordinal Nash social welfare function

We now provide a characterization of the *ON* social welfare function over the domains investigated in previous sections. In the case where $n = 2$, the axiomatization is carried out for the set $\mathcal{D}^{\mathrm{UD}}$, as stated in Theorem 1. In the case where $n > 2$, the axiomatization is restricted to domains contained in $\mathcal{D}^{C} \cap \mathcal{D}^{\mathrm{UD}}(\succsim_k)$ for some $k \in \mathbf{N}$ and $\succsim_k \in \mathcal{P}$, as required by Propositions 6, 7 and 8. A full characterization of the set A_W as defined in Proposition 1 appears in the appendix.

Theorem 2. *Consider a domain $\mathcal{D}$ satisfying the domain restrictions* **CMP**, $\mathcal{D} \subseteq \mathcal{D}^{C}$ *and* $\mathcal{D} \setminus \mathcal{D}^{\mathrm{CM}} \subseteq \mathcal{D}^{\mathrm{UD}}(\succsim_k)$ *for some* $k \in \mathbf{N}$ *and* $\succsim_k \in \mathcal{P}$.

(1) *ON is the unique social welfare function satisfying* **WO**, **PAR**, **ANM** *and* **IIA-NEU**
(2) *If* $n = 2$, *then* $\mathcal{D} \subseteq \mathcal{D}^{\mathrm{UD}}$, e.g. $\mathcal{D} = \mathcal{D}^{\mathrm{UD}}$
(3) If $n > 2$, *then* $\mathcal{D} \subseteq \mathcal{D}^{C} \cap \mathcal{D}^{\mathrm{UD}}(\succsim_k)$, e.g. $\mathcal{D} = \mathcal{D}^{\mathrm{CM}} \cap \mathcal{D}^{\mathrm{SSLC}} \cap \mathcal{D}^{\mathrm{UD}}(\succsim_k)$
(4) *For every* $\succsim \in \mathcal{D}$, *there exist origin biseparable representations* $\langle g v_k \rangle_{k=1}^{n}$ *such that for every* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} R \mathbf{y}$ *if, and only if*, $\prod_k v_k(\mathbf{x}) \geq \prod_k v_k(\mathbf{y})$.

Proof. The case of $n = 2$ is shown by Theorem 1. We provide a proof for the case of $n > 2$. Let A_W be the set defined in Proposition 1 and let $\mathbf{s} \in \mathbb{R}^n_+$. Suppose first that there exist $i, j \in \mathbf{N}$ for which $s_k = 1$ for each $k \in \mathbf{N} \setminus \{i, j\}$. Then $\mathbf{s} \in A_W$ if, and only if, $s_i s_j \geq 1$. Furthermore, for any $\succsim \in \mathcal{D}$, $R = W(\succsim)$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that there exist $i, j \in \mathbf{N}$ for which $\mathbf{x} \sim_k \mathbf{y}$ for each $k \in \mathbf{N} \setminus \{i, j\}$, $\mathbf{x} R \mathbf{y}$ if, and only if, $\mathbf{y} \gg \mathbf{x}^0$ implies $u_i(\mathbf{x}; \mathbf{y}) u_j(\mathbf{x}; \mathbf{y}) \geq 1$. The proof is similar to the one given for Proposition 2 applied for individuals i, j instead of 1, 2, two profiles for which only i, j are permuted, vectors $\mathbf{t}, \tilde{\mathbf{s}}$ for which all components except i, j equal to 1 and states $\mathbf{x}, \mathbf{y}, \mathbf{z}$ for which $\mathbf{x} \sim_k \mathbf{y} \sim_k \mathbf{z}$ for each $k \in \mathbf{N} \setminus \{i, j\}$. Hence, $\mathcal{D} \cap \mathcal{D}^{\mathrm{CM}} \subseteq \mathcal{D}^{\mathrm{UD}}$ by Lemma A.1(in the Appendix A) and Proposition 6.

Let $\succsim \in \mathcal{D}$, $R = W(\succsim)$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Suppose that $\mathbf{x} R' \mathbf{y}$ where $R' = ON(\succsim)$. By Lemma A.1, there exist sequences $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^{m}, \{i_l \in \mathbf{N}\}_{l=2}^{m}, \{j_l \in \mathbf{N}\}_{l=2}^{m}$ such that $\mathbf{z}^1 = \mathbf{x}$, $\mathbf{z}^m = \mathbf{y}$ and for every $2 \leq l \leq m$, $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ for each $k \in \mathbf{N} \setminus \{i_l, j_l\}$ and $\mathbf{z}^l \gg \mathbf{x}^0$ implies $u_{i_l}(\mathbf{z}^{l-1}; \mathbf{z}^l) u_{j_l}(\mathbf{z}^{l-1}; \mathbf{z}^l) \geq 1$. Therefore for every $2 \leq l \leq m$, $\mathbf{z}^{l-1} R \mathbf{z}^l$. Thus $\mathbf{x} R \mathbf{y}$ by transitivity. Suppose now that $\mathbf{y} P' \mathbf{x}$, where P' is the asymmetric part of R'. By Lemma A.1, there exist sequences $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^{m}, \{i_l \in \mathbf{N}\}_{l=2}^{m}, \{j_l \in \mathbf{N}\}_{l=2}^{m}$ such that $\mathbf{z}^1 = \mathbf{y}$, $\mathbf{z}^m = \mathbf{x}$ and there exist $2 \leq \hat{l} \leq m$ such that $\mathbf{z}^l \gg \mathbf{x}^0$ implies $u_{i_{\hat{l}}}(\mathbf{z}^{\hat{l}-1}; \mathbf{z}^{\hat{l}}) u_{j_{\hat{l}}}(\mathbf{z}^{\hat{l}-1}; \mathbf{z}^{\hat{l}}) > 1$. Therefore $\mathbf{z}^{\hat{l}-1} P \mathbf{z}^{\hat{l}}$. Thus $\mathbf{y} P \mathbf{x}$ by transitivity. By completeness of R', for any $\succsim \in \mathcal{D}$, $R = W(\succsim)$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} R \mathbf{y}$ if, and only if, $\mathbf{x} R' \mathbf{y}$. Hence, $W(\succsim) = ON(\succsim)$, proving one direction of (1). Thus (4) follows from Proposition 7. Part (3) of the theorem, as well as ON satisfying the axioms, follow from Propositions 6, 8 and 9 and the claims given for **PAR** and **ANM** in the proof of Theorem 1. □

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Appendix A

Lemma 1. Let $\succsim \in \mathcal{P}^n$ and $R = ON(\succsim)$. For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

(1) The followings are equivalent:
  (a) $\mathbf{x} R \mathbf{y}$
  (b) There exist $m \geq 2$ and sequences $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^{m}$, $\{i_l \in \mathbf{N}\}_{l=2}^{m}$, $\{j_l \in \mathbf{N}\}_{l=2}^{m}$ such that $\mathbf{z}^1 = \mathbf{x}$, $\mathbf{z}^m = \mathbf{y}$ and for every $2 \leq l \leq m$, $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ for each $k \in \mathbf{N} \setminus \{i_l, j_l\}$ and $\mathbf{z}^l \gg \mathbf{x}^0$ implies $u_{i_l}(\mathbf{z}^{l-1}; \mathbf{z}^l) u_{j_l}(\mathbf{z}^{l-1}; \mathbf{z}^l) \geq 1$. Moreover, if $\#\{k \in \mathbf{N} | x_k \neq y_k\} \leq 2$, then this condition holds for $m = 2$
(2) $\#\{k \in \mathbf{N} | x_k \neq y_k\} \leq 2$ and not $\mathbf{y} \gg \mathbf{x}^0$ implies $\mathbf{x} R \mathbf{y}$
(3) $\#\{k \in \mathbf{N} | x_k \neq y_k\} \leq 2$ and not $\mathbf{x} \gg \mathbf{x}^0$ and $\mathbf{x} R \mathbf{y}$ *implies not* $\mathbf{y} \gg \mathbf{x}^0$.

Proof. By definition, for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x}R\mathbf{y}$ if, and only if, there exist $m \geq 2$ and sequences $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$, $\{j_l \in \mathbf{N}\}_{l=2}^m$ such that $\mathbf{z}^1 = \mathbf{x}, \mathbf{z}^m = \mathbf{y}$ and for every $2 \leq l \leq m$, $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ for each $k \in \mathbf{N} \setminus \{i_l, j_l\}$ and either $\mathbf{z}^{l-1} \succsim_k \mathbf{z}^l$ for $k \in \{i_l, j_l\}$, or there exist $p_l, q_l \in [0, 1]$ such that $p_l \mathbf{z}^{l-1} \sim_{i_l} \mathbf{z}^l, q_l \mathbf{z}^l \sim_{j_l} \mathbf{z}^{l-1}$ and $p_l \leq q_l$. Moreover, if $\#\{k \in \mathbf{N} | x_k \neq y_k\} \leq 2$, then this condition holds for $m = 2$. For any $\mathbf{w}, \hat{\mathbf{w}} \in \mathbf{X}$ and $i, j \in \mathbf{N}$ such that $\mathbf{w} \sim_k \hat{\mathbf{w}}$ for each $k \in \mathbf{N} \setminus \{i, j\}$, either $\mathbf{w} \succsim_k \hat{\mathbf{w}}$ for both i, j, or there exist $p, q \in [0, 1]$ such that $p\mathbf{w} \sim_i \hat{\mathbf{w}}$, $q\hat{\mathbf{w}} \sim_j \mathbf{w}$, where by definition $p = u_i(\hat{\mathbf{w}}; \mathbf{w}), q = u_j(\mathbf{w}; \hat{\mathbf{w}})$. Therefore, if $\hat{\mathbf{w}} \gg \mathbf{x}^0$, then $u_i(\mathbf{w}; \hat{\mathbf{w}}) u_j(\mathbf{w}; \hat{\mathbf{w}}) \geq 1$ if, and only if, either $\mathbf{w} \succsim_k \hat{\mathbf{w}}$ for both i, j, or $p \leq q$. If not $\hat{\mathbf{w}} \gg \mathbf{x}^0$, then either $\mathbf{w} \succsim_k \hat{\mathbf{w}}$ for both i, j, or $0 = p \leq q$. Hence, condition (1)(a) holds if, and only if, condition (1)(b) holds. Suppose that $\#\{k \in \mathbf{N} | x_k \neq y_k\} \leq 2$, then (2) follows immediately and (3) also holds since $\mathbf{y} \gg \mathbf{x}^0$ implies $\prod_k u_k(\mathbf{x}; \mathbf{y}) = 0$ and thus not $\mathbf{x}R\mathbf{y}$, a contradiction. □

Proof of Proposition 6. First we show that (4) implies (1). Suppose there exist functions $g, v_1, \ldots, v_n$ such that for any $k \in \mathbf{N}$, gv_k are origin biseparable representations of $\succsim_k$. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$ such that $\mathbf{x}R\mathbf{y}R\mathbf{z}$. Then there exist $l_1, l_2 \geq 1$ such that $\mathbf{x}R^{l_1}\mathbf{y}R^{l_2}\mathbf{z}$, which implies $\mathbf{x}R^m\mathbf{z}$, where $m = l_1 + l_2$ and $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$ and $\{j_l \in \mathbf{N}\}_{l=2}^m$ are the corresponding sequences. If $\#\{k \in \mathbf{N} | x_k \neq z_k\} > 2$, or $\mathbf{x} \gtrapprox \mathbf{z}$, or ($\#\{k \in \mathbf{N} | x_k \neq z_k\} \leq 2$ and not $\mathbf{z} \gg \mathbf{x}^0$), then $\mathbf{x}R\mathbf{z}$ by Lemma A.1. Otherwise, for every $2 \leq l \leq m$, $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ for each $k \in \mathbf{N} \setminus \{i_l, j_l\}$. Moreover, if not $\mathbf{z}^l \gg \mathbf{x}^0$, then $\prod_k v_k(\mathbf{z}^{l-1}) \geq \prod_k v_k(\mathbf{z}^l) = 0$. If $\mathbf{z}^{l-1}, \mathbf{z}^l \gg \mathbf{x}^0$, then either $u_k(\mathbf{z}^{l-1}; \mathbf{z}^l) \geq 1$ for $k \in \{i_l, j_l\}$, or

$$g^{-1}(v_{i_l}(\mathbf{z}^l)/v_{i_l}(\mathbf{z}^{l-1})) = u_{i_l}(\mathbf{z}^l; \mathbf{z}^{l-1}) \leq u_{j_l}(\mathbf{z}^{l-1}; \mathbf{z}^l) = g^{-1}(v_{j_l}(\mathbf{z}^{l-1})/v_{j_l}(\mathbf{z}^l)).$$

Therefore, $\prod_k v_k(\mathbf{z}^{l-1}) \geq \prod_k v_k(\mathbf{z}^l)$. Thus $\prod_k v_k(\mathbf{x}) \geq v_k(\mathbf{z})$. Let $i, j \in \mathbf{N}$ such that $\mathbf{z} \succsim_i \mathbf{x}$ and $\mathbf{x} \succsim_j \mathbf{z}$ and $\mathbf{x} \sim_k \mathbf{z}$ for each $k \in \mathbf{N} \setminus \{i, j\}$. Then $[v_i(\mathbf{x})/v_i(\mathbf{z})][v_j(\mathbf{x})/v_j(\mathbf{z})] \geq 1$ and therefore $u_i(\mathbf{x}; \mathbf{z}) = g^{-1}(v_i(\mathbf{x})/v_i(\mathbf{z})) \geq g^{-1}(v_j(\mathbf{z})/v_j(\mathbf{x})) = u_j(\mathbf{z}; \mathbf{x})$. Thus $\mathbf{x}R^1\mathbf{z}$ by Lemma A.1 and therefore $\mathbf{x}R\mathbf{z}$. Hence R is transitive.

The remainder of the proof is the same as the corresponding part in the proof of Proposition 4, when the proof for individual 2 is repeated for all $k \in \mathbf{N} \setminus \{1\}$. The only exception appears in the proof that (1) implies (3), where in addition it is required that $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$ are chosen such that $\mathbf{x} \sim_i \mathbf{y} \sim_i \mathbf{z}$ for every $i \in \mathbf{N} \setminus \{k, 1\}$. □

Proof of Proposition 7. Suppose $\mathbf{x}R\mathbf{y}$. By Lemma A.1, there exist $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$, $\{j_l \in \mathbf{N}\}_{l=2}^m$ such that for every $2 \leq l \leq m$, $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ for each $k \in \mathbf{N} \setminus \{i_l, j_l\}$. Moreover, if not $\mathbf{z}^l \gg \mathbf{x}^0$, then $\prod_k v_k(\mathbf{z}^{l-1}) \geq \prod_k v_k(\mathbf{z}^l) = 0$. If $\mathbf{z}^l \gg \mathbf{x}^0$, then either $u_k(\mathbf{z}^{l-1}; \mathbf{z}^l) \geq 1$ for $k \in \{i_l, j_l\}$, or $g^{-1}(v_{i_l}(\mathbf{z}^l)/v_{i_l}(\mathbf{z}^{l-1})) \leq g^{-1}(v_{j_l}(\mathbf{z}^{l-1})/v_{j_l}(\mathbf{z}^l))$. Therefore, $\prod_k v_k(\mathbf{z}^{l-1}) \geq \prod_k v_k(\mathbf{z}^l)$. Thus $\prod_k v_k(\mathbf{x}) \geq \prod_k v_k(\mathbf{y})$. To prove the converse, suppose $\mathbf{y}P\mathbf{x}$. Then there exist $2 \leq \hat{l} \leq m$ such that $v_k(\mathbf{z}^{\hat{l}-1}) > \prod_k v_k(\mathbf{z}^{\hat{l}})$ and thus $\prod_k v_k(\mathbf{y}) > \prod_k v_k(\mathbf{x})$. Therefore $\prod_k v_k(\mathbf{x}) \geq \prod_k v_k(\mathbf{y})$ implies $\mathbf{x}R\mathbf{y}$ by completeness of R. □

Corollary 2. (To Theorem 2). Let $n > 2$. Let W satisfy **PAR**, **ANM** and **IIA-NEU** under the domain restrictions **CMP** and $\mathcal{D} \subseteq \mathcal{D}^C \cap \mathcal{D}^{\mathrm{UD}}(\succsim_k)$ for some $k \in \mathbf{N}$. Let $\succsim_k \in \mathcal{P}$. Let A_W be the corresponding set as in Proposition 1 and let the function $h_g : \mathbb{R}_+ \to \mathbb{R}_+$ be defined such that

$$h_g(r) = \begin{cases} g(r) & \text{if} \quad r \leq 1 \\ [g(r^{-1})]^{-1} & \text{if} \quad r > 1 \end{cases}$$

For any $\mathbf{s} \in \mathbb{R}_+^n$, $\mathbf{s} \in A_W$ if, and only if, $\prod_k h_g(s_k) \geq 1$.

Proof. Let $\mathbf{s} \in \mathbb{R}_+^n$. Suppose that $\mathbf{s}$ satisfies $s_k = 1$ for every $k \in \mathbf{N} \setminus \{i, j\}$. If $\mathbf{s} \geq \mathbf{1}$, then $\prod_k s_k \geq 1$ and $\prod_k h_g(s_k) \geq 1$. If not $\mathbf{s} \geq \mathbf{1}$, then we can assume without loss of generality that $s_i < 1$. Thus $\prod_k s_k \geq 1$ if, and only if, $h_g(s_i) = g(s_i) \geq g(1/s_j) = 1/h_g(s_j)$ if, and only if, $\prod_k h_g(s_k) \geq 1$. Hence $\mathbf{s} \in A_W$ if, and only if, $\prod_k h_g(s_k) \geq 1$. Suppose that $\mathbf{s}$ satisfy $\#\{k \in \mathbf{N} | s_k \neq 1\} > 2$. Let $\succsim \in \mathcal{D}$, $R = W(\succsim)$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $\mathbf{y} \gg \mathbf{x}^0$ and $\mathbf{s} = \mathbf{u}(\mathbf{x}; \mathbf{y})$. For every $k \in \mathbf{N}$, $h_g(s_k) = h_g[u_k(\mathbf{x}; \mathbf{y})] = v_k(\mathbf{x})/v_k(\mathbf{y})$, where gv_k are origin biseparable representations of $\succsim_k$. Then $\prod_k h_g(s_k) \geq 1 \Leftrightarrow \prod_k v_k(\mathbf{x}) \geq \prod_k v_k(\mathbf{y}) \Leftrightarrow \mathbf{x}R\mathbf{y} \Leftrightarrow \mathbf{s} \in A_W$. Hence $\mathbf{s} \in A_W$ if, and only if, $\prod_k h_g(s_k) \geq 1$. (Note that this condition coincides with the requirement $s_1 s_2 \geq 1$ that characterized the set A_W in the case $n = 2$.) □

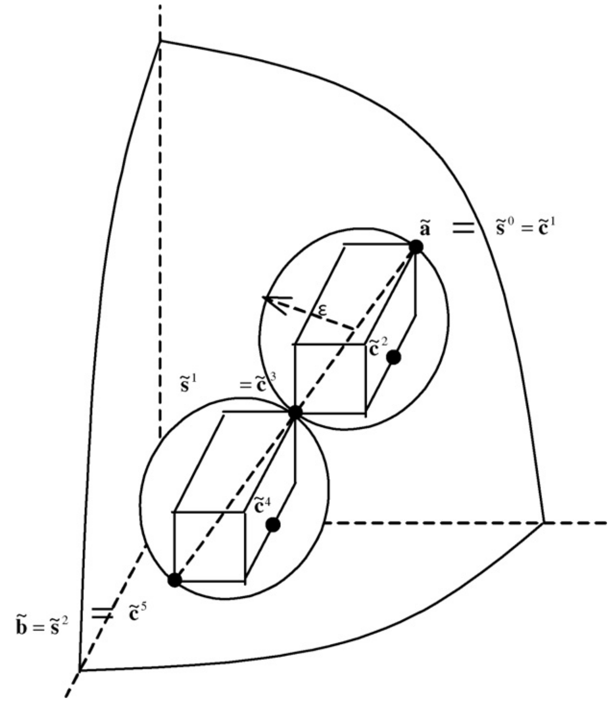

Fig. 2. Construction of the sequences $\{\tilde{\mathbf{c}}^l \in \tilde{C}\}_{l=1}^5$, $\{i_l \in \mathbf{N}\}_{l=2}^5$, $\{j_l \in \mathbf{N}\}_{l=2}^5$

Lemma 2. Let $C \subseteq \mathbb{R}_+^n$ be a 0-comprehensive[12] set such that the set $\{\langle \log c_k \rangle_{k=1}^n | \mathbf{0} < \mathbf{c} \in \mathbf{C}\}$ has a smooth boundary and is strictly convex. Let $\mathbf{a}, \mathbf{b} \in C$. $\prod_k a_k \geq \prod_k b_k$ if, and only if, there exist $m \geq 2$ and sequences $\{\mathbf{c}^l \in C\}_{l=1}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$, $\{j_l \in \mathbf{N}\}_{l=2}^m$ such that $\mathbf{c}^1 = \mathbf{a}$, $\mathbf{c}^m = \mathbf{b}$ and for every $2 \leq l \leq m$, $c_k^{l-1} = c_k^l$ for each $k \in \mathbf{N} \setminus \{i_l, j_l\}$ and $c_{i_l}^{l-1} c_{j_l}^{l-1} \geq c_{i_l}^l c_{j_l}^l$. Moreover, if $\#\{k \in \mathbf{N} | a_k \neq b_k\} \leq 2$, then this condition holds for $m = 2$. Furthermore, $\prod_k a_k > \prod_k b_k$ if, and only if, in addition there exist $2 \leq l \leq m$ such that $c_{i_l}^{l-1} c_{j_l}^{l-1} > c_{i_l}^l c_{j_l}^l$.

Proof. Clearly, if there exist such sequences $\{\mathbf{c}^l \in C\}_{l=1}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$, $\{j_l \in \mathbf{N}\}_{l=2}^m$, then for every $2 \leq l \leq m$, $\prod_k c_k^{l-1} \geq \prod_k c_k^l$. Thus $\prod_k a_k \geq \prod_k b_k$. If in addition there exist $2 \leq h \leq m$ such that $c_{i_h}^{h-1} c_{j_h}^{h-1} > c_{i_h}^h c_{j_h}^h$, then clearly $\prod_k a_k > \prod_k b_k$.

To prove the converse, suppose $\prod_k a_k \geq \prod_k b_k$. Let $\mathbf{c}^1 = \mathbf{a}$ and let the sequences $\{\mathbf{c}^l \in C\}_{l=2}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$, $\{j_l \in \mathbf{N}\}_{l=1}^m$ be constructed as follows, considering several cases.

(1) Suppose $\#\{k \in \mathbf{N} | a_k \neq b_k\} \leq 2$, then let $m = 2$, $\mathbf{c}^2 = \mathbf{b}$ and $i_2, j_2 \in \mathbf{N}$ such that $a_k = b_k$ for each $k \in \mathbf{N} \setminus \{i_2, j_2\}$. Thus the required conditions are satisfied.
(2) Hereafter assume that $\#\{k \in \mathbf{N} | a_k \neq b_k\} > 2$. Suppose $\prod_k a_k > \prod_k b_k$. Since C is 0-comprehensive, there exist $\mathbf{c}^2 \in C$, $i_2, j_2 \in \mathbf{N}$ such that $c_k^2 = c_k^1$ for each $k \in \mathbf{N} \setminus \{i_2, j_2\}$, $c_{i_2}^2 < c_{i_2}^1$, $c_{j_2}^2 \leq c_{j_2}^1$ and $\prod_k c_k^2 = \prod_k b_k$. Thus $c_{i_2}^1 c_{j_2}^1 > c_{i_2}^2 c_{j_2}^2$. The remaining sequences are constructed as in the case where $\prod_k a_k = \prod_k b_k$.
(3) Hereafter assume that $\prod_k a_k = \prod_k b_k$. Suppose $\prod_k a_k = 0$. Then there exist $i, j \in \mathbf{N}$ such that $a_i = b_j = 0$. Let $m = \#\{k \in \mathbf{N} \setminus \{j\} | a_k \neq b_k\} + 1$. For every $2 \leq l \leq m$, let $j_l = j$, $i_l \in \mathbf{N}$ such that $c_{i_l}^{l-1} \neq b_{i_l}$. Let $\mathbf{c}^l \in C$ such that $c_k^l = c_k^{l-1}$ for each $k \in \mathbf{N} \setminus \{i_l, j_l\}$, $c_{i_l}^l = b_{i_l}$ and $c_{j_l}^l = 0$. Clearly, these sequences satisfy the required conditions.
(4) Hereafter assume that $\prod_k a_k = \prod_k b_k > 0$. Suppose that $\mathbf{a}, \mathbf{b}$ are interior points of C. Let $\tilde{C} = \{\langle \log r_k \rangle_{k=1}^n | \mathbf{0} < \mathbf{r} \in \mathbf{C}\}$. Note that $\tilde{C}$ is a closed and strictly convex set. Let $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}^l$ be the interior points in $\tilde{C}$ corresponding to $\mathbf{a}, \mathbf{b}, \mathbf{c}^l$ in C. Then there exists $0 < \delta < \|\tilde{\mathbf{b}} - \tilde{\mathbf{a}}\|$, such that for every $\tilde{\mathbf{r}} \in \mathbb{R}_+^n$ for which $|\tilde{\mathbf{r}}| < \delta$, $\tilde{\mathbf{a}} + \tilde{\mathbf{r}} \in \tilde{C}$ and $\tilde{\mathbf{b}} + \tilde{\mathbf{r}} \in \tilde{C}$. Let

[12] $\forall \mathbf{a}, \mathbf{b} \in \mathbb{R}_+^n$, $\mathbf{b} \leq a \in C \Rightarrow \mathbf{b} \in C$.

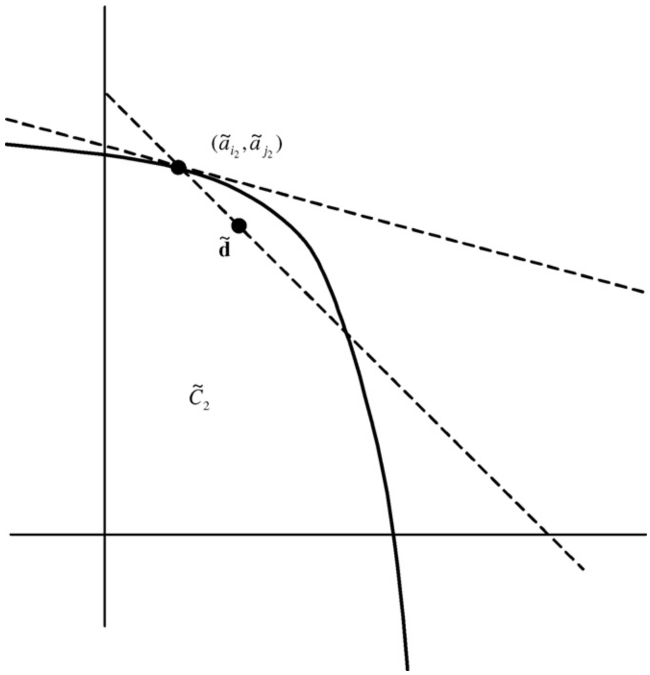

Fig. 3. Existence of an interior point $\tilde{\mathbf{d}} \in \tilde{C}_2$.

$m_1 = \lceil |\tilde{\mathbf{b}} - \tilde{\mathbf{a}}| / 2\delta \rceil$, $\varepsilon = |\tilde{\mathbf{b}} - \tilde{\mathbf{a}}| / 2m_1 \le \delta$ and for every $0 \le t \le m_1$, let $\tilde{\mathbf{s}}^t = \tilde{\mathbf{a}} + t/m_1(\tilde{\mathbf{b}} - \tilde{\mathbf{a}})$. Since $\sum_k s_k^0 = \sum s_k^1$, $\tilde{\mathbf{s}}^0 \ne \tilde{\mathbf{s}}^1$ and $\mathbf{N}$ is finite, there exist $m_2 > 2$ and extension of the sequences $i_l, j_l, \tilde{\mathbf{c}}^l$ (see Fig. 2) such that for every $2 \le l \le m_2$, $\tilde{c}_{i_l}^{l-1} < \tilde{s}_{i_l}^1$, $\tilde{c}_{j_l}^{l-1} > \tilde{s}_{j_l}^1$, $\tilde{c}_k^l = \tilde{c}_k^{l-1}$ for each $k \in \mathbf{N} \setminus \{i_l, j_l\}$, $\alpha_l \equiv \min\{\tilde{s}_{i_l}^1 - \tilde{c}_{i_l}^{l-1}, \tilde{c}_{j_l}^{l-1} - \tilde{s}_{j_l}^1\} > 0$, $\tilde{c}_{i_l}^l = \tilde{c}_{i_l}^{l-1} + \alpha_l$, $\tilde{c}_{j_l}^l = \tilde{c}_{j_l}^{l-1} - \alpha_l$ and $\tilde{\mathbf{c}}^{m_2} = \tilde{\mathbf{s}}^1$.

Then for every $2 \le l \le m_2$ and $k \in \mathbf{N}$, $\min\{\tilde{s}_k^0, \tilde{s}_k^1\} \le \tilde{c}_k^l \le \max\{\tilde{s}_k^0, \tilde{s}_k^1\}$, thus

$$\left|\tilde{\mathbf{c}}^l - (\tilde{\mathbf{s}}^0 + \tilde{\mathbf{s}}^1)/2\right| \le \left|\tilde{\mathbf{s}}^0 - \tilde{\mathbf{s}}^1\right| / 2 = \varepsilon.$$

Let $m_3 = m_1(m_2 - 1) + 1$ and extend the sequences $i_l, j_l, \tilde{\mathbf{c}}^l$ such that for every $2 \le t \le m_1$ and $2 \le l \le m_2$, $i_{(t-1)(m_2-1)+l} = i_l$, $j_{(t-1)(m_2-1)+l} = j_l$ and $\tilde{\mathbf{c}}^{(t-1)(m_2-1)+l} = \tilde{\mathbf{c}}^l - \tilde{\mathbf{s}}^1 + \tilde{\mathbf{s}}^t = \tilde{\mathbf{c}}^l + [(t-1)/m_1](\tilde{\mathbf{b}} - \tilde{\mathbf{a}})$. Thus $|\tilde{\mathbf{c}}^{(t-1)(m_2-1)+l} - (\tilde{\mathbf{s}}^{t-1} + \tilde{\mathbf{s}}^t)/2| = |\tilde{\mathbf{c}}^l - (\tilde{\mathbf{s}}^0 + \tilde{\mathbf{s}}^1)/2| \le \varepsilon$. Then $\tilde{\mathbf{c}}^{m_3} = \tilde{\mathbf{b}}$ and for every $2 \le l \le m_3$,there exist $1 \le t_l \le m_1$ such that $|\tilde{\mathbf{c}}^l - (\tilde{\mathbf{s}}^{t_l-1} + \tilde{\mathbf{s}}^{t_l})/2| \le \varepsilon$. Let $\tilde{\mathbf{r}}^l = \tilde{\mathbf{c}}^l - (\tilde{\mathbf{s}}^{t_l-1} + \tilde{\mathbf{s}}^{t_l})/2$. Then $\tilde{\mathbf{c}}^l = [(2t_l - 1)/2m_1](\tilde{\mathbf{b}} + \tilde{\mathbf{r}}^l) + (1 - (2t_l - 1)/2m_1)(\tilde{\mathbf{a}} + \tilde{\mathbf{r}}^l) \in \tilde{C}$ by convexity of $\tilde{C}$. For every $2 \le l \le m_3$, let $\mathbf{c}^l \in C$ correspond to $\tilde{\mathbf{c}}^l$ in $\tilde{C}$. Then the required conditions are satisfied.

(5) Hereafter assume that either $\mathbf{a}$ or $\mathbf{b}$ are on the boundary ∂C of C. Suppose that $\mathbf{a}$ is on the boundary ∂C and $\mathbf{b}$ is an interior point of C. Construct $\mathbf{c}^2, i_2, j_2$ as follows: by strict convexity of $\tilde{C}$ and its smooth boundary $\partial\tilde{C}$ at $\tilde{\mathbf{a}}$, there exists a unique (up to positive scalar multiplication) $\mathbf{t} \in \mathbb{R}_+^n$ such that for each $\tilde{\mathbf{r}} \in \tilde{C} \setminus \{\tilde{\mathbf{a}}\}$, $\sum_k t_k(\tilde{r}_k - \tilde{a}_k) < 0$. Suppose there exists $\lambda \ne 0$ such that for every $k \in \mathbf{N}$, $t_k = \lambda$. Since $\sum_k(\tilde{b}_k - \tilde{a}_k) = 0$, it follows that $\sum_k t_k(\tilde{b}_k - \tilde{a}_k) = 0$. Since $\tilde{\mathbf{a}} \ne \tilde{\mathbf{b}}$, then $\tilde{\mathbf{b}} \notin \tilde{C}$, a contradiction. Thus, there exist $i_2, j_2 \in \mathbf{N}$ such that $t_{i_2} < t_{j_2}$. Let $\tilde{C}_2 = \{\langle \tilde{r}_{i_2}, \tilde{r}_{j_2}\rangle | \tilde{\mathbf{r}} \in \tilde{C}, \forall k \in \mathbf{N} \setminus \{i_2, j_2\}, \tilde{r}_k = \tilde{a}_k\} \subseteq \mathbb{R}^2$. Then $\tilde{C}_2$ is a closed and strictly convex set with smooth boundary $\partial\tilde{C}_2$ at $(\tilde{a}_{i_2}, \tilde{a}_{j_2})$. Therefore, (t_{i_2}, t_{j_2}) satisfies uniquely (up to positive scalar multiplication) for each $\tilde{\mathbf{r}} \in \tilde{C}_2 \setminus \{\tilde{a}_{i_2}, \tilde{a}_{j_2}\}$,

$$t_{i_2}(\tilde{r}_1 - \tilde{a}_{i_2}) + t_{j_2}(\tilde{r}_2 - \tilde{a}_{j_2}) < 0.$$

Since (1,1) is not a scalar multiple of (t_{i_2}, t_{j_2}), the set $\tilde{D}_2 = \{\tilde{\mathbf{r}} \in \tilde{C}_2 \setminus \{(\tilde{a}_{i_2}, \tilde{a}_{j_2})\} | (\tilde{r}_1 - \tilde{a}_{i_2}) + (\tilde{r}_2 - \tilde{a}_{j_2}) \geq 0\}$ is not empty. Moreover, there exist $\varepsilon > 0$ and $\tilde{\mathbf{d}} \equiv (\tilde{a}_{i_2} + \varepsilon, \tilde{a}_{j_2} - \varepsilon) \in \tilde{D}_2$, such that $\tilde{\mathbf{d}}$ is an interior point of $\tilde{C}_2$ (see Fig. 3).

Let $\tilde{\mathbf{c}}^2 \in \tilde{C}$ satisfy $\tilde{c}^2_k = \tilde{a}_k$ for each $k \in \mathbf{N} \setminus \{i_2, j_2\}$, $\tilde{c}^2_{i_2} = \tilde{d}_1$ and $\tilde{c}^2_{j_2} = \tilde{d}_2$. Let $\mathbf{c}^2 \in C$ be the corresponding point to $\tilde{\mathbf{c}}^2 \in \tilde{C}$. Clearly, $\mathbf{c}^2$ is an interior point of C. Note that $c^2_{i_2} c^2_{j_2} = c^1_{i_2} c^1_{j_2}$. The sequences $i_l, j_l, \mathbf{c}^l$ can then be extended as in case (4) to satisfy the required conditions.

(6) Assume now that $\mathbf{b}$ is on the boundary ∂C of C and $\mathbf{a}$ is an interior point of C. Similarly to case (5), there exists $\hat{\mathrm{c}}$, an interior point of C, and $i, j \in \mathbf{N}$ such that $\hat{b}_k = b_k$ for each $k \in \mathbf{N} \setminus \{i, j\}$ and $\hat{b}_i \hat{b}_j = b_i b_j$. Thus the sequences $i_l, j_l, \mathbf{c}^l$ will satisfy the required conditions when extended as in case (4) for $\mathbf{a}$ and $\hat{\mathrm{b}}$. □

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