2

Tensors

As mentioned in the introduction, all laws of continuum mechanics must be formulated in terms of quantities that are independent of coordinates. It is the purpose of this chapter to introduce such mathematical entities. We shall begin by introducing a short-hand notation - the indicial notation – in Part A of this chapter, which will be followed by the concept of tensors introduced as a linear transformation in Part B. The basic field operations needed for continuum formulations are presented in Part C and their representations in curvilinear coordinates in Part D.

Part A The Indicial Notation

2A1 Summation Convention, Dummy Indices

Consider the sum

$$s = a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n \tag{2A1.1}$$

We can write the above equation in a compact form by using the summation sign:

$$s = \sum_{i=1}^{n} a_i x_i \tag{2A1.2}$$

It is obvious that the following equations have exactly the same meaning as Eq. (2A1.2)

$$s = \sum_{j=1}^{n} a_j x_j \tag{2A1.3}$$

$$s = \sum_{m=1}^{n} a_m x_m \tag{2A1.4}$$

etc.

The index i in Eq. (2A1.2), or j in Eq. (2A1.3), or m in Eq. (2A1.4) is a dummy index in the sense that the sum is independent of the letter used.

We can further simplify the writing of Eq.(2A1.1) if we adopt the following convention: Whenever an index is repeated once, it is a dummy index indicating a summation with the index running through the integers 1,2, ..., n.

This convention is known as Einstein's summation convention. Using the convention, Eq. (2A1.1) shortens to

$$s = a_i x_i \tag{2A1.5}$$

We also note that

$$a_i x_i = a_m x_m = a_j x_j = \ldots \tag{2A1.6}$$

It is emphasized that expressions such as $a_i b_i x_i$ are not defined within this convention. That is, an index should *never* be repeated more than once when the summation convention is used. Therefore, an expression of the form

$$\sum_{i=1}^{n} a_i b_i x_i$$

must retain its summation sign.

In the following we shall always take n to be 3 so that, for example,

$$a_i x_i = a_m x_m = a_1 x_1 + a_2 x_2 + a_3 x_3$$

$$a_{ii} = a_{mm} = a_{11} + a_{22} + a_{33}$$

$$a_i \mathbf{e}_i = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3$$

The summation convention obviously can be used to express a double sum, a triple sum, etc. For example, we can write

$$\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i x_j \tag{2A1.7}$$

simply as

$$a_{ij} x_i x_j \tag{2A1.8}$$

Expanding in full, the expression (2A1.8) gives a sum of nine terms, i.e.,

$$\begin{aligned} a_{ij} x_i x_j = {} & a_{11} x_1 x_1 + a_{12} x_1 x_2 + a_{13} x_1 x_3 + a_{21} x_2 x_1 + a_{22} x_2 x_2 \\ & + a_{23} x_2 x_3 + a_{31} x_3 x_1 + a_{32} x_3 x_2 + a_{33} x_3 x_3 \end{aligned} \tag{2A1.9}$$

For beginners, it is probably better to perform the above expansion in two steps, first, sum over i and then sum over j (or vice versa), i.e.,

$$a_{ij} x_i x_j = a_{1j} x_1 x_j + a_{2j} x_2 x_j + a_{3j} x_3 x_j$$

where

$$a_{1j}x_1x_j = a_{11}x_1x_1 + a_{12}x_1x_2 + a_{13}x_1x_3$$

etc.

Similarly, the triple sum

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a_{ijk}x_ix_jx_k \tag{2A1.10}$$

will simply be written as

$$a_{ijk}x_ix_jx_k \tag{2A1.11}$$

The expression (2A1.11) represents the sum of 27 terms.

We emphasize again that expressions such as $a_{ii}x_ix_jx_j$ or $a_{ijk}x_ix_ix_jx_k$ are not defined in the summation convention, they do not represent

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ii}x_ix_jx_j \quad \text{or} \quad \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a_{ijk}x_ix_ix_jx_k$$

2A2 Free Indices

Consider the following system of three equations

$$\begin{aligned} x'_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ x'_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ x'_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \tag{2A2.1}$$

Using the summation convention, Eqs. (2A2.1) can be written as

$$\begin{aligned} x'_1 &= a_{1m}x_m \\ x'_2 &= a_{2m}x_m \\ x'_3 &= a_{3m}x_m \end{aligned} \tag{2A2.2}$$

which can be shortened into

$$x'_i = a_{im}x_m, \quad i = 1,2,3 \tag{2A2.3}$$

An index which appears only once in each term of an equation such as the index i in Eq. (2A2.3) is called a "free index." A free index takes on the integral number 1, 2, or 3 *one at a time*. Thus Eq. (2A2.3) is short-hand for three equations each having a sum of three terms on its right-hand side [i.e., Eqs. (2A2.1)].

A further example is given by

$$\mathbf{e}_i' = Q_{mi}\mathbf{e}_m, \quad i = 1,2,3 \tag{2A2.4}$$

representing

$$\begin{aligned}
\mathbf{e}_1' &= Q_{11}\mathbf{e}_1 + Q_{21}\mathbf{e}_2 + Q_{31}\mathbf{e}_3 \\
\mathbf{e}_2' &= Q_{12}\mathbf{e}_1 + Q_{22}\mathbf{e}_2 + Q_{32}\mathbf{e}_3 \\
\mathbf{e}_3' &= Q_{13}\mathbf{e}_1 + Q_{23}\mathbf{e}_2 + Q_{33}\mathbf{e}_3
\end{aligned} \tag{2A2.5}$$

We note that $x_j' = a_{jm}x_m, j=1,2,3$, is the same as Eq. (2A2.3) and $\mathbf{e}_j' = Q_{mj}\mathbf{e}_m,\ j=1,2,3$ is the same as Eq. (2A2.4). However,

$$a_i = b_j$$

is a meaningless equation. *The free index appearing in every term of an equation must be the same.* Thus the following equations are meaningful

$$a_i + k_i = c_i$$

$$a_i + b_i c_j d_j = 0$$

If there are two free indices appearing in an equation such as

$$T_{ij} = A_{im}A_{jm} \quad i = 1,2,3; j = 1,2,3 \tag{2A2.6}$$

then the equation is a short-hand writing of 9 equations; each has a sum of 3 terms on the right-hand side. In fact,

$$\begin{aligned}
T_{11} &= A_{1m}A_{1m} = A_{11}A_{11} + A_{12}A_{12} + A_{13}A_{13} \\
T_{12} &= A_{1m}A_{2m} = A_{11}A_{21} + A_{12}A_{22} + A_{13}A_{23} \\
T_{13} &= A_{1m}A_{3m} = A_{11}A_{31} + A_{12}A_{32} + A_{13}A_{33} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
T_{33} &= A_{3m}A_{3m} = A_{31}A_{31} + A_{32}A_{32} + A_{33}A_{33}
\end{aligned}$$

Again, equations such as

$$T_{ij} = T_{ik}$$

have no meaning.

2A3 Kronecker Delta

The Kronecker delta, denoted by δ_{ij}, is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \tag{2A3.1}$$

That is,

$$\delta_{11} = \delta_{22} = \delta_{33} = 1$$

$$\delta_{12} = \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0$$

In other words, the matrix of the Kronecker delta is the identity matrix, i.e.,

$$[\delta_{ij}] = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2A3.2}$$

We note the following:

$$\text{(a)}\ \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3 \tag{2A3.3}$$

$$\begin{aligned} \text{(b)}\ \delta_{1m}a_m &= \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 = a_1 \\ \delta_{2m}a_m &= \delta_{21}a_1 + \delta_{22}a_2 + \delta_{23}a_3 = a_2 \\ \delta_{3m}a_m &= \delta_{31}a_1 + \delta_{32}a_2 + \delta_{33}a_3 = a_3 \end{aligned}$$

Or, in general

$$\delta_{im}a_m = a_i \tag{2A3.4}$$

$$\begin{aligned} \text{(c)}\ \delta_{1m}T_{mj} &= \delta_{11}T_{1j} + \delta_{12}T_{2j} + \delta_{13}T_{3j} = T_{1j} \\ \delta_{2m}T_{mj} &= T_{2j} \\ \delta_{3m}T_{mj} &= T_{3j} \end{aligned}$$

or, in general

$$\delta_{im}T_{mj} = T_{ij} \tag{2A3.5}$$

In particular,

$$\delta_{im}\delta_{mj} = \delta_{ij} \tag{2A3.6}$$

$$\delta_{im}\delta_{mn}\delta_{nj} = \delta_{ij}$$

etc.

(d) If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit vectors perpendicular to each other, then

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{2A3.7}$$

2A4 Permutation Symbol

The permutation symbol, denoted by ε_{ijk} is defined by

$$\varepsilon_{ijk} = \begin{Bmatrix} +1 \\ -1 \\ 0 \end{Bmatrix} \equiv \text{according to whether i,j,k} \begin{Bmatrix} \text{form an even} \\ \text{form an odd} \\ \text{do not form a} \end{Bmatrix} \text{permutation of 1,2,3} \tag{2A4.1}$$

i.e.,

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$$
$$\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$$
$$\varepsilon_{111} = \varepsilon_{112} = \cdots = 0$$

We note that

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji} \tag{2A4.2}$$

If $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ form a right-handed triad, then

$$\mathbf{e}_1\times\mathbf{e}_2 = \mathbf{e}_3,\ \mathbf{e}_2\times\mathbf{e}_3 = \mathbf{e}_1,\ \mathbf{e}_2\times\mathbf{e}_1 = -\mathbf{e}_3,\ \mathbf{e}_1\times\mathbf{e}_1 = 0, \ldots$$

which can be written for short as

$$\mathbf{e}_i\times\mathbf{e}_j = \varepsilon_{ijk}\mathbf{e}_k = \varepsilon_{jki}\mathbf{e}_k = \varepsilon_{kij}\mathbf{e}_k \tag{2A4.3}$$

Now, if $\mathbf{a} = a_i\mathbf{e}_i$, and $\mathbf{b} = b_i\mathbf{e}_i$, then

$$\mathbf{a}\times\mathbf{b} = (a_i\mathbf{e}_i)\times(b_j\mathbf{e}_j) = a_ib_j(\mathbf{e}_i\times\mathbf{e}_j) = a_ib_j\varepsilon_{ijk}\mathbf{e}_k$$

i.e.,

$$\mathbf{a}\times\mathbf{b} = a_ib_j\varepsilon_{ijk}\mathbf{e}_k \tag{2A4.4}$$

The following useful identity can be proven (see Prob. 2A7)

$$\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \tag{2A4.5}$$

2A5 Manipulations with the Indicial Notation

(a) *Substitution*

If

$$a_i = U_{im}b_m \tag{i}$$

and

$$b_i = V_{im}c_m \tag{ii}$$

then, in order to substitute the b_i's in (ii) into (i) we first change the free index in (ii) from i to m and the dummy index m to some other letter, say n so that

$$b_m = V_{mn}c_n \tag{iii}$$

Now, (i) and (iii) give

$$a_i = U_{im}V_{mn}c_n \tag{iv}$$

Note (iv) represents three equations each having the sum of nine terms on its right-hand side.

(b) *Multiplication*

If

$$p = a_m b_m \tag{i}$$

and

$$q = c_m d_m \tag{ii}$$

then,

$$pq = a_m b_m c_n d_n \tag{iii}$$

It is important to note that $pq \neq a_m b_m c_m d_m$. In fact, the right hand side of this expression is not even defined in the summation convention and further it is obvious that

$$pq \neq \sum_{m=1}^{3} a_m b_m c_m d_m.$$

Since the dot product of vectors is distributive, therefore, if $\mathbf{a} = a_i \mathbf{e}_i$ and $\mathbf{b} = b_i \mathbf{e}_i$, then

$$\mathbf{a} \cdot \mathbf{b} = (a_i \mathbf{e}_i) \cdot (b_j \mathbf{e}_j) = a_i b_j (\mathbf{e}_i \cdot \mathbf{e}_j) \tag{iv}$$

In particular, if $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit vectors perpendicular to one another, then $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ so that

$$\mathbf{a} \cdot \mathbf{b} = a_i b_j \delta_{ij} = a_i b_i = a_j b_j = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{v}$$

(c) *Factoring*

If

$$T_{ij} n_j - \lambda n_i = 0 \tag{i}$$

then, using the Kronecker delta, we can write

$$n_i = \delta_{ij} n_j \tag{ii}$$

so that (i) becomes

$$T_{ij} n_j - \lambda \delta_{ij} n_j = 0 \tag{iii}$$

Thus,

$$(T_{ij} - \lambda \delta_{ij}) n_j = 0 \tag{iv}$$

(d) *Contraction*

The operation of identifying two indices and so summing on them is known as contraction. For example, T_{ii} is the contraction of T_{ij},

$$T_{ii} = T_{11} + T_{22} + T_{33} \tag{i}$$

If

$$T_{ij} = \lambda\theta\delta_{ij}+2\mu E_{ij} \tag{ii}$$

then

$$T_{ii} = \lambda\theta\delta_{ii}+2\mu E_{ii} = 3\lambda\theta+2\mu E_{ii} \tag{iii}$$

Part B Tensors

2B1 Tensor – A Linear Transformation

Let **T** be a transformation, which transforms any vector into another vector. If **T** transforms **a** into **c** and **b** into **d**, we write $\mathbf{Ta} = \mathbf{c}$ and $\mathbf{Tb} = \mathbf{d}$.

If **T** has the following linear properties:

$$\mathbf{T}(\mathbf{a}+\mathbf{b}) = \mathbf{Ta}+\mathbf{Tb} \tag{2B1.1a}$$

$$\mathbf{T}(\alpha\mathbf{a}) = \alpha\mathbf{Ta} \tag{2B1.1b}$$

where **a** and **b** are two arbitrary vectors and α is an arbitrary scalar then **T** is called a **linear transformation.** It is also called a **second-order tensor** or simply a **tensor.**† An alternative and equivalent definition of a linear transformation is given by the single linear property:

$$\mathbf{T}(\alpha\mathbf{a}+\beta\mathbf{b}) = \alpha\mathbf{Ta}+\beta\mathbf{Tb} \tag{2B1.2}$$

where **a** and **b** are two arbitrary vectors and α and β are arbitrary scalars.

If two tensors, **T** and **S**, transform any arbitrary vector **a** in an identical way, then these tensors are equal to each other, i.e., $\mathbf{Ta}=\mathbf{Sa} \rightarrow \mathbf{T}=\mathbf{S}$.

Example 2B1.1

Let **T** be a transformation which transforms every vector into a fixed vector **n**. Is this transformation a tensor?

Solution. Let **a** and **b** be any two vectors, then by the definition of **T**,

$$\mathbf{Ta} = \mathbf{n}, \quad \mathbf{Tb} = \mathbf{n} \quad \text{and} \quad \mathbf{T}(\mathbf{a}+\mathbf{b}) = \mathbf{n}$$

Clearly,

$$\mathbf{T}(\mathbf{a}+\mathbf{b}) \neq \mathbf{Ta}+\mathbf{Tb}$$

Thus, **T** is not a linear transformation. In other words, it is not a tensor.

† Scalars and vectors are sometimes called the zeroth and first order tensor, respectively. Even though they can also be defined algebraically, in terms of certain operational rules, we choose not to do so. The geometrical concept of scalars and vectors, which we assume that the students are familiar with, is quite sufficient for our purpose.

Example 2B1.2

Let $\mathbf{T}$ be a transformation which transforms every vector into a vector that is k times the original vector. Is this transformation a tensor?

Solution. Let $\mathbf{a}$ and $\mathbf{b}$ be arbitrary vectors and α and β be arbitrary scalars, then by the definition of $\mathbf{T}$,

$$\mathbf{Ta} = k\mathbf{a}, \quad \mathbf{Tb} = k\mathbf{b}, \text{ and } \mathbf{T}(\alpha\mathbf{a}+\beta\mathbf{b}) = k(\alpha\mathbf{a}+\beta\mathbf{b})$$

Clearly,

$$\mathbf{T}(\alpha\mathbf{a}+\beta\mathbf{b}) = \alpha(k\mathbf{a})+\beta(k\mathbf{b}) = \alpha\mathbf{Ta}+\beta\mathbf{Tb}$$

Thus, by Eq. (2B1.2), $\mathbf{T}$ is a linear transformation. In other words, it is a tensor.

In the previous example, if $k=0$ then the tensor $\mathbf{T}$ transforms all vectors into zero. This tensor is the zero tensor and is symbolized by $\mathbf{0}$.

Example 2B1.3

Consider a transformation $\mathbf{T}$ that transforms every vector into its mirror image with respect to a fixed plane. Is $\mathbf{T}$ a tensor?

Solution. Consider a parallelogram in space with its sides represented by vectors $\mathbf{a}$ and $\mathbf{b}$ and its diagonal represented the resultant $\mathbf{a}+\mathbf{b}$. Since the parallelogram remains a parallelogram after the reflection, the diagonal (the resultant vector) of the reflected parallelogram is clearly both $\mathbf{T}(\mathbf{a}+\mathbf{b})$, the reflected $(\mathbf{a}+\mathbf{b})$, and $\mathbf{Ta}+\mathbf{Tb}$, the sum of the reflected $\mathbf{a}$ and the reflected $\mathbf{b}$. That is, $\mathbf{T}(\mathbf{a}+\mathbf{b}) = \mathbf{Ta}+\mathbf{Tb}$. Also, for an arbitrary scalar α, the reflection of $\alpha\mathbf{a}$ is obviously the same as α times the reflection of $\mathbf{a}$ (i.e., $\mathbf{T}(\alpha\mathbf{a}) = \alpha\mathbf{Ta}$) because both vectors have the same magnitude given by α times the magnitude of $\mathbf{a}$ and the same direction. Thus, by Eqs. (2B1.1), $\mathbf{T}$ is a tensor.

Example 2B1.4

When a rigid body undergoes a rotation about some axis, vectors drawn in the rigid body in general change their directions. That is, the rotation transforms vectors drawn in the rigid body into other vectors. Denote this transformation by $\mathbf{R}$. Is $\mathbf{R}$ a tensor?

Solution. Consider a parallelogram embedded in the rigid body with its sides representing vectors $\mathbf{a}$ and $\mathbf{b}$ and its diagonal representing the resultant $\mathbf{a}+\mathbf{b}$. Since the parallelogram remains a parallelogram after a rotation about any axis, the diagonal (the resultant vector) of the rotated parallelogram is clearly both $\mathbf{R}(\mathbf{a}+\mathbf{b})$, the rotated $(\mathbf{a}+\mathbf{b})$, and $\mathbf{Ra}+\mathbf{Rb}$, the sum of the rotated $\mathbf{a}$ and the rotated $\mathbf{b}$. That is $\mathbf{R}(\mathbf{a}+\mathbf{b}) = \mathbf{Ra}+\mathbf{Rb}$. A similar argument as that used in the previous example leads to $\mathbf{R}(\alpha\mathbf{a}) = \alpha\mathbf{Ra}$. Thus, $\mathbf{R}$ is a tensor.

Example 2B1.5

Let **T** be a tensor that transforms the specific vectors **a** and **b** according to

$$\mathbf{Ta} = \mathbf{a}+2\mathbf{b}, \quad \mathbf{Tb} = \mathbf{a}-\mathbf{b}$$

Given a vector $\mathbf{c} = 2\mathbf{a}+\mathbf{b}$, find $\mathbf{Tc}$.

Solution. Using the linearity property of tensors

$$\mathbf{Tc} = \mathbf{T}(2\mathbf{a}+\mathbf{b}) = 2\mathbf{Ta}+\mathbf{Tb} = 2(\mathbf{a}+2\mathbf{b})+(\mathbf{a}-\mathbf{b}) = 3\mathbf{a}+3\mathbf{b}$$

2B2 Components of a Tensor

The components of a vector depend on the base vectors used to describe the components. This will also be true for tensors. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be unit vectors in the direction of the x_1-, x_2-, x_3-axes respectively, of a rectangular Cartesian coordinate system. Under a transformation **T**, these vectors, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ become $\mathbf{Te}_1$, $\mathbf{Te}_2$, and $\mathbf{Te}_3$. Each of these $\mathbf{Te}_i$ ($i=1,2,3$), being a vector, can be written as:

$$\begin{aligned}
\mathbf{Te}_1 &= T_{11}\mathbf{e}_1+T_{21}\mathbf{e}_2+T_{31}\mathbf{e}_3 \\
\mathbf{Te}_2 &= T_{12}\mathbf{e}_1+T_{22}\mathbf{e}_2+T_{32}\mathbf{e}_3 \\
\mathbf{Te}_3 &= T_{13}\mathbf{e}_1+T_{23}\mathbf{e}_2+T_{33}\mathbf{e}_3
\end{aligned} \tag{2B2.1a}$$

or

$$\mathbf{Te}_i = T_{ji}\mathbf{e}_j \tag{2B2.1b}$$

It is clear from Eqs. (2B2.1a) that

$$T_{11} = \mathbf{e}_1\cdot\mathbf{Te}_1, \; T_{12} = \mathbf{e}_1\cdot\mathbf{Te}_2, \; T_{21} = \mathbf{e}_2\cdot\mathbf{Te}_1, \; \ldots$$

or in general

$$T_{ij} = \mathbf{e}_i\cdot\mathbf{Te}_j \tag{2B2.2}$$

The components T_{ij} in the above equations are defined as the components of the tensor **T**. These components can be put in a matrix as follows:

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

This matrix is called the matrix of the tensor **T** with respect to the set of base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ or $\{\mathbf{e}_i\}$ for short. We note that, because of the way we have chosen to denote the components of transformation of the base vectors, the elements of the first column are components of the vector $\mathbf{Te}_1$, those in the second column are the components of the vector $\mathbf{Te}_2$, and those in the third column are the components of $\mathbf{Te}_3$.

Example 2B2.1

Obtain the matrix for the tensor **T** which transforms the base vectors as follows:

$$\mathbf{T}\mathbf{e}_1 = 4\mathbf{e}_1+\mathbf{e}_2$$
$$\mathbf{T}\mathbf{e}_2 = 2\mathbf{e}_1+3\mathbf{e}_3$$
$$\mathbf{T}\mathbf{e}_3 = -\mathbf{e}_1+3\mathbf{e}_2+\mathbf{e}_3$$

Solution. By Eq. (2B2.1a) it is clear that:

$$[\mathrm{T}] = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

Example 2B2.2

Let **T** transform every vector into its mirror image with respect to a fixed plane. If $\mathbf{e}_1$ is normal to the reflection plane ($\mathbf{e}_2$ and $\mathbf{e}_3$ are parallel to this plane), find a matrix of **T**.

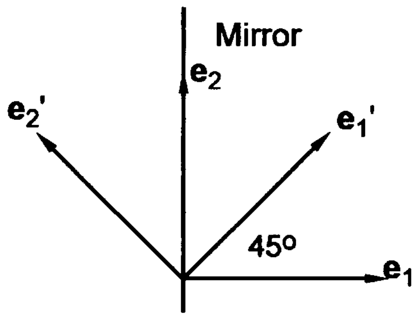

Fig. 2B.1

Solution. Since the normal to the reflection plane is transformed into its negative and vectors parallel to the plane are not altered:

$$\mathbf{T}\mathbf{e}_1 = -\mathbf{e}_1$$
$$\mathbf{T}\mathbf{e}_2 = \mathbf{e}_2$$
$$\mathbf{T}\mathbf{e}_3 = \mathbf{e}_3$$

Thus,

$$[\mathbf{T}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}_i}$$

We note that this is only one of the infinitely many matrices of the tensor **T**, each depends on a particular choice of base vectors. In the above matrix, the choice of $\mathbf{e}_i$ is indicated at the bottom right corner of the matrix. If we choose $\mathbf{e}_1'$ and $\mathbf{e}_2'$ to be on a plane perpendicular to the mirror with each making 45° with the mirror as shown in Fig. 2B.1 and $\mathbf{e}_3'$ points straight out from the paper. Then we have

$$\mathbf{Te}_1' = \mathbf{e}_2'$$

$$\mathbf{Te}_2' = \mathbf{e}_1'$$

$$\mathbf{Te}_3' = \mathbf{e}_3'$$

Thus, with respect to $\{\mathbf{e}_i'\}$, the matrix of the tensor is

$$[\mathbf{T}]' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}_i'}$$

Throughout this book, we shall denote the matrix of a tensor **T** with respect to the basis $\mathbf{e}_i$ by either $[\mathbf{T}]$ or $[T_{ij}]$ and with respect to the basis $\mathbf{e}_i'$ by either $[\mathbf{T}]'$ or $[T_{ij}']$ The last two matrices should not be confused with $[\mathbf{T}']$, which represents the matrix of the tensor $\mathbf{T}'$ with respect to the basis $\mathbf{e}_i$.

Example 2B2.3

Let **R** correspond to a right-hand rotation of a rigid body about the x_3-axis by an angle θ. Find a matrix of **R**.

Solution. From Fig. 2B.2 it is clear that

$$\mathbf{Re}_1 = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$$

$$\mathbf{Re}_2 = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2$$

$$\mathbf{Re}_3 = \mathbf{e}_3$$

Thus,

$$[\mathbf{R}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}_i}$$

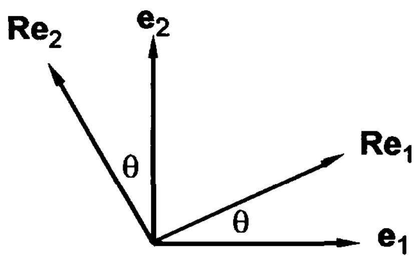

Fig. 2B.2

2B3 Components of a Transformed Vector

Given the vector **a** and the tensor **T**, we wish to compute the components of **b**=**Ta** from the components of **a** and the components of **T**. Let the components of **a** with respect to $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be $[a_1, a_2, a_3]$, i.e.,

$$\mathbf{a} = a_1\mathbf{e}_1+a_2\mathbf{e}_2+a_3\mathbf{e}_3 \tag{i}$$

then

$$\mathbf{b} = \mathbf{Ta} = \mathbf{T}(a_1\mathbf{e}_1+a_2\mathbf{e}_2+a_3\mathbf{e}_3) = a_1\mathbf{Te}_1+a_2\mathbf{Te}_2+a_3\mathbf{Te}_3 \tag{ii}$$

Thus,

$$\begin{aligned} b_1 &= \mathbf{e}_1\cdot\mathbf{b} = a_1(\mathbf{e}_1\cdot\mathbf{Te}_1)+a_2(\mathbf{e}_1\cdot\mathbf{Te}_2)+a_3(\mathbf{e}_1\cdot\mathbf{Te}_3) \\ b_2 &= \mathbf{e}_2\cdot\mathbf{b} = a_1(\mathbf{e}_2\cdot\mathbf{Te}_1)+a_2(\mathbf{e}_2\cdot\mathbf{Te}_2)+a_3(\mathbf{e}_2\cdot\mathbf{Te}_3) \\ b_3 &= \mathbf{e}_3\cdot\mathbf{b} = a_1(\mathbf{e}_3\cdot\mathbf{Te}_1)+a_2(\mathbf{e}_3\cdot\mathbf{Te}_2)+a_3(\mathbf{e}_3\cdot\mathbf{Te}_3) \end{aligned} \tag{iii}$$

By Eq. (2B2.2), we have,

$$\begin{aligned} b_1 &= T_{11}a_1+T_{12}a_2+T_{13}a_3 \\ b_2 &= T_{21}a_1+T_{22}a_2+T_{23}a_3 \\ b_3 &= T_{31}a_1+T_{32}a_2+T_{33}a_3 \end{aligned} \tag{2B3.1a}$$

We can write the above three equations in matrix form as:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{2B3.1b}$$

or

$$[\mathbf{b}] = [\mathbf{T}][\mathbf{a}] \tag{2B3.1c}$$

We can concisely derive Eq. (2B3.1a) using indicial notation as follows: From $\mathbf{a} = a_i\mathbf{e}_i$, we get $\mathbf{Ta} = \mathbf{T}a_i\mathbf{e}_i = a_i\mathbf{Te}_i$. Since $\mathbf{Te}_i = T_{ji}\mathbf{e}_j$, (Eq. (2B2.1b)), therefore,

$$b_k = \mathbf{b}\cdot\mathbf{e}_k = \mathbf{Ta}\cdot\mathbf{e}_k = a_iT_{ji}\mathbf{e}_j\cdot\mathbf{e}_k = a_iT_{ji}\delta_{jk} = a_iT_{ki}$$

i.e.,

$$b_k = T_{ki}a_i \tag{2B3.1d}$$

Eq. (2B3.1d) is nothing but Eq. (2B3.1a) in indicial notation. We see that for the tensorial equation $\mathbf{b} = \mathbf{Ta}$, there corresponds a matrix equation of exactly the same form, i.e., $[\mathbf{b}] = [\mathbf{T}][\mathbf{a}]$. This is the reason we adopted the convention that $\mathbf{Te}_1 = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3$, etc. If we had adopted the convention $\mathbf{Te}_1 = T_{11}\mathbf{e}_1 + T_{12}\mathbf{e}_2 + T_{13}\mathbf{e}_3$, etc., then we would have obtained $[\mathbf{b}] = [\mathbf{T}]^T[\mathbf{a}]$ for the tensorial equation $\mathbf{b} = \mathrm{T}\mathbf{a}$, which would not be as natural.

Example 2B3.1

Given that a tensor $\mathbf{T}$ which transforms the base vectors as follows:

$$\mathbf{Te}_1 = 2\mathbf{e}_1 - 6\mathbf{e}_2 + 4\mathbf{e}_3$$
$$\mathbf{Te}_2 = 3\mathbf{e}_1 + 4\mathbf{e}_2 - \mathbf{e}_3$$
$$\mathbf{Te}_3 = -2\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3$$

How does this tensor transform the vector $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$?

Solution. Using Eq. (2B3.1b)

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -2 \\ -6 & 4 & 1 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

or

$$\mathbf{b} = 2\mathbf{e}_1 + 5\mathbf{e}_2 + 8\mathbf{e}_3$$

2B4 Sum of Tensors

Let $\mathbf{T}$ and $\mathbf{S}$ be two tensors and $\mathbf{a}$ be an arbitrary vector. The sum of $\mathbf{T}$ and $\mathbf{S}$, denoted by $\mathbf{T} + \mathbf{S}$, is defined by:

$$(\mathbf{T} + \mathbf{S})\mathbf{a} = \mathbf{Ta} + \mathbf{Sa} \tag{2B4.1}$$

It is easily seen that by this definition $\mathbf{T}+\mathbf{S}$ is indeed a tensor.

To find the components of $\mathbf{T}+\mathbf{S}$, let

$$\mathbf{W}=\mathbf{T}+\mathbf{S} \tag{2B4.2a}$$

Using Eqs. (2B2.2) and (2B4.1), the components of $\mathbf{W}$ are obtained to be

$$W_{ij} = \mathbf{e}_i\cdot(\mathbf{T}+\mathbf{S})\mathbf{e}_j = \mathbf{e}_i\cdot\mathbf{T}\mathbf{e}_j+\mathbf{e}_i\cdot\mathbf{S}\mathbf{e}_j$$

i.e.,

$$W_{ij} = T_{ij}+S_{ij} \tag{2B4.2b}$$

In matrix notation, we have

$$[\mathbf{W}] = [\mathbf{T}]+[\mathbf{S}] \tag{2B4.2c}$$

2B5 Product of Two Tensors

Let $\mathbf{T}$ and $\mathbf{S}$ be two tensors and $\mathbf{a}$ be an arbitrary vector, then $\mathbf{TS}$ and $\mathbf{ST}$ are defined to be the transformations (easily seen to be tensors)

$$(\mathbf{TS})\mathbf{a} = \mathbf{T}(\mathbf{S}\mathbf{a}) \tag{2B5.1}$$

and

$$(\mathbf{ST})\mathbf{a} = \mathbf{S}(\mathbf{T}\mathbf{a}) \tag{2B5.2}$$

Thus the components of $\mathbf{TS}$ are

$$(\mathbf{TS})_{ij}=\mathbf{e}_i\cdot(\mathbf{TS})\mathbf{e}_j=\mathbf{e}_i\cdot\mathbf{T}(\mathbf{S}\mathbf{e}_j)=\mathbf{e}_i\cdot\mathbf{T}S_{mj}\mathbf{e}_m=S_{mj}\mathbf{e}_i\cdot\mathbf{T}\mathbf{e}_m=T_{im}S_{mj}$$

i.e.,

$$(\mathbf{TS})_{ij} = T_{im}S_{mj} \tag{2B5.3}$$

Similarly,

$$(\mathbf{ST})_{ij} = S_{im}T_{mj} \tag{2B5.4}$$

In fact, Eq. (2B5.3) is equivalent to the matrix equation:

$$[\mathbf{TS}] = [\mathbf{T}][\mathbf{S}] \tag{2B5.5}$$

whereas, Eq. (2B5.4) is equivalent to the matrix equation:

$$[\mathbf{ST}] = [\mathbf{S}][\mathbf{T}] \tag{2B5.6}$$

The two matrix products are in general different. Thus, it is clear that in general, the tensor product is not commutative (i.e., $\mathbf{TS} \neq \mathbf{ST}$).

If $\mathbf{T}$,$\mathbf{S}$, and $\mathbf{V}$ are three tensors, then

$$(\mathbf{T}(\mathbf{SV}))\mathbf{a} = \mathbf{T}((\mathbf{SV})\mathbf{a}) = \mathbf{T}(\mathbf{S}(\mathbf{V}\mathbf{a}))$$

and

$$(\mathbf{TS})(\mathbf{Va}) = \mathbf{T}(\mathbf{S}(\mathbf{Va}))$$

i.e.,

$$\mathbf{T}(\mathbf{SV}) = (\mathbf{TS})\mathbf{V} \tag{2B5.7}$$

Thus, the tensor product is associative. It is, therefore, natural to define the integral positive powers of a transformation by these simple products, so that

$$\mathbf{T}^2 = \mathbf{TT}, \quad \mathbf{T}^3 = \mathbf{TTT}, \quad .... \tag{2B5.8}$$

Example 2B5.1

(a)Let **R** correspond to a 90° right-hand rigid body rotation about the x_3-axis. Find the matrix of **R**.

(b)Let **S** correspond to a 90° right-hand rigid body rotation about the x_1-axis. Find the matrix of **S**.

(c)Find the matrix of the tensor that corresponds to the rotation (a) then (b).

(d)Find the matrix of the tensor that corresponds to the rotation (b) then (a).

(e)Consider a point P whose initial coordinates are (1,1,0). Find the new position of this point after the rotations of part (c). Also find the new position of this point after the rotations of part (d).

Solution. (a) For this rotation the transformation of the base vectors is given by

$$\mathbf{Re}_1 = \mathbf{e}_2$$

$$\mathbf{Re}_2 = -\mathbf{e}_1$$

$$\mathbf{Re}_3 = \mathbf{e}_3$$

so that,

$$[\mathbf{R}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)In a similar manner to (a) the transformation of the base vectors is given by

$$\mathbf{Se}_1 = \mathbf{e}_1$$

$$\mathbf{Se}_2 = \mathbf{e}_3$$

$$\mathbf{Se}_3 = -\mathbf{e}_2$$

so that,

$$[\mathbf{S}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

(c)Since $\mathbf{S}(\mathbf{Ra}) = (\mathbf{SR})\mathbf{a}$, the resultant rotation is given by the single transformation **SR** whose components are given by the matrix

$$[\mathbf{SR}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

(d)In a manner similar to (c) the resultant rotation is given by the single transformation **RS** whose components are given by the matrix

$$[\mathbf{RS}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(e)Let $\mathbf{r}$ be the initial position of the point P. Let $\mathbf{r}^*$ and $\mathbf{r}^{**}$ be the rotated position of P after the rotations of part (c) and part (d) respectively. Then

$$[\mathbf{r}^*] = [\mathbf{SR}][\mathbf{r}] = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

i.e.,

$$\mathbf{r}^* = -\mathbf{e}_1 + \mathbf{e}_3$$

and

$$[\mathbf{r}^{**}] = [\mathbf{RS}][\mathbf{r}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

i.e.,

$$\mathbf{r}^{**} = \mathbf{e}_2 + \mathbf{e}_3$$

This example further illustrates that the order of rotations is important.

2B6 Transpose of a Tensor

The transpose of a tensor $\mathbf{T}$, denoted by $\mathbf{T}^T$, is defined to be the tensor which satisfies the following identity for all vectors **a** and **b**:

$$\mathbf{a} \cdot \mathbf{Tb} = \mathbf{b} \cdot \mathbf{T}^T \mathbf{a} \tag{2B6.1}$$

It can be easily seen that $\mathbf{T}^T$ is a tensor. From the above definition, we have

$$\mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j = \mathbf{e}_j \cdot \mathbf{T}^T \mathbf{e}_i$$

Thus,

$$T_{ij} = T^T_{ji} \tag{2B6.2}$$

or

$$[\mathbf{T}^T] = [\mathbf{T}]^T$$

i.e., the matrix of $\mathbf{T}^T$ is the transpose of the matrix of $\mathbf{T}$.

We also note that by Eq. (2B6.1), we have

$$\mathbf{a}\cdot\mathbf{T}^T\mathbf{b} = \mathbf{b}\cdot(\mathbf{T}^T)^T\mathbf{a}$$

Thus, $\mathbf{b}\cdot\mathbf{Ta} = \mathbf{b}\cdot(\mathbf{T}^T)^T\mathbf{a}$ for any $\mathbf{a}$ and $\mathbf{b}$, so that

$$\mathbf{T} = (\mathbf{T}^T)^T \tag{2B6.3}$$

It can also be established that (see Prob. 2B13)

$$(\mathbf{TS})^T = \mathbf{S}^T\mathbf{T}^T \tag{2B6.4}$$

That is, the transpose of a product of the tensors is equal to the product of transposed tensors in reverse order. More generally,

$$(\mathbf{ABC}...\mathbf{D})^T = \mathbf{D}^T...\mathbf{C}^T\mathbf{B}^T\mathbf{A}^T \tag{2B6.5}$$

2B7 Dyadic Product of Two Vectors

The dyadic product of vectors $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{ab}$, is defined to be the transformation which transforms an arbitrary vector $\mathbf{c}$ according to the rule:

$$(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}) \tag{2B7.1}$$

Now, for any $\mathbf{c}$, $\mathbf{d}$, α and β, we have, from the above definition:

$$(\mathbf{ab})(\alpha\mathbf{c}+\beta\mathbf{d}) = \mathbf{a}(\mathbf{b}\cdot(\alpha\mathbf{c}+\beta\mathbf{d})) = \mathbf{a}((\alpha\mathbf{b}\cdot\mathbf{c})+(\beta\mathbf{b}\cdot\mathbf{d})) = \alpha(\mathbf{ab})\mathbf{c}+\beta(\mathbf{ab})\mathbf{d}$$

Thus, $\mathbf{ab}$ is a tensor. Letting $\mathbf{W}=\mathbf{ab}$, then the components of $\mathbf{W}$ are:

$$W_{ij} = \mathbf{e}_i\cdot\mathbf{We}_j = \mathbf{e}_i\cdot(\mathbf{ab})\mathbf{e}_j = \mathbf{e}_i\cdot\mathbf{a}(\mathbf{b}\cdot\mathbf{e}_j) = a_ib_j$$

i.e.,

$$W_{ij} = a_ib_j \tag{2B7.2a}$$

In matrix notation, Eq. (2B7.2a) is

$$[\mathbf{W}] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [b_1, b_2, b_3] = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} \tag{2B7.2b}$$

In particular, the components of the dyadic product of the base vectors $\mathbf{e}_i$ are:

$$[\mathbf{e}_1\mathbf{e}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; [\mathbf{e}_1\mathbf{e}_2] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; ...$$

Thus, it is clear that any tensor $\mathbf{T}$ can be expressed as:

$$\mathbf{T} = T_{11}\mathbf{e}_1\mathbf{e}_1 + T_{12}\mathbf{e}_1\mathbf{e}_2 + \ldots + T_{33}\mathbf{e}_3\mathbf{e}_3 \tag{2B7.3a}$$

i.e.,

$$\mathbf{T} = T_{ij}\mathbf{e}_i\mathbf{e}_j \tag{2B7.3b}$$

We note that another commonly used notation for the dyadic product of **a** and **b** is $\mathbf{a}\otimes\mathbf{b}$.

2B8 Trace of a Tensor

The trace of any dyad **ab** is defined to be a scalar given by $\mathbf{a}\cdot\mathbf{b}$. That is,

$$\text{tr } \mathbf{ab} = \mathbf{a}\cdot\mathbf{b} \tag{2B8.1}$$

Furthermore, the trace is defined to be a linear operator that satisfies the relation:

$$\text{tr}(\alpha\mathbf{ab} + \beta\mathbf{cd}) = \alpha\text{tr } \mathbf{ab} + \beta\text{tr } \mathbf{cd} \tag{2B8.2}$$

Using Eq. (2B7.3b), the trace of **T** can, therefore, be obtained as

$$\text{tr } \mathbf{T} = \text{tr}(T_{ij}\mathbf{e}_i\mathbf{e}_j) = T_{ij}\text{tr}(\mathbf{e}_i\mathbf{e}_j) = T_{ij}\mathbf{e}_i\cdot\mathbf{e}_j = T_{ij}\delta_{ij} = T_{ii}$$

that is,

$$\text{tr } \mathbf{T} = T_{ii} = T_{11} + T_{22} + T_{33} = \text{sum of diagonal elements} \tag{2B8.3}$$

It is obvious that

$$\text{tr } \mathbf{T}^{\mathbf{T}} = \text{tr } \mathbf{T} \tag{2B8.4}$$

Example 2B8.1

Show that for any second-order tensor **A** and **B**

$$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA}) \tag{2B8.5}$$

Solution. Let $\mathbf{C}=\mathbf{AB}$, then $C_{ij}=A_{im}B_{mj}$. Thus,

$$\text{tr } \mathbf{AB} = \text{tr } \mathbf{C} = C_{ii} = A_{im}B_{mi} \tag{i}$$

Let $\mathbf{D}=\mathbf{BA}$, then $D_{ij}=B_{im}A_{mj}$, and

$$\text{tr } \mathbf{BA} = \text{tr } \mathbf{D} = D_{ii} = B_{im}A_{mi} \tag{ii}$$

But $B_{im}A_{mi}=B_{mi}A_{im}$ (change of dummy indices), that is

$$\text{tr } \mathbf{BA} = \text{tr } \mathbf{AB} \tag{iii}$$

2B9 Identity Tensor and Tensor Inverse

The linear transformation which transforms every vector into itself is called an identity tensor. Denoting this special tensor by **I**, we have, for any vector **a**,

$$\mathbf{Ia} = \mathbf{a} \tag{2B9.1}$$

and in particular,

$$\mathbf{Ie}_1 = \mathbf{e}_1$$
$$\mathbf{Ie}_2 = \mathbf{e}_2$$
$$\mathbf{Ie}_3 = \mathbf{e}_3$$

Thus, the components of the identity tensor are:

$$I_{ij} = \mathbf{e}_i \cdot \mathbf{Ie}_j = \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{2B9.2a}$$

i.e.,

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2B9.2b}$$

It is obvious that the identity matrix is the matrix of **I** for *all* rectangular Cartesian coordinates and that **TI** = **IT** = **T** for any tensor **T**. We also note that if **Ta** = **a** for any arbitrary **a**, then **T** = **I**.

Example 2B9.1

Write the tensor **T**, defined by the equation **Ta** = k**a**, where k is a constant and **a** is arbitrary, in terms of the identity tensor and find its components.

Solution. Using Eq. (2B9.1) we can write k **a** as k**Ia** so that **Ta** = k**a** becomes

$$\mathbf{Ta} = k\mathbf{Ia}$$

and since **a** is arbitrary

$$\mathbf{T} = k\mathbf{I}$$

The components of this tensor are clearly,

$$T_{ij} = k\delta_{ij}$$

Given a tensor **T**, if a tensor **S** exists such that **ST**=**I** then we call **S** the inverse of **T** or $\mathbf{S}=\mathbf{T}^{-1}$. (Note: With $\mathbf{T}^{-1}\mathbf{T}=\mathbf{T}^{-1+1}=\mathbf{T}^{o}=\mathbf{I}$, the zeroth power of a tensor is the identity tensor). To find the components of the inverse of a tensor **T** is to find the inverse of the matrix of **T**. From the study of matrices we know that the inverse exists as long as $\det\mathbf{T}\neq 0$ (that is, **T** is non-singular) and in this case, $[\mathbf{T}]^{-1}[\mathbf{T}] = [\mathbf{T}][\mathbf{T}]^{-1} = [\mathbf{I}]$. Thus, the inverse of a tensor satisfies the following reciprocal relation:

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{T}\mathbf{T}^{-1} = \mathbf{I} \tag{2B9.3}$$

We can easily show (see Prob. 2B15) that for the tensor inverse the following relations are satisfied,

$$(\mathbf{T}^T)^{-1} = (\mathbf{T}^{-1})^T \tag{2B9.4}$$

and

$$(\mathbf{S}\mathbf{T})^{-1} = \mathbf{T}^{-1}\mathbf{S}^{-1} \tag{2B9.5}$$

We note that if the inverse exists then we have the reciprocal relation that

$$\mathbf{Ta} = \mathbf{b} \quad \text{and} \quad \mathbf{a} = \mathbf{T}^{-1}\mathbf{b}$$

This indicates that when a tensor is invertible there is a one to one mapping of vectors $\mathbf{a}$ and $\mathbf{b}$. On the other hand, if a tensor $\mathbf{T}$ does not have an inverse, then, for a given $\mathbf{b}$, there are in general more than one $\mathbf{a}$ which transforms into $\mathbf{b}$. For example, consider the singular tensor $\mathbf{T} = \mathbf{cd}$ (the dyadic product of $\mathbf{c}$ and $\mathbf{d}$, which does not have an inverse because its determinant is zero), we have

$$\mathbf{Ta} = \mathbf{c}(\mathbf{d}\cdot\mathbf{a}) \equiv \mathbf{b}$$

Now, let $\mathbf{h}$ be any vector perpendicular to $\mathbf{d}$ (i.e., $\mathbf{d}\cdot\mathbf{h} = 0$), then

$$\mathbf{T}(\mathbf{a} + \mathbf{h}) = \mathbf{c}(\mathbf{d}\cdot\mathbf{a}) = \mathbf{b}$$

That is, all vectors $\mathbf{a} + \mathbf{h}$ transform under $\mathbf{T}$ into the same vector $\mathbf{b}$.

2B10 Orthogonal Tensor

An orthogonal tensor is a linear transformation, under which the transformed vectors preserve their lengths and angles. Let $\mathbf{Q}$ denote an orthogonal tensor, then by definition, $|\mathbf{Qa}| = |\mathbf{a}|$ and $\cos(\mathbf{a},\mathbf{b}) = \cos(\mathbf{Qa},\mathbf{Qb})$ for any $\mathbf{a}$ and $\mathbf{b}$, Thus,

$$\mathbf{Qa}\cdot\mathbf{Qb} = \mathbf{a}\cdot\mathbf{b} \tag{2B10.1}$$

for any $\mathbf{a}$ and $\mathbf{b}$.

Using the definitions of the transpose and the product of tensors:

$$(\mathbf{Qa})\cdot(\mathbf{Qb}) = \mathbf{b}\cdot\mathbf{Q}^T(\mathbf{Qa}) = \mathbf{b}\cdot(\mathbf{Q}^T\mathbf{Q})\mathbf{a} \tag{i}$$

Therefore,

$$\mathbf{b}\cdot(\mathbf{Q}^T\mathbf{Q})\mathbf{a} = \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{Ia} \tag{ii}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are arbitrary, it follows that

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{iii}$$

This means that $\mathbf{Q}^{-1}=\mathbf{Q}^T$ and from Eq. (2B9.3),

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I} \tag{2B10.2a}$$

In matrix notation, Eqs. (2B10.2a) take the form:

$$[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{Q}]^T[\mathbf{Q}] = [\mathbf{I}] \tag{2B10.2b}$$

and in subscript notation, these equations take the form:

$$Q_{im}Q_{jm} = Q_{mi}Q_{mj} = \delta_{ij} \tag{2B10.2c}$$

Example 2B10.1

The tensor given in Example 2B2.2, being a reflection, is obviously an orthogonal tensor. Verify that $[\mathbf{T}][\mathbf{T}]^T = [\mathbf{I}]$ for the $[\mathbf{T}]$ in that example. Also, find the determinant of $[\mathbf{T}]$.

Solution. Using the matrix of Example 2B7.1:

$$[\mathbf{T}][\mathbf{T}]^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The determinant of $[\mathbf{T}]$ is

$$|\mathbf{T}| = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

Example 2B10.2

The tensor given in Example 2B2.3, being a rigid body rotation, is obviously an orthogonal tensor. Verify that $[\mathbf{R}][\mathbf{R}]^T = [\mathbf{I}]$ for the $[\mathbf{R}]$ in that example. Also find the determinant of $[\mathbf{R}]$.

Solution. It is clear that

$$[\mathbf{R}][\mathbf{R}]^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det[\mathbf{R}] \equiv |\mathbf{R}| = \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = +1$$

The determinant of the matrix of any orthogonal tensor $\mathbf{Q}$ is easily shown to be equal to either $+1$ or -1. In fact,

$$[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{I}]$$

therefore,

$$|[\mathbf{Q}][\mathbf{Q}]^T| = |\mathbf{Q}||\mathbf{Q}^T| = |\mathbf{I}|$$

Now, $|\mathbf{Q}| = |\mathbf{Q}^T|$, and $|\mathbf{I}| = 1$, therefore, $|\mathbf{Q}|^2 = 1$, thus

$$|\mathbf{Q}| = \pm 1 \tag{2B10.3}$$

From the previous examples we can see that the value of $+1$ corresponds to *rotation* and -1 corresponds to *reflection*.

2B11 Transformation Matrix Between Two Rectangular Cartesian Coordinate Systems.

Suppose $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_i'\}$ are unit vectors corresponding to two rectangular Cartesian coordinate systems (see Fig. 2B.3). It is clear that $\{\mathbf{e}_i\}$ can be made to coincide with $\{\mathbf{e}_i'\}$ through either a rigid body rotation (if both bases are same handed) or a rotation followed by a reflection (if different handed). That is $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_i'\}$ can be related by an orthogonal tensor $\mathbf{Q}$ through the equations

$$\mathbf{e}_i' = \mathbf{Q}\mathbf{e}_i = Q_{mi}\mathbf{e}_m \tag{2B11.1a}$$

i.e.,

$$\begin{aligned} \mathbf{e}_1' &= Q_{11}\mathbf{e}_1 + Q_{21}\mathbf{e}_2 + Q_{31}\mathbf{e}_3 \\ \mathbf{e}_2' &= Q_{12}\mathbf{e}_1 + Q_{22}\mathbf{e}_2 + Q_{32}\mathbf{e}_3 \\ \mathbf{e}_3' &= Q_{13}\mathbf{e}_1 + Q_{23}\mathbf{e}_2 + Q_{33}\mathbf{e}_3 \end{aligned} \tag{2B11.1b}$$

where

$$Q_{im}Q_{jm} = Q_{mi}Q_{mj} = \delta_{ij}$$

or

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$$

We note that $Q_{11} = \mathbf{e}_1 \cdot \mathbf{Q}\mathbf{e}_1 = \mathbf{e}_1 \cdot \mathbf{e}_1' =$ cosine of the angle between $\mathbf{e}_1$ and $\mathbf{e}_1'$, $Q_{12} = \mathbf{e}_1 \cdot \mathbf{Q}\mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_2' =$ cosine of the angle between $\mathbf{e}_1$ and $\mathbf{e}_2'$, etc. In general, $Q_{ij} =$ cosine of the angle between $\mathbf{e}_i$ and $\mathbf{e}_j'$ which may be written:

$$Q_{ij} = \cos(\mathbf{e}_i, \mathbf{e}_j') = \mathbf{e}_i \cdot \mathbf{e}_j' \tag{2B11.2}$$

The matrix of these directional cosines, i.e., the matrix

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

is called the transformation matrix between $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_i'\}$. Using this matrix, we shall obtain, in the following sections, the relationship between the two sets of components, with respect to these two sets of base vectors, of either a vector or a tensor.

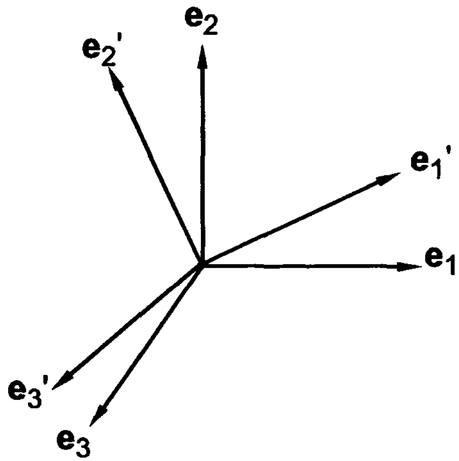

Fig. 2B.3

Example 2B11.1

Let $\{\mathbf{e}_i'\}$ be obtained by rotating the basis $\{\mathbf{e}_i\}$ about the $\mathbf{e}_3$ axis through 30^o as shown in Fig. 2B.4. We note that in this figure, $\mathbf{e}_3$ and $\mathbf{e}_3'$ coincide.

Solution. We can obtain the transformation matrix in two ways.

(i) Using Eq. (2B11.2), we have

$$Q_{11}=\cos(\mathbf{e}_1,\mathbf{e}_1')=\cos 30^o=\frac{\sqrt{3}}{2},\ Q_{12}=\cos(\mathbf{e}_1,\mathbf{e}_2')=\cos 120^o=-\frac{1}{2},\ Q_{13}=\cos(\mathbf{e}_1,\mathbf{e}_3')=\cos 90^o=0$$

$$Q_{21}=\cos(\mathbf{e}_2,\mathbf{e}_1')=\cos 60^o=\frac{1}{2},\ Q_{22}=\cos(\mathbf{e}_2,\mathbf{e}_2')=\cos 30^o=\frac{\sqrt{3}}{2},\ Q_{23}=\cos(\mathbf{e}_2,\mathbf{e}_3')=\cos 90^o=0$$

$$Q_{31}=\cos(\mathbf{e}_3,\mathbf{e}_1')=\cos 90^o=0,\ Q_{32}=\cos(\mathbf{e}_3,\mathbf{e}_2')=\cos 90^o=0,\ Q_{33}=\cos(\mathbf{e}_3,\mathbf{e}_3')=\cos 0^o=1$$

(ii) It is easier to simply look at Fig. 2B.4 and decompose each of the $\mathbf{e}_i'$'s into its components in the $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ directions, i.e.,

$$\mathbf{e}_1' = \frac{\sqrt{3}}{2}\mathbf{e}_1+\frac{1}{2}\mathbf{e}_2$$

$$\mathbf{e}_2' = -\frac{1}{2}\mathbf{e}_1+\frac{\sqrt{3}}{2}\mathbf{e}_2$$

$$\mathbf{e}_3' = \mathbf{e}_3$$

Thus, by either method, the transformation matrix is

$$[\mathbf{Q}] = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

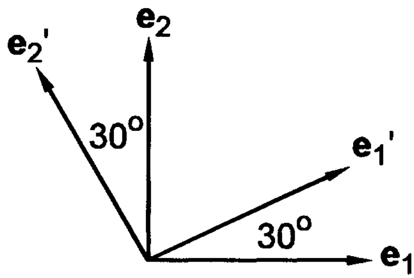

Fig. 2B.4

2B12 Transformation Laws for Cartesian Components of Vectors

Consider any vector **a**, then the components of **a** with respect to $\{\mathbf{e}_i\}$ are

$$a_i = \mathbf{a}\cdot\mathbf{e}_i$$

and its components with respect to $\{\mathbf{e}_i'\}$are

$$a_i' = \mathbf{a}\cdot\mathbf{e}_i'$$

Now, $\mathbf{e}_i' = Q_{mi}\mathbf{e}_m$, [Eq. (2B11.1a)], therefore,

$$a_i' = \mathbf{a}\cdot Q_{mi}\mathbf{e}_m = Q_{mi}(\mathbf{a}\cdot\mathbf{e}_m)$$

i.e.,

$$a_i' = Q_{mi}a_m \qquad \text{(2B12.1a)}$$

In matrix notation, Eq. (2B12.1a) is

$$\begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{2B12.1b}$$

or

$$[\mathbf{a}]' = [\mathbf{Q}]^T[\mathbf{a}] \tag{2B12.1c}$$

Equation (2B12.1) is the transformation law relating components of the same vector with respect to different rectangular Cartesian unit bases. It is very important to note that in Eq. (2B12.1c), $[\mathbf{a}]'$ denote the matrix of the vector $\mathbf{a}$ with respect to the primed basis $\mathbf{e}_i'$ and $[\mathbf{a}]$ denote that with respect to the unprimed basis $\mathbf{e}_i$. Eq. (2B12.1) is not the same as $\mathbf{a}'=\mathbf{Q}^T\mathbf{a}$. The distinction is that $[\mathbf{a}]$ and $[\mathbf{a}]'$ are matrices of the same vector, where $\mathbf{a}$ and $\mathbf{a}'$ are two different vectors; $\mathbf{a}'$ being the transformed vector of $\mathbf{a}$ (through the transformation $\mathbf{Q}^T$).

If we premultiply Eq. (2B12.1c) with $[\mathbf{Q}]$, we get

$$[\mathbf{a}] = [\mathbf{Q}][\mathbf{a}]' \tag{2B12.2a}$$

The indicial notation equation for Eq.(2B12.2a) is

$$a_i = Q_{im}a_m' \tag{2B12.2b}$$

Example 2B12.1

Given that the components of a vector $\mathbf{a}$ with respect to $\{\mathbf{e}_i\}$ are given by (2,0,0), (i.e., $\mathbf{a}=2\mathbf{e}_1$), find its components with respect to $\{\mathbf{e}_i'\}$, where the $\mathbf{e}_i'$ axes are obtained by a 90° counter-clockwise rotation of the $\mathbf{e}_i$ axes about the $\mathbf{e}_3$-axis.

Solution. The answer to the question is obvious from Fig. 2B.5, that is

$$\mathbf{a} = 2\mathbf{e}_1 = -2\mathbf{e}_2'$$

We can also obtain the answer by using Eq. (2B12.2a). First we find the transformation matrix. With $\mathbf{e}_1' = \mathbf{e}_2$, $\mathbf{e}_2' = -\mathbf{e}_1$ and $\mathbf{e}_3' = \mathbf{e}_3$, by Eq. (2B11.1b), we have

$$[\mathbf{Q}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$[\mathbf{a}]' = [\mathbf{Q}]^T[\mathbf{a}] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

i.e.,

$$\mathbf{a} = -2\mathbf{e}_2'$$

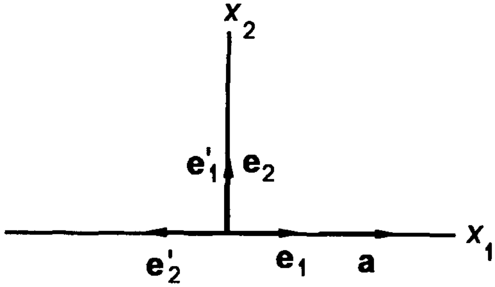

Fig. 2B.5

2B13 Transformation Law for Cartesian Components of a Tensor

Consider any tensor **T**, then the components of **T** with respect to the basis $\{\mathbf{e}_i\}$are:

$$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j$$

Its components with respect to $\{\mathbf{e}'_i\}$are:

$$T'_{ij} = \mathbf{e}'_i \cdot \mathbf{T}\mathbf{e}'_j$$

With $\mathbf{e}'_i = Q_{mi}\mathbf{e}_m$,

$$T'_{ij} = Q_{mi}\mathbf{e}_m \cdot \mathbf{T}Q_{nj}\mathbf{e}_n = Q_{mi}Q_{nj}(\mathbf{e}_m \cdot \mathbf{T}\mathbf{e}_n)$$

i.e.,

$$T'_{ij} = Q_{mi}Q_{nj}T_{mn} \tag{2B13.1a}$$

In matrix notation, Eq. (2B13.1a) reads

$$\begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \tag{2B13.1b}$$

or

$$[\mathbf{T}]' = [\mathbf{Q}]^T[\mathbf{T}][\mathbf{Q}] \tag{2B13.1c}$$

We can also express the unprimed components in terms of the primed components. Indeed, premultiply Eq. (2B13.1c) with [Q] and postmultiply it with $[\mathbf{Q}]^T$, we obtain, since $[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{Q}]^T[\mathbf{Q}] = [\mathbf{I}]$,

$$[\mathbf{T}] = [\mathbf{Q}][\mathbf{T}]'[\mathbf{Q}]^T \tag{2B13.2a}$$

Using indicial notation, Eq. (2B13.2a) reads

$$T_{ij} = Q_{im}Q_{jn}T'_{mn} \tag{2B13.2b}$$

Equations (2B13.1& 2B13.2) are the transformation laws relating the components of the same tensor with respect to different Cartesian unit bases. It is important to note that in these equations, [T] and [T]'are different matrices of the same tensor **T**. We note that the equation $[\mathbf{T}]' = [\mathbf{Q}]^T[\mathbf{T}][\mathbf{Q}]$ differs from the equation $\mathbf{T}' = \mathbf{Q}^T\mathbf{T}\mathbf{Q}$ in that the former relates the components of the same tensor **T** whereas the latter relates the two different tensors **T** and **T**′.

Example 2B13.1

Given the matrix of a tensor **T** in respect to the basis $\{\mathbf{e}_i\}$:

$$[\mathbf{T}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find $[\mathbf{T}]_{\mathbf{e}'_i}$, i.e., find the matrix of **T** with respect to the $\{\mathbf{e}'_i\}$ basis, where $\{\mathbf{e}'_i\}$ is obtained by rotating $\{\mathbf{e}_i\}$ about $\mathbf{e}_3$ through 90°. (see Fig. 2B.5).

Solution. Since $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = -\mathbf{e}_1$ and $\mathbf{e}'_3 = \mathbf{e}_3$, by Eq. (2B11.1b), we have

$$[\mathbf{Q}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, Eq. (2B13.1c) gives

$$[\mathbf{T}]' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e., $T'_{11} = 2,\ T'_{12} = -1,\ T'_{13} = 0, T'_{21} = -1$, etc.

Example 2B13.2

Given a tensor **T** and its components T_{ij} and T'_{ij} with respect to two sets of bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$. Show that T_{ii} is invariant with respect to this change of bases, i.e., $T_{ii} = T'_{ii}$.

Solution. The primed components are related to the unprimed components by Eq. (2B13.1a)

$$T'_{ij} = Q_{mi}Q_{nj}T_{mn}$$

Thus,

$$T'_{ii} = Q_{mi}Q_{ni}T_{mn}$$

But, $Q_{mi}Q_{ni} = \delta_{mn}$ (Eq. (2B10.2c)), therefore,

$$T'_{ii} = \delta_{mn}T_{mn} = T_{mm}$$

i.e.,

$$T'_{11}+T'_{22}+T'_{33} = T_{11}+T_{22}+T_{33}$$

We see from Example 2B13.1, that we can calculate all nine components of a tensor **T** with respect to $\mathbf{e}'_i$ from the matrix $[\mathbf{T}]_{\mathbf{e}_i}$, by using Eq. (2B13.1c). However, there are often times when we need only a few components. Then it is more convenient to use the Eq. (2B2.2) ($T'_{ij} = \mathbf{e}'_i \cdot \mathbf{T}\mathbf{e}'_j$) which defines each of the specific components.

In matrix form this equation is written as:

$$T'_{ij} = [\mathbf{e}'_i]^T[\mathbf{T}][\mathbf{e}'_j] \tag{2B13.4}$$

where $[\mathbf{e}']^T$ denotes a row matrix whose elements are the components of $\mathbf{e}'_i$ with respect to the basis $\{\mathbf{e}_i\}$.

Example 2B13.3

Obtain T'_{12} for the tensor **T** and the bases $\mathbf{e}_i$ and $\mathbf{e}'_i$ given in Example 2B13.1

Solution. Since $\mathbf{e}'_1 = \mathbf{e}_2$, and $\mathbf{e}'_2 = -\mathbf{e}_1$, thus

$$T'_{12} = \mathbf{e}'_1 \cdot \mathbf{T}\mathbf{e}'_2 = \mathbf{e}_2 \cdot \mathbf{T}(-\mathbf{e}_1) = -\mathbf{e}_2 \cdot \mathbf{T}\mathbf{e}_1 = -T_{21} = -1$$

Alternatively, using Eq. (2B13.4)

$$T'_{12} = [\mathbf{e}'_1]^T[\mathbf{T}][\mathbf{e}'_2] = [0, 1, 0]\begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = [0, 1, 0]\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = -1$$

2B14 Defining Tensors by Transformation Laws

Equations (2B12.1) or (2B13.1) state that when the components of a vector or a tensor with respect to $\{\mathbf{e}_i\}$ are known, then its components with respect to any $\{\mathbf{e}'_i\}$ are uniquely determined from them. In other words, the components a_i or T_{ij} with respect to one set of $\{\mathbf{e}_i\}$

completely characterizes a vector or a tensor. Thus, it is perfectly meaningful to use a statement such as "consider a tensor T_{ij}" meaning consider the tensor **T** whose components with respect to some set of $\{\mathbf{e}_i\}$ are T_{ij}. In fact, an alternative way of defining a tensor is through the use of transformation laws relating the components of a tensor with respect to different bases. Confining ourselves to only rectangular Cartesian coordinate systems and using unit vectors along positive coordinate directions as base vectors, we now define Cartesian components of tensors of different orders in terms of their transformation laws in the following where the primed quantities are referred to basis $\{\mathbf{e}_i'\}$ and unprimed quantities to basis $\{\mathbf{e}_i\}$, the $\mathbf{e}_i'$ and $\mathbf{e}_i$ are related by $\mathbf{e}_i'=\mathbf{Q}\mathbf{e}_i$, $\mathbf{Q}$ being an orthogonal transformation

$$\alpha' = \alpha \qquad \textbf{zeroth-order tensor(or scalar)}$$

$$a_i' = Q_{mi}a_m \qquad \textbf{first-order tensor (or vector)}$$

$$T_{ij}' = Q_{mi}Q_{nj}T_{mn} \qquad \textbf{second-order tensor(or tensor)}$$

$$T_{ijk}' = Q_{mi}Q_{nj}Q_{rk}T_{mnr} \qquad \textbf{third-order tensor}$$

etc.

Using the above transformation laws, one can easily establish the following three rules (a)the addition rule (b) the multiplication rule and (c) the quotient rule.

(a)The addition rule:

If T_{ij} and S_{ij} are components of any two tensors, then $T_{ij}+S_{ij}$ are components of a tensor. Similarly if T_{ijk} and S_{ijk} are components of any two third order tensors, then $T_{ijk}+S_{ijk}$ are components of a third order tensor.

To prove this rule, we note that since $T_{ijk}'=Q_{mi}Q_{nj}Q_{rk}T_{mnr}$ and $S_{ijk}'=Q_{mi}Q_{nj}Q_{rk}S_{mnr}$ we have,

$$T_{ijk}'+S_{ijk}' = Q_{mi}Q_{nj}Q_{rk}T_{mnr}+Q_{mi}Q_{nj}Q_{rk}S_{mnr} = Q_{mi}Q_{nj}Q_{rk}(T_{mnr}+S_{mnr})$$

Letting $W_{ijk}' = T_{ijk}'+S_{ijk}'$ and $W_{mnr}=T_{mnr}+S_{mnr}$, we have,

$$W_{ijk}' = Q_{mi}Q_{nj}Q_{rk}W_{mnr}$$

i.e, W_{ijk} are components of a third order tensor.

(b)The multiplication rule:

Let a_i be components of any vector and T_{ij} be components of any tensor. We can form many kinds of products from these components. Examples are (a)a_ia_i (b)$a_ia_ja_k$ (c) $T_{ij}T_{kl}$, etc. It can be proved that each of these products are components of a tensor, whose order is equal to the number of the free indices. For example, a_ia_i is a scalar (zeroth order tensor), $a_ia_ja_k$ are components of a third order tensor, $T_{ij}T_{kl}$ are components of a fourth order tensor.

To prove that $T_{ij}T_{kl}$ are components of a fourth-order tensor, let $M_{ijkl}=T_{ij}T_{kl}$, then

$$M'_{ijkl} = T'_{ij}T'_{kl} = Q_{mi}Q_{nj}T_{mn}Q_{rk}Q_{sl}T_{rs} = Q_{mi}Q_{nj}Q_{rk}Q_{sl}T_{mn}T_{rs}$$

i.e.,

$$M'_{ijkl} = Q_{mi}Q_{nj}Q_{rk}Q_{sl}M_{mnrs}$$

which is the transformation law for a fourth order tensor.

It is quite clear from the proof given above that the order of the tensor whose components are obtained from the multiplication of components of tensors is determined by the number of free indices; no free index corresponds to a scalar, one free index corresponds to a vector, two free indices correspond a second-order tensor, etc.

(c) The quotient rule:

If a_i are components of an arbitrary vector and T_{ij} are components of an arbitrary tensor and $a_i = T_{ij}b_j$ for all coordinates, then b_i are components of a vector. To prove this, we note that since a_i are components of a vector, and T_{ij} are components of a second-order tensor, therefore,

$$a_i = Q_{im}a'_m \tag{i}$$

and

$$T_{ij} = Q_{im}Q_{jn}T'_{mn} \tag{ii}$$

Now, substituting Eqs. (i) and (ii) into the equation $a_i = T_{ij}b_j$, we have

$$Q_{im}a'_m = Q_{im}Q_{jn}T'_{mn}b'_j \tag{iii}$$

But, the equation $a_i = T_{ij}b_j$ is true for all coordinates, thus, we also have

$$a'_m = T'_{mn}b'_n \tag{iv}$$

Thus, Eq. (iii) becomes

$$Q_{im}T'_{mn}b'_n = Q_{im}Q_{jn}T'_{mn}b'_j \tag{v}$$

Multiplying the above equation with Q_{ik} and noting that $Q_{ik}Q_{im} = \delta_{km}$, we get

$$T'_{kn}b'_n = Q_{jn}T'_{kn}b_j \tag{vi}$$

i.e.,

$$T'_{kn}(b'_n - Q_{jn}b_j) = 0 \tag{vii}$$

Since the above equation is to be true for any tensor **T**, therefore, the parenthesis must be identically zero. Thus,

$$b'_n = Q_{jn}b_j \tag{viii}$$

This is the transformation law for the components of a vector. Thus, b_i are components of a vector.

Another example which will be important later when we discuss the relationship between stress and strain for an elastic body is the following: If T_{ij} and E_{ij} are components of arbitrary second order tensors **T** and **E** then

$$T_{ij} = C_{ijkl}E_{kl}$$

for all coordinates, then C_{ijkl} are components of a fourth order tensor. The proof for this example follows that of the previous example.

2B15 Symmetric and Antisymmetric Tensors

A tensor is said to be symmetric if $\mathbf{T} = \mathbf{T}^T$. Thus, the components of a symmetric tensor have the property,

$$T_{ij} = T_{ij}^T = T_{ji} \tag{2B15.1}$$

i.e.,

$$T_{12} = T_{21}, \quad T_{13} = T_{31}, \quad T_{23} = T_{32}$$

A tensor is said to be antisymmetic if $\mathbf{T} = -\mathbf{T}^T$. Thus, the components of an antisymmetric tensor have the property

$$T_{ij} = -T_{ij}^T = -T_{ji} \tag{2B15.2}$$

i.e.,

$$T_{11} = T_{22} = T_{33} = 0$$

and

$$T_{12} = -T_{21}, \quad T_{13} = -T_{31}, \quad T_{23} = -T_{32}.$$

Any tensor **T** can always be decomposed into the sum of a symmetric tensor and an antisymmetric tensor. In fact,

$$\mathbf{T} = \mathbf{T}^S + \mathbf{T}^A \tag{2B15.3}$$

where

$$\mathbf{T}^S = \frac{\mathbf{T}+\mathbf{T}^T}{2} \quad \text{is symmetric}$$

and

$$\mathbf{T}^A = \frac{\mathbf{T}-\mathbf{T}^T}{2} \quad \text{is antisymmetric}$$

It is not difficult to prove that the decomposition is unique (see Prob. 2B27)

Example 2B15.1

Show that if $\mathbf{T}$ is symmetric and $\mathbf{W}$ is antisymmetric, then $\mathrm{tr}(\mathbf{TW})=0$.

Solution. We have, [see Example 2B8.4]

$$\mathrm{tr}(\mathbf{TW})=\mathrm{tr}(\mathbf{TW})^T=\mathrm{tr}(\mathbf{W}^T\mathbf{T}^T) \tag{i}$$

Since $\mathbf{T}$ is symmetric and W is antisymmetric, therefore, by definition, $\mathbf{T}=\mathbf{T}^T$, $\mathbf{W}=-\mathbf{W}^T$. Thus, (see Example 2B8.1)

$$\mathrm{tr}(\mathbf{TW})=-\mathrm{tr}(\mathbf{WT})=-\mathrm{tr}(\mathbf{TW}) \tag{ii}$$

Consequently, $2\mathrm{tr}(\mathbf{TW})=0$. That is,

$$\mathrm{tr}(\mathbf{TW})=0 \tag{iii}$$

2B16 The Dual Vector of an Antisymmetric Tensor

The diagonal elements of an antisymmetric tensor are always zero, and, of the six non-diagonal elements, only three are independent, because $T_{12}=-T_{12}$, $T_{13}=-T_{31}$ and $T_{23}=-T_{32}$. Thus, an antisymmetric tensor has really only three components, just like a vector. Indeed, it does behavior like a vector. More specifically, for every antisymmetric tensor T, there corresponds a vector $\mathbf{t}^A$, such that for every vector $\mathbf{a}$ the transformed vector, $\mathbf{Ta}$, can be obtained from the cross product of $\mathbf{t}^A$ with $\mathbf{a}$. That is,

$$\mathbf{Ta}=\mathbf{t}^A\times\mathbf{a} \tag{2B16.1}$$

This vector, $\mathbf{t}^A$, is called the **dual vector** (or **axial vector**) of the antisymmetric tensor. The form of the dual vector is given below:

From Eq.(2B16.1), we have, since $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=\mathbf{b}\cdot\mathbf{c}\times\mathbf{a}$,

$$T_{12}=\mathbf{e}_1\cdot\mathbf{Te}_2=\mathbf{e}_1\cdot\mathbf{t}^A\times\mathbf{e}_2=\mathbf{t}^A\cdot\mathbf{e}_2\times\mathbf{e}_1=-\mathbf{t}^A\cdot\mathbf{e}_3=-t_3^A$$
$$T_{31}=\mathbf{e}_3\cdot\mathbf{Te}_1=\mathbf{e}_3\cdot\mathbf{t}^A\times\mathbf{e}_1=\mathbf{t}^A\cdot\mathbf{e}_1\times\mathbf{e}_3=-\mathbf{t}^A\cdot\mathbf{e}_2=-t_2^A$$
$$T_{23}=\mathbf{e}_2\cdot\mathbf{Te}_3=\mathbf{e}_2\cdot\mathbf{t}^A\times\mathbf{e}_3=\mathbf{t}^A\cdot\mathbf{e}_3\times\mathbf{e}_2=-\mathbf{t}^A\cdot\mathbf{e}_1=-t_1^A$$

Similar derivations will give $T_{21}=t_3^A$, $T_{13}=t_2^A$,$T_{32}=t_1^A$ and $T_{11}=T_{22}=T_{33}=0$. Thus, *only* an antisymmetric tensor has a dual vector defined by Eq.(2B16.1). It is given by:

$$\mathbf{t}^A=-(T_{23}\mathbf{e}_1+T_{31}\mathbf{e}_2+T_{12}\mathbf{e}_3)=(T_{32}\mathbf{e}_1+T_{13}\mathbf{e}_2+T_{21}\mathbf{e}_3) \tag{2B16.2a}$$

or, in indicial notation

$$2\mathbf{t}^A=-\varepsilon_{ijk}T_{jk}\mathbf{e}_i \tag{2B16.2b}$$

Example 2B16.1

Given

$$[\mathbf{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(a)Decompose the tensor into a symmetric and an antisymmetric part.

(b)Find the dual vector for the antisymmetric part.

(c)Verify $\mathbf{T}^A\mathbf{a} = \mathbf{t}^A \times \mathbf{a}$ for $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_3$.

Solution. (a) $[\mathbf{T}] = [\mathbf{T}^S] + [\mathbf{T}^A]$, where

$$[\mathbf{T}^S] = \frac{[\mathbf{T}]+[\mathbf{T}]^T}{2} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$[\mathbf{T}^A] = \frac{[\mathbf{T}]-[\mathbf{T}]^T}{2} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

(b)The dual vector of $\mathbf{T}^A$ is

$$\mathbf{t}^A = -(T^A_{23}\mathbf{e}_1 + T^A_{31}\mathbf{e}_2 + T^A_{12}\mathbf{e}_3) = -(0\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3) = \mathbf{e}_2 + \mathbf{e}_3.$$

(c) Let $\mathbf{b} = \mathbf{T}^A\mathbf{a}$, then

$$[\mathbf{b}] = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

i.e.,

$$\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$$

On the other hand,

$$\mathbf{t}^A \times \mathbf{a} = (\mathbf{e}_2 + \mathbf{e}_3) \times (\mathbf{e}_1 + \mathbf{e}_3) = -\mathbf{e}_3 + \mathbf{e}_1 + \mathbf{e}_2 = \mathbf{b}$$

Example 2B16.2

Given that $\mathbf{R}$ is a rotation tensor and that $\mathbf{m}$ is a unit vector in the direction of the axis of rotation, prove that the dual vector $\mathbf{q}$ of $\mathbf{R}^A$ is parallel to $\mathbf{m}$.

Solution. Since $\mathbf{m}$ is parallel to the axis of rotation, therefore,

$$\mathbf{Rm} = \mathbf{m} \tag{i}$$

Thus, $(\mathbf{R}^T\mathbf{R})\mathbf{m} = \mathbf{R}^T\mathbf{m}$. Since $\mathbf{R}^T\mathbf{R} = \mathbf{I}$, we have

$$\mathbf{R}^T\mathbf{m} = \mathbf{m} \tag{ii}$$

Thus, (i) and (ii) gives

$$(\mathbf{R}-\mathbf{R}^T)\mathbf{m} = \mathbf{0} \tag{iii}$$

But $(\mathbf{R}-\mathbf{R}^T)\mathbf{m} = 2\mathbf{q}\times\mathbf{m}$, where $\mathbf{q}$ is the dual vector of $\mathbf{R}^A$. Thus,

$$\mathbf{q}\times\mathbf{m} = \mathbf{0} \tag{iv}$$

i.e., $\mathbf{q}$ is parallel to $\mathbf{m}$. We note that it can be shown (see Prob. 2B29 or Prob. 2B36) that if θ denotes the right-hand rotation angle, then

$$\mathbf{q} = (\sin\theta)\mathbf{m} \tag{2B16.3}$$

2B17 Eigenvalues and Eigenvectors of a Tensor

Consider a tensor $\mathbf{T}$. If $\mathbf{a}$ is a vector which transforms under $\mathbf{T}$ into a vector parallel to itself, i.e.,

$$\mathbf{Ta} = \lambda\mathbf{a} \tag{2B17.1}$$

then $\mathbf{a}$ is an **eigenvector** and λ is the corresponding **eigenvalue**.

If $\mathbf{a}$ is an eigenvector with corresponding eigenvalue λ of the linear transformation $\mathbf{T}$, then any vector parallel to $\mathbf{a}$ is also an eigenvector with the same eigenvalue λ. In fact, for any scalar α,

$$\mathbf{T}(\alpha\mathbf{a}) = \alpha\mathbf{Ta} = \alpha(\lambda\mathbf{a}) = \lambda(\alpha\mathbf{a}) \tag{i}$$

Thus, an eigenvector, as defined by Eq. (2B17.1), has an arbitrary length. For definiteness, we shall agree that all eigenvectors sought will be of unit length.

A tensor may have infinitely many eigenvectors. In fact, since $\mathbf{Ia}=\mathbf{a}$, any vector is an eigenvector for the identity tensor $\mathbf{I}$, with eigenvalues all equal to unity. For the tensor $\beta\mathbf{I}$, the same is true, except that the eigenvalues are all equal to β.

Some tensors have eigenvectors in only one direction. For example, for any rotation tensor, which effects a rigid body rotation about an axis through an angle not equal to integral multiples of π, only those vectors which are parallel to the axis of rotation will remain parallel to themselves.

Let $\mathbf{n}$ be a unit eigenvector, then

$$\mathbf{Tn} = \lambda\mathbf{n} = \lambda\mathbf{In} \tag{2B17.2}$$

Thus,

$$(\mathbf{T}-\lambda\mathbf{I})\mathbf{n} = \mathbf{0} \tag{2B17.3a}$$

Let $\mathbf{n} = \alpha_i \mathbf{e}_i$, then in component form

$$(T_{ij} - \lambda\delta_{ij})\alpha_j = 0 \tag{2B17.3b}$$

In long form, we have

$$\begin{aligned}
(T_{11}-\lambda)\alpha_1 + T_{12}\alpha_2 + T_{13}\alpha_3 &= 0 \\
T_{21}\alpha_1 + (T_{22}-\lambda)\alpha_2 + T_{23}\alpha_3 &= 0 \\
T_{31}\alpha_1 + T_{32}\alpha_2 + (T_{33}-\lambda)\alpha_3 &= 0
\end{aligned} \tag{2B17.3c}$$

Equations (2B17.3c) are a system of linear homogeneous equations in α_1, α_2, and α_3. Obviously, regardless of the values of λ, a solution for this system is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. This is know as the **trivial solution.** This solution simply states the obvious fact that $\mathbf{a} = \mathbf{0}$ satisfies the equation $\mathbf{Ta} = \lambda\mathbf{a}$, independent of the value of λ. To find the nontrivial eigenvectors for $\mathbf{T}$, we note that a homogeneous system of equations admits nontrivial solution only if the determinant of its coefficients vanishes. That is

$$|\mathbf{T} - \lambda\mathbf{I}| = 0 \tag{2B17.4a}$$

i.e.,

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = 0 \tag{2B17.4b}$$

For a given $\mathbf{T}$, the above equation is a cubic equation in λ. It is called the **characteristic equation** of $\mathbf{T}$. The roots of this **characteristic equation** are the **eigenvalues** of $\mathbf{T}$.

Equations (2B17.3), together with the equation

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \tag{2B17.5}$$

allow us to obtain eigenvectors of unit length. The following examples illustrate how eigenvectors and eigenvalues of a tensor can be obtained.

Example 2B17.1

If, with respect to some basis $\{\mathbf{e}_i\}$, the matrix of $\mathbf{T}$ is

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

find the eigenvalues and eigenvectors for this tensor.

Solution. We note that this tensor is $2\mathbf{I}$, so that $\mathbf{Ta} = 2\mathbf{Ia} = 2\mathbf{a}$, for any vector $\mathbf{a}$. Therefore, by the definition of eigenvector,(see Eq. (2B17.1)), any direction is a direction for an eigenvector. The eigenvalues for all the directions are the same, which is 2. However, we can also use Eq. (2B17.3) to find the eigenvalues and Eqs. (2B17.4) to find the eigenvectors. Indeed, Eq. (2B17.3) gives, for this tensor the following characteristic equation:

$$(2-\lambda)^3 = 0.$$

So we have a triple root $\lambda=2$. Substituting $\lambda=2$ in Eqs. (2B17.3c), we obtain

$$(2-2)\alpha_1 = 0$$

$$(2-2)\alpha_2 = 0$$

$$(2-2)\alpha_3 = 0$$

Thus, all three equations are automatically satisfied for arbitrary values of α_1, α_2, and α_3, so that vectors in all directions are eigenvectors. We can choose any three directions as the three independent eigenvectors. In particular, we can choose the basis $\{\mathbf{e}_i\}$ as a set of linearly independent eigenvectors.

Example 2B17.2

Show that if $T_{21}=T_{31}=0$, then $\pm\mathbf{e}_1$ is an eigenvector of $\mathbf{T}$ with eigenvalue T_{11}.

Solution. From $\mathbf{T}\mathbf{e}_1=T_{11}\mathbf{e}_1+T_{21}\mathbf{e}_2+T_{31}\mathbf{e}_3$, we have

$$\mathbf{T}\mathbf{e}_1 = T_{11}\mathbf{e}_1 \text{ and } \mathbf{T}(-\mathbf{e}_1) = T_{11}(-\mathbf{e}_1)$$

Thus, by definition, Eq. (2B17.1), $\pm\mathbf{e}_1$ are eigenvectors with T_{11} as its eigenvalue. Similarly, if $T_{12}=T_{32}=0$, then $\pm\mathbf{e}_2$ are eigenvectors with corresponding eigenvalue T_{22} and if $T_{13}=T_{23}=0$, then $\pm\mathbf{e}_3$ are eigenvectors with corresponding eigenvalue T_{33}.

Example 2B17.3

Given that

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Find the eigenvalues and their corresponding eigenvectors.

Solution. The characteristic equation is

$$(2-\lambda)^2(3-\lambda) = 0$$

Thus, $\lambda_1=3$, $\lambda_2=\lambda_3=2$. (note the ordering of the eigenvalues is arbitrary). These results are obvious in view of Example 2B17.2. In fact, that example also tells us that the eigenvector corresponding to $\lambda_1=3$ is $\mathbf{e}_3$ and eigenvectors corresponding to $\lambda_2=\lambda_3=2$ are $\mathbf{e}_1$ and $\mathbf{e}_2$. However, there are actually infinitely many eigenvectors corresponding to the double root. In fact, since $\mathbf{T}\mathbf{e}_1=2\mathbf{e}_1$ and $\mathbf{T}\mathbf{e}_2=2\mathbf{e}_2$, therefore,

$$\mathbf{T}(\alpha\mathbf{e}_1+\beta\mathbf{e}_2) = \alpha\mathbf{T}\mathbf{e}_1+\beta\mathbf{T}\mathbf{e}_2 = 2\alpha\mathbf{e}_1+2\beta\mathbf{e}_2=2(\alpha\mathbf{e}_1+\beta\mathbf{e}_2)$$

i.e., $\alpha\mathbf{e}_1+\beta\mathbf{e}_2$ is an eigenvector with eigenvalue 2. This fact can also be obtained from Eqs.(2B17.3c). With $\lambda=2$ these equations give

$$0\alpha_1=0$$

$$0\alpha_2=0$$

$$\alpha_3=0$$

Thus, α_1 and α_2 are arbitrary and $\alpha_3=0$ so that any vector perpendicular to $\mathbf{e}_3$, i.e., $\mathbf{n}=\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2$ is an eigenvector.

Example 2B17.4

Find the eigenvalues and eigenvectors for the tensor

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

Solution. The characteristic equation gives

$$[\mathbf{T}-\lambda\mathbf{I}] = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 4 & -3-\lambda \end{bmatrix} = (2-\lambda)(\lambda^2-25) = 0$$

Thus, there are three distinct eigenvalues, $\lambda_1=2$, $\lambda_2=5$ and $\lambda_3=-5$.

Corresponding to $\lambda_1=2$, Eqs. (2B17.3c) give

$$0\alpha_1 = 0$$

$$\alpha_2+4\alpha_3 = 0$$

$$4\alpha_2-5\alpha_3 = 0$$

and Eq. (2B17.5) gives

$$\alpha_1^2+\alpha_2^2+\alpha_3^2=1$$

Thus, $\alpha_2=\alpha_3=0$ and $\alpha_1=\pm1$, so that the eigenvector corresponding to $\lambda_1=2$ is $\mathbf{n}_1=\pm\mathbf{e}_1$. We note that from the Example 2B17.2, this eigenvalue 2 and the corresponding eigenvector $\pm\mathbf{e}_1$ can be written down by inspection without computation.

Corresponding to $\lambda_2=5$, we have

$$3\alpha_1 = 0$$

$$-2\alpha_2+4\alpha_3=0$$

$$4\alpha_2-8\alpha_3=0$$

Thus (note the second and third equations are the same),

$$\alpha_1=0,\quad \alpha_2=\pm 2/\sqrt{5},\quad \alpha_3=\pm 1/\sqrt{5}$$

and the eigenvector corresponding to $\lambda_2=5$ is

$$\mathbf{n}_2=\pm\frac{1}{\sqrt{5}}(2\mathbf{e}_2+\mathbf{e}_3)$$

Corresponding to $\lambda_3=-5$, similar computations give

$$\mathbf{n}_3=\pm\frac{1}{\sqrt{5}}(-\mathbf{e}_2+2\mathbf{e}_3)$$

All the examples given above have three eigenvalues that are real. It can be shown that if a tensor is real (i.e., with real components) and symmetric, then all its eigenvalues are real. If a tensor is real but not symmetric, then two of the eigenvalues may be complex conjugates. The following example illustrates this possibility.

Example 2B17.5

Find the eigenvalues and eigenvectors for the rotation tensor $\mathbf{R}$ corresponding to a 90° rotation about the $\mathbf{e}_3$-axis (see Example 2B5.1(a)).

Solution. The characteristic equation is

$$\begin{vmatrix} 0-\lambda & -1 & 0 \\ 1 & 0-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}=0$$

i.e.,

$$\lambda^2(1-\lambda)+(1-\lambda)=(1-\lambda)(\lambda^2+1)=0$$

Thus, only one eigenvalue is real, namely $\lambda_1=1$, the other two are imaginary, $\lambda_{2,3}=\pm\sqrt{-1}$. Correspondingly, there is only one real eigenvector. Only real eigenvectors are of interest to us, we shall therefore compute only the eigenvector corresponding to $\lambda_1=1$.

From

$$(0-1)\alpha_1-\alpha_2=0$$

$$\alpha_1-\alpha_2=0$$

$$(1-1)\alpha_3=0$$

and

$$\alpha_1^2+\alpha_2^2+\alpha_3^2=1$$

We obtain $\alpha_1=0$, $\alpha_2=0$, $\alpha_3=\pm 1$, i.e., $\mathbf{n}=\pm\mathbf{e}_3$, which, of course, is parallel to the axis of rotation.

2B18 Principal Values and Principal Directions of Real Symmetric tensors

In the following chapters, we shall encounter several tensors (stress tensor, strain tensor, rate of deformation tensor, etc.) which are symmetric, for which the following theorem, stated without proof, is important: "the eigenvalues of any real symmetric tensor are all real." Thus, for a real symmetric tensor, there always exist at least three real eigenvectors which we shall also call the **principal directions**. The corresponding eigenvalues are called the **principal values**. We now prove that there always exist three principal directions which are mutually perpendicular.

Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be two eigenvectors corresponding to the eigenvalues λ_1 and λ_2 respectively of a tensor $\mathbf{T}$. Then

$$\mathbf{T}\mathbf{n}_1 = \lambda_1\mathbf{n}_1 \tag{i}$$

and

$$\mathbf{T}\mathbf{n}_2 = \lambda_2\mathbf{n}_2 \tag{ii}$$

Thus,

$$\mathbf{n}_2\cdot\mathbf{T}\mathbf{n}_1 = \lambda_1\mathbf{n}_1\cdot\mathbf{n}_2 \tag{iii}$$

$$\mathbf{n}_1\cdot\mathbf{T}\mathbf{n}_2 = \lambda_2\mathbf{n}_2\cdot\mathbf{n}_1 \tag{iv}$$

The definition of the transpose of $\mathbf{T}$ gives $\mathbf{n}_1\cdot\mathbf{T}\mathbf{n}_2 = \mathbf{n}_2\cdot\mathbf{T}^T\mathbf{n}_1$, thus for a symmetric tensor $\mathbf{T}$, $\mathbf{T}=\mathbf{T}^\mathbf{T}$, so that $\mathbf{n}_1\cdot\mathbf{T}\mathbf{n}_2 = \mathbf{n}_2\cdot\mathbf{T}\mathbf{n}_1$. Thus, from Eqs. (iii) and (iv), we have

$$(\lambda_1-\lambda_2)(\mathbf{n}_1\cdot\mathbf{n}_2) = 0 \tag{v}$$

It follows that if λ_1 is not equal to λ_2, then $\mathbf{n}_1\cdot\mathbf{n}_2 = 0$, i.e., $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular to each other. We have thus proven that if the eigenvalues are all distinct, then the *three principal directions* are *mutually perpendicular*.

Next, let us suppose that $\mathbf{n}_1$ and $\mathbf{n}_2$ are two eigenvectors corresponding to the same eigenvalue λ. Then, by definition, $\mathbf{T}\mathbf{n}_1 = \lambda\mathbf{n}_1$ and $\mathbf{T}\mathbf{n}_2 = \lambda\mathbf{n}_2$ so that for any α, and β, $\mathbf{T}(\alpha\mathbf{n}_1+\beta\mathbf{n}_2)=\alpha\mathbf{T}\mathbf{n}_1+\beta\mathbf{T}\mathbf{n}_2=\lambda(\alpha\mathbf{n}_1+\beta\mathbf{n}_2)$. That is $\alpha\mathbf{n}_1+\beta\mathbf{n}_2$ is also an eigenvector with the same eigenvalue λ. In other words, if there are two distinct eigenvectors with the same eigenvalue, then, there are infinitely many eigenvectors (which forms a plane) with the same eigenvalue. This situation arises when the characteristic equation has a repeated root. Suppose the characteristic equation has roots λ_1 and $\lambda_2=\lambda_3=\lambda$ (λ_1 distinct from λ). Let $\mathbf{n}_1$ be the eigenvector corresponding to λ_1, then $\mathbf{n}_1$ is perpendicular to any eigenvector of λ. Now, corresponding to λ, the equations

$$(T_{11}-\lambda)\alpha_1+T_{12}\alpha_2+T_{13}\alpha_3 = 0 \tag{2B18.1a}$$

$$T_{21}\alpha_1+(T_{22}-\lambda)\alpha_2+T_{23}\alpha_3 = 0 \tag{2B18.1b}$$

$$T_{31}\alpha_1+T_{32}\alpha_2+(T_{33}-\lambda)\alpha_3 = 0 \tag{2B18.1c}$$

degenerate to one independent equation (see Example 2B17.3) so that there are infinitely many eigenvectors lying on the plane whose normal is $\mathbf{n}_1$. Therefore, though not unique, *there again exist three mutually perpendicular principal directions.*

In the case of a triple root, the above three equations will be automatically satisfied for whatever values of $(\alpha_1,\alpha_2,\alpha_3)$ so that any vector is an eigenvector (see Example 2B17.1).

Thus, for every real symmetric tensor, there always exists at least one triad of principal directions which are mutually perpendicular.

2B19 Matrix of a Tensor with Respect to Principal Directions

We have shown that for a real symmetric tensor, there always exist three principal directions which are mutually perpendicular. Let $\mathbf{n}_1,\mathbf{n}_2$ and $\mathbf{n}_3$ be unit vectors in these directions. Then using $\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3$ as base vectors, the components of the tensor are

$$\begin{aligned}
T_{11} &= \mathbf{n}_1\cdot\mathbf{T}\mathbf{n}_1 = \mathbf{n}_1\cdot\lambda_1\mathbf{n}_1 = \lambda_1\\
T_{22} &= \mathbf{n}_2\cdot\mathbf{T}\mathbf{n}_2 = \mathbf{n}_2\cdot\lambda_2\mathbf{n}_2 = \lambda_2\\
T_{33} &= \mathbf{n}_3\cdot\mathbf{T}\mathbf{n}_3 = \mathbf{n}_3\cdot\lambda_3\mathbf{n}_3 = \lambda_3\\
T_{12} &= \mathbf{n}_1\cdot\mathbf{T}\mathbf{n}_2 = \mathbf{n}_1\cdot\lambda_2\mathbf{n}_2 = \lambda_2(\mathbf{n}_1\cdot\mathbf{n}_2) = 0 = T_{21}\\
T_{13} &= \mathbf{n}_1\cdot\mathbf{T}\mathbf{n}_3 = \mathbf{n}_1\cdot\lambda_3\mathbf{n}_3 = \lambda_3(\mathbf{n}_1\cdot\mathbf{n}_3) = 0 = T_{31}\\
T_{23} &= \mathbf{n}_2\cdot\mathbf{T}\mathbf{n}_3 = \mathbf{n}_2\cdot\lambda_3\mathbf{n}_3 = \lambda_3(\mathbf{n}_2\cdot\mathbf{n}_3) = 0 = T_{32}
\end{aligned}$$

That is

$$[\mathbf{T}]_{\mathbf{n}_i} = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix} \tag{2B19.1}$$

Thus, the matrix is diagonal and the diagonal elements are the eigenvalues of **T**.

We now show that the principal values of a tensor **T** include the maximum and minimum values that the diagonal elements of any matrix of **T** can have.

First, for any unit vector $\mathbf{e}_1' = \alpha\mathbf{n}_1+\beta\mathbf{n}_2+\gamma\mathbf{n}_3$,

$$T_{11}' = \mathbf{e}_1'\cdot T\mathbf{e}_1{}' = [\alpha\ \beta\ \gamma]\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$$

i.e.,

$$T_{11}' = \lambda_1\alpha^2+\lambda_2\beta^2+\lambda_3\gamma^2$$

Without loss of generality, let

$$\lambda_1 \geq \lambda_2 \geq \lambda_3$$

then noting that $\alpha^2+\beta^2+\gamma^2 = 1$, we have

$$\lambda_1 = \lambda_1(\alpha^2+\beta^2+\gamma^2) \geq \lambda_1\alpha^2+\lambda_2\beta^2+\lambda_3\gamma^2$$

i.e.,

$$\lambda_1 \geq T'_{11}$$

Also,

$$\lambda_1\alpha^2+\lambda_2\beta^2+\lambda_3\gamma^2 \geq \lambda_3(\alpha^2+\beta^2+\gamma^2) = \lambda_3$$

i.e.,

$$T'_{11} \geq \lambda_3$$

Thus, the $\begin{Bmatrix}\text{maximum}\\ \text{minimum}\end{Bmatrix}$ value of the principal values of **T** is the $\begin{Bmatrix}\text{maximum}\\ \text{minimum}\end{Bmatrix}$ value of the diagonal elements of all [T] of **T**.

2B20 Principal Scalar Invariants of a Tensor

The characteristic equation of a tensor **T**, $|T_{ij}-\lambda\delta_{ij}|=0$ is a cubic equation in λ. It can be written as

$$\lambda^3-I_1\lambda^2+I_2\lambda-I_3 = 0 \tag{2B20.1}$$

where

$$I_1 = T_{11}+T_{22}+T_{33} = T_{ii} = \operatorname{tr}\mathbf{T}$$

$$I_2 = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} = \frac{1}{2}(T_{ii}T_{jj}-T_{ij}T_{ji}) = \frac{1}{2}[(\operatorname{tr}\mathbf{T})^2-\operatorname{tr}(\mathbf{T}^2)]$$

$$I_3 = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = \det[\mathbf{T}]$$

Since by definition, the eigenvalues of **T** do *not* depend on the choices of the base vectors, therefore the coefficients of Eq. (2B20.1) will not depend on any particular choice of basis. They are called the **principal scalar invariants** of **T**.

We note that, in terms of the eigenvalues of **T** which are the roots of Eq.(2B20.1), the I_i's take the simpler form

$$I_1 = \lambda_1+\lambda_2+\lambda_3$$

$$I_2 = \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1$$

$$I_3 = \lambda_1 \lambda_2 \lambda_3 \tag{2B20.2}$$

Example 2B20.1

For the tensor of Example 2B17.4, first find the principal scalar invariants and then evaluate the eigenvalues using Eq. (2B20.1).

Solution. The matrix of **T** is

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

$$I_1 = 2+3-3 = 2$$

$$I_2 = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} = -25$$

$$I_3 = |\mathbf{T}| = 2\begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} = -50$$

These values give the characteristic equation

$$\lambda_3 - 2\lambda_2 - 25\lambda + 50 = 0$$

or,

$$(\lambda-2)(\lambda-5)(\lambda+5) = 0$$

Thus, the eigenvalues are $\lambda = 2, 5, -5$ as previously determined.

Part C Tensor Calculus

2C1 Tensor-valued functions of a Scalar

Let $\mathbf{T}=\mathbf{T}(t)$ be a tensor-valued function of a scalar t (such as time). The derivative of $\mathbf{T}$ with respect to t is defined to be a second-order tensor given by

$$\frac{d\mathbf{T}}{dt}=\lim_{\Delta t\to 0}\frac{\mathbf{T}(t+\Delta t)-\mathbf{T}(t)}{\Delta t} \tag{2C1.1}$$

The following identities can be easily established [only Eq. (2C1.2d) will be proven here]:

$$\frac{d}{dt}(\mathbf{T}+\mathbf{S})=\frac{d\mathbf{T}}{dt}+\frac{d\mathbf{S}}{dt} \tag{2C1.2a}$$

$$\frac{d}{dt}(\alpha(t)\mathbf{T})=\frac{d\alpha}{dt}\mathbf{T}+\alpha\frac{d\mathbf{T}}{dt} \tag{2C1.2b}$$

$$\frac{d}{dt}(\mathbf{TS})=\frac{d\mathbf{T}}{dt}\mathbf{S}+\mathbf{T}\frac{d\mathbf{S}}{dt} \tag{2C1.2c}$$

$$\frac{d}{dt}(\mathbf{Ta})=\frac{d\mathbf{T}}{dt}\mathbf{a}+\mathbf{T}\frac{d\mathbf{a}}{dt} \tag{2C1.2d}$$

$$\frac{d}{dt}(\mathbf{T}^T)=\left(\frac{d\mathbf{T}}{dt}\right)^T \tag{2C1.2e}$$

To prove Eq. (2C1.2d), we use the definition (2C1.1)

$$\begin{aligned}\frac{d}{dt}(\mathbf{Ta})&=\lim_{\Delta t\to 0}\frac{\mathbf{T}(t+\Delta t)\mathbf{a}(t+\Delta t)-\mathbf{T}(t)\mathbf{a}(t)}{\Delta t}\\&=\lim_{\Delta t\to 0}\frac{\mathbf{T}(t+\Delta t)\mathbf{a}(t+\Delta t)-\mathbf{T}(t)\mathbf{a}(t)+\mathbf{T}(t)\mathbf{a}(t+\Delta t)-\mathbf{T}(t)\mathbf{a}(t+\Delta t)}{\Delta t}\\&=\lim_{\Delta t\to 0}\frac{[\mathbf{T}(t+\Delta t)-\mathbf{T}(t)]\mathbf{a}(t+\Delta t)}{\Delta t}+\lim_{\Delta t\to 0}\frac{\mathbf{T}(t)[\mathbf{a}(t+\Delta t)-\mathbf{a}(t)]}{\Delta t}\end{aligned}$$

Thus,

$$\frac{d}{dt}(\mathbf{Ta})=\frac{d\mathbf{T}}{dt}\mathbf{a}+\mathbf{T}\frac{d\mathbf{a}}{dt}$$

Example 2C1.1

Show that in Cartesian coordinates the components of $d\mathbf{T}/dt$, i.e., $(d\mathbf{T}/dt)_{ij}$ are given by the derivatives of the components, dT_{ij}/dt.

Solution.

$$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j \tag{i}$$

Since the base vectors are fixed,

$$\frac{d\mathbf{e}_1}{dt} = \frac{d\mathbf{e}_2}{dt} = \frac{d\mathbf{e}_2}{dt} = 0 \tag{ii}$$

Therefore,

$$\frac{dT_{ij}}{dt} = \mathbf{e}_i \cdot \frac{d}{dt}(\mathbf{T}\mathbf{e}_j) = \mathbf{e}_i \cdot \frac{d\mathbf{T}}{dt}\mathbf{e}_j = \left(\frac{d\mathbf{T}}{dt}\right)_{ij} \tag{iii}$$

Example 2C1.2

Show that for an orthogonal tensor $\mathbf{Q}(t)$, $(d\mathbf{Q}/dt)\mathbf{Q}^T$ is an antisymmetric tensor.

Solution. Since $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$, we have

$$\mathbf{Q}\frac{d\mathbf{Q}^T}{dt} + \frac{d\mathbf{Q}}{dt}\mathbf{Q}^T = 0 \tag{i}$$

That is

$$\mathbf{Q}\frac{d\mathbf{Q}^T}{dt} = -\frac{d\mathbf{Q}}{dt}\mathbf{Q}^T \tag{ii}$$

Since

$$\frac{d\mathbf{Q}^T}{dt} = \left(\frac{d\mathbf{Q}}{dt}\right)^T \qquad \text{[see Eq. (2C1.2e)]}$$

Therefore,

$$\mathbf{Q}\left(\frac{d\mathbf{Q}}{dt}\right)^T = -\frac{d\mathbf{Q}}{dt}\mathbf{Q}^T \tag{iii}$$

But

$$\mathbf{Q}\left(\frac{d\mathbf{Q}}{dt}\right)^T = \left(\frac{d\mathbf{Q}}{dt}\mathbf{Q}^T\right)^T \qquad \text{[see Eq. (2B6.4)]}$$

therefore,

$$\left(\frac{d\mathbf{Q}}{dt}\mathbf{Q}^T\right)^T = -\frac{d\mathbf{Q}}{dt}\mathbf{Q}^T \tag{iv}$$

Example 2C1.3

A time-dependent rigid body rotation about a fixed point can be represented by a rotation tensor $\mathbf{R}(t)$, so that a position vector $\mathbf{r}_o$ is transformed through rotation into $\mathbf{r}(t)=\mathbf{R}(t)\mathbf{r}_o$. Derive the equation

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega}\times\mathbf{r} \tag{i}$$

where $\boldsymbol{\omega}$ is the dual vector of the antisymmetric tensor $\frac{d\mathbf{R}}{dt}\mathbf{R}^T$.

Solution. From $\mathbf{r}(t)=\mathbf{R}(t)\mathbf{r}_o$

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{R}}{dt}\mathbf{r}_o = \frac{d\mathbf{R}}{dt}\mathbf{R}^T\mathbf{r} \tag{ii}$$

But, $\frac{d\mathbf{R}}{dt}\mathbf{R}^T$ is an antisymmetric tensor (see Example 2C1.2) so that

$$\frac{d\mathbf{r}}{dt} = \left(\frac{d\mathbf{R}}{dt}\mathbf{R}^T\right)\mathbf{r} = \boldsymbol{\omega}\times\mathbf{r} \tag{iii}$$

where $\boldsymbol{\omega}$ is the dual vector of $\frac{d\mathbf{R}}{dt}\mathbf{R}^T$.

From the well-known equation in rigid body kinematics, we can identify $\boldsymbol{\omega}$ as the angular velocity of the body.

2C2 Scalar Field, Gradient of a Scalar Function

Let $\phi(\mathbf{r})$ be a scalar-valued function of the position vector $\mathbf{r}$. That is, for each position $\mathbf{r}$, $\phi(\mathbf{r})$ gives the value of a scalar, such as density, temperature or electric potential at the point. In other words, $\phi(\mathbf{r})$ describes a scalar field. Associated with a scalar field, there is a vector field, called the **gradient** of ϕ, which is of considerable importance. The gradient of ϕ at a point $\mathbf{r}$ is defined to be a vector, denoted by (grad ϕ), or by $\nabla\phi$ such that its dot product with $d\mathbf{r}$ gives the difference of the values of the scalar at $\mathbf{r}+d\mathbf{r}$ and $\mathbf{r}$, i.e.,

$$d\phi = \phi(\mathbf{r}+d\mathbf{r})-\phi(\mathbf{r}) \equiv \nabla\phi\cdot d\mathbf{r} \tag{2C2.1}$$

If dr denotes the magnitude of $d\mathbf{r}$, and $\mathbf{e}$ the unit vector in the direction of $d\mathbf{r}$ (note: $\mathbf{e}=d\mathbf{r}/dr$), then the above equation gives, for $d\mathbf{r}$ in the $\mathbf{e}$ direction,

$$\frac{d\phi}{dr} = \nabla\phi \cdot \mathbf{e} \tag{2C2.2}$$

That is, the component of $\nabla\phi$ in the direction of $\mathbf{e}$ gives the rate of change of ϕ in that direction (the directional derivative). In particular, the components of $\nabla\phi$ in the $\mathbf{e}_1$ direction is given by

$$\left(\frac{d\phi}{dr}\right)_{\text{in the } \mathbf{e}_1 \text{ direction}} \equiv \frac{\partial\phi}{\partial x_1} = \nabla\phi \cdot \mathbf{e}_1 = (\nabla\phi)_1$$

Similarly,

$$\left(\frac{d\phi}{dr}\right)_{\text{in the } \mathbf{e}_2 \text{ direction}} \equiv \frac{\partial\phi}{\partial x_2} = \nabla\phi \cdot \mathbf{e}_2 = (\nabla\phi)_2$$

$$\left(\frac{d\phi}{dr}\right)_{\text{in the } \mathbf{e}_3 \text{ direction}} \equiv \frac{\partial\phi}{\partial x_3} = \nabla\phi \cdot \mathbf{e}_3 = (\nabla\phi)_3$$

Therefore, the Cartesian components of $\nabla\phi$ are $\dfrac{\partial\phi}{\partial x_i}$, that is,

$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3 \tag{2C2.3}$$

The gradient vector has a simple geometrical interpretation. For example, if $\phi(\mathbf{r})$ describes a temperature field, then, on a surface of constant temperature (i.e., isothermal surface), ϕ = a constant. Let $\mathbf{r}$ be a point on this surface. Then, for any and all neighboring point $\mathbf{r}+d\mathbf{r}$ on the same isothermal surface, $d\phi=0$. Thus, $\nabla\phi \cdot d\mathbf{r}=0$. In other words, $\nabla\phi$ is a vector, perpendicular to the surface at the point $\mathbf{r}$. On the other hand, the dot product $\nabla\phi \cdot d\mathbf{r}$ is a maximum when $d\mathbf{r}$ is in the same direction as $\nabla\phi$. In other words, $\nabla\phi$ is greatest if $d\mathbf{r}$ is normal to the surface of constant ϕ and in this case,

$$\frac{d\phi}{dr} = |\nabla\phi|, \quad \text{for } d\mathbf{r} \text{ in the normal direction.}$$

Example 2C2.1

If $\phi=x_1x_2+x_3$, find a unit vector $\mathbf{n}$ normal to the surface of a constant ϕ passing through (2,1,0).

Solution. We have

$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3 = x_2\mathbf{e}_1 + x_1\mathbf{e}_2 + \mathbf{e}_3$$

At the point (2,1,0), $\nabla\phi=\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3$. Thus,

$$\mathbf{n} = \frac{1}{\sqrt{6}}(\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3)$$

Example 2C2.2

If $\mathbf{q}$ denotes the heat flux vector (rate of heat flow/area), the Fourier heat conduction law states that

$$\mathbf{q} = -k\nabla\theta$$

where θ is the temperature field and k is the thermal conductivity. If $\theta=2(x_1^2+x_2^2)$, find θ at $A(1,0)$ and $B(1/\sqrt{2},\ 1/\sqrt{2})$. Sketch curves of constant θ (isotherms) and indicate the vectors $\mathbf{q}$ at the two points.

Solution. Since,

$$\nabla\theta = \frac{\partial\theta}{\partial x_1}\mathbf{e}_1+\frac{\partial\theta}{\partial x_2}\mathbf{e}_2+\frac{\partial\theta}{\partial x_3}\mathbf{e}_3 = 4x_1\mathbf{e}_1+4x_2\mathbf{e}_2$$

therefore,

$$\mathbf{q} = -4k(x_1\mathbf{e}_1+x_2\mathbf{e}_2)$$

At point A,

$$\mathbf{q}_A = -4k\mathbf{e}_1$$

and at point B,

$$\mathbf{q}_B = -2\sqrt{2}\,k(\mathbf{e}_1+\mathbf{e}_2)$$

Clearly, the isotherm, Fig.2C.1, are circles and the heat flux is an inward radial vector.

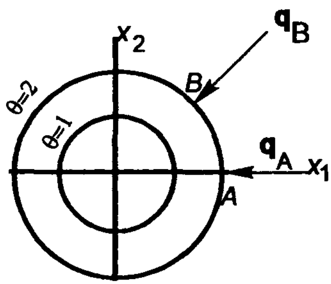

Fig. 2C.1

Example 2C2.3

A more general heat conduction law can be given in the following form:

$$\mathbf{q} = -\mathbf{K}\nabla\theta$$

where **K** is a tensor known as thermal conductivity tensor.

(a)What tensor **K** corresponds to the Fourier heat conduction law mentioned in the previous example?

(b)If it is known that **K** is symmetric, show that there are at least three directions in which heat flow is normal to the surface of constant temperature.

(c)If $\theta = 2x_1+3x_2$ and

$$[\mathbf{K}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

find **q**.

Solution.

(a)Clearly, $\mathbf{K}=k\mathbf{I}$, so that $\mathbf{q}=-k\mathbf{I}\nabla\theta=-k\nabla\theta$

(b)For symmetric **K**, we know from Section 2B.18 that there exist at least three principal directions $\mathbf{n}_1$,$\mathbf{n}_2$ and $\mathbf{n}_3$ such that

$$\mathbf{K}\mathbf{n}_1 = k_1\mathbf{n}_1$$

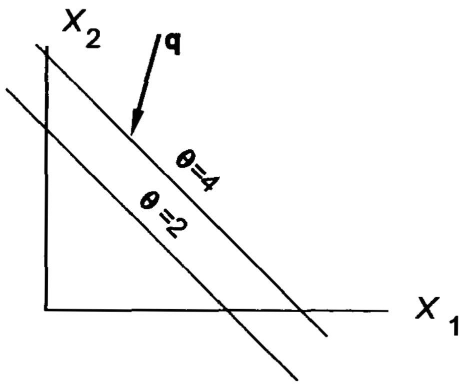

Fig. 2C.2

$$\mathbf{K}\mathbf{n}_2 = k_2\mathbf{n}_2$$

$$\mathbf{K}\mathbf{n}_3 = k_3\mathbf{n}_3$$

where k_1,k_2 and k_3 are eigenvalues of **K**. Thus, for $\nabla\theta$ in the direction of $\mathbf{n}_1$,

$$\mathbf{q}_1 = -\mathbf{K}\nabla\theta = -\mathbf{K}|\nabla\theta|\mathbf{n}_1 = -|\nabla\theta|\mathbf{K}\mathbf{n}_1 = -k_1|\nabla\theta|\mathbf{n}_1$$

But $\mathbf{n}_1$, being in the same direction as $\nabla\theta$, is perpendicular to the surface of constant θ. Thus, $\mathbf{q}_1$ is normal to the surface of constant temperature. Similarly, $\mathbf{q}_2$ is normal to the surface of constant temperature., etc. We note that if k_1,k_2 and k_3 are all distinct, the equations indicate that different thermal conductivities in the three principal directions.

(c)Since $\theta = 2x_1+3x_2$, we have

$$[\mathbf{q}] = -\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 0 \end{bmatrix}$$

i.e.,

$$\mathbf{q} = -\mathbf{e}_1 - 4\mathbf{e}_2$$

which is clearly in a different direction from the normal.

2C3 Vector Field, Gradient of a Vector Field

Let $\mathbf{v}(\mathbf{r})$ be a vector-valued function of position, describing, for example, a displacement or a velocity field. Associated with $\mathbf{v}(\mathbf{r})$, there is a tensor field, called the gradient of $\mathbf{v}$, which is of considerable importance. The **gradient** of $\mathbf{v}$ (denoted by $\nabla\mathbf{v}$ or grad $\mathbf{v}$) is defined to be the second-order tensor which, when operating on $d\mathbf{r}$ gives the difference of $\mathbf{v}$ at $\mathbf{r}+d\mathbf{r}$ and $\mathbf{r}$. That is,

$$d\mathbf{v} = \mathbf{v}(\mathbf{r}+d\mathbf{r}) - \mathbf{v}(\mathbf{r}) \equiv (\nabla\mathbf{v})d\mathbf{r} \tag{2C3.1}$$

Again, let dr denote $|d\mathbf{r}|$ and $\mathbf{e}$ denote $d\mathbf{r}/dr$, we have

$$\left(\frac{d\mathbf{v}}{dr}\right)_{\text{in } \mathbf{e} \text{ direction}} = (\nabla\mathbf{v})\mathbf{e} \tag{2C3.2}$$

Thus, the second-order tensor $(\nabla\mathbf{v})$ transforms the unit vector $\mathbf{e}$ into the vector describing the rate of change $\mathbf{v}$ in that direction.

Since

$$\left(\frac{d\mathbf{v}}{dr}\right)_{\text{in } \mathbf{e}_1 \text{ direction}} \equiv \frac{\partial\mathbf{v}}{\partial x_1} = (\nabla\mathbf{v})\mathbf{e}_1$$

thus, in Cartesian coordinates,

$$(\nabla\mathbf{v})_{11} = \mathbf{e}_1\cdot(\nabla\mathbf{v})\mathbf{e}_1 = \mathbf{e}_1\cdot\frac{\partial\mathbf{v}}{\partial x_1} = \frac{\partial}{\partial x_1}(\mathbf{e}_1\cdot\mathbf{v})$$

That is,

$$(\nabla\mathbf{v})_{11} = \frac{\partial v_1}{\partial x_1}$$

Or, in general

$$\left(\frac{d\mathbf{v}}{dr}\right)_{\text{in } \mathbf{e}_j \text{ direction}} \equiv \frac{\partial \mathbf{v}}{\partial x_j} = (\nabla \mathbf{v})\mathbf{e}_j \tag{2C3.3}$$

thus,

$$(\nabla \mathbf{v})_{ij} = \mathbf{e}_i \cdot (\nabla \mathbf{v})\mathbf{e}_j = \mathbf{e}_i \cdot \frac{\partial \mathbf{v}}{\partial x_j} = \frac{\partial}{\partial x_j}(\mathbf{e}_i \cdot \mathbf{v}) \tag{2C3.4}$$

so that the Cartesian components of $(\nabla \mathbf{v})$ are

$$(\nabla \mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j} \tag{2C3.5a}$$

That is,

$$[\nabla \mathbf{v}] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} \tag{2C3.5b}$$

A geometrical interpretation of $\nabla \mathbf{v}$ will be given later in connection with the kinematics of deformation.

2C4 Divergence of a Vector Field and Divergence of a Tensor Field.

Let $\mathbf{v}(\mathbf{r})$ be a vector field. The **divergence** of $\mathbf{v}(\mathbf{r})$ is defined to be a scalar field given by the trace of the gradient of $\mathbf{v}$. That is,

$$\text{div}\mathbf{v} \equiv \text{tr}(\nabla \mathbf{v}) \tag{2C4.1}$$

With reference to rectangular Cartesian basis, the diagonal elements of $\nabla \mathbf{v}$ are $\frac{\partial v_1}{\partial x_1}, \frac{\partial v_2}{\partial x_2}$ and $\frac{\partial v_3}{\partial x_3}$. Thus

$$\text{div}\mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial v_m}{\partial x_m} \tag{2C4.2}$$

Let $\mathbf{T}(\mathbf{r})$ be a second order tensor field. The **divergence** of $\mathbf{T}$ is defined to be a vector field, denoted by div $\mathbf{T}$, such that for any vector $\mathbf{a}$

$$(\text{div}\mathbf{T}) \cdot \mathbf{a} \equiv \text{div}(\mathbf{T}^T\mathbf{a}) - \text{tr}(\mathbf{T}^T(\nabla \mathbf{a})) \tag{2C4.3}$$

To find the Cartesian components of the vector div **T**, let **b** = div **T**, then (note $\nabla \mathbf{e}_i = 0$ for Cartesian coordinates), from Eq. (2C4.3),

$$b_i = \mathbf{b}\cdot\mathbf{e}_i = \operatorname{div}(\mathbf{T}^T\mathbf{e}_i) - \operatorname{tr}(\mathbf{T}^T\nabla\mathbf{e}_i) = \operatorname{div}(T_{im}\mathbf{e}_m) - 0 = \frac{\partial T_{im}}{\partial x_m} \tag{2C4.4}$$

In other words,

$$\operatorname{div}\mathbf{T} = \frac{\partial T_{im}}{\partial x_m}\mathbf{e}_i \tag{2C4.5}$$

Example 2C4.1

If $\alpha = \alpha(\mathbf{r})$ and $\mathbf{a} = \mathbf{a}(\mathbf{r})$, show that $\operatorname{div}(\alpha\mathbf{a}) = \alpha\operatorname{div}\mathbf{a} + (\nabla\alpha)\cdot\mathbf{a}$.

Solution. Let $\mathbf{b} = \alpha\mathbf{a}$. Then $b_i = \alpha a_i$ and

$$\begin{aligned}\operatorname{div}\mathbf{b} &= \frac{\partial b_i}{\partial x_i} = \alpha\frac{\partial a_i}{\partial x_i} + \frac{\partial \alpha}{\partial x_i}a_i \\ &= \alpha\operatorname{div}\mathbf{a} + (\nabla\alpha)\cdot\mathbf{a}\end{aligned}$$

Example 2C4.2

Given $\alpha(\mathbf{r})$ and $\mathbf{T}(\mathbf{r})$, show that

$$\operatorname{div}(\alpha\mathbf{T}) = \mathbf{T}(\nabla\alpha) + \alpha\operatorname{div}\mathbf{T}$$

Solution. We have, from Eq. (2C4.5),

$$\operatorname{div}(\alpha\mathbf{T}) = \frac{\partial}{\partial x_j}(\alpha T_{ij})\mathbf{e}_i = \frac{\partial \alpha}{\partial x_j}T_{ij}\mathbf{e}_i + \alpha\frac{\partial T_{ij}}{\partial x_j}\mathbf{e}_i$$

But

$$\frac{\partial \alpha}{\partial x_j}T_{ij}\mathbf{e}_i = \mathbf{T}(\nabla\alpha)$$

and

$$\alpha\frac{\partial T_{ij}}{\partial x_j}\mathbf{e}_i = \alpha\operatorname{div}\mathbf{T}$$

Thus, the desired result follows.

2C5 Curl of a Vector Field

Let $\mathbf{v}(\mathbf{r})$ be a vector field. The **curl** of **v** is defined to be the vector field given by twice the dual vector of the antisymmetric part of $(\nabla\mathbf{v})$. That is

$$\operatorname{curl}\mathbf{v} \equiv 2\mathbf{t}^A \tag{2C5.1}$$

where $\mathbf{t}^A$ is the dual vector of $(\nabla\mathbf{v})^A$.

In a rectangular Cartesian basis,

$$[\nabla\mathbf{v}]^A = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2}-\frac{\partial v_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3}-\frac{\partial v_3}{\partial x_1}\right) \\ -\frac{1}{2}\left(\frac{\partial v_1}{\partial x_2}-\frac{\partial v_2}{\partial x_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3}-\frac{\partial v_3}{\partial x_2}\right) \\ -\frac{1}{2}\left(\frac{\partial v_1}{\partial x_3}-\frac{\partial v_3}{\partial x_1}\right) & -\frac{1}{2}\left(\frac{\partial v_2}{\partial x_3}-\frac{\partial v_3}{\partial x_2}\right) & 0 \end{bmatrix}$$

Thus, the curl of $\mathbf{v}$ is given by [see Eq. (2B16.2)]

$$\operatorname{curl}\mathbf{v} = 2\mathbf{t}^A = \left(\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}\right)\mathbf{e}_1+\left(\frac{\partial v_1}{\partial x_3}-\frac{\partial v_3}{\partial x_1}\right)\mathbf{e}_2+\left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_3 \tag{2C5.2}$$

Part D Curvilinear Coordinates

2D1 Polar Coordinates

In this section, the invariant definitions of ∇f, $\nabla \mathbf{v}$, div$\mathbf{v}$ and div$\mathbf{T}$ will be utilized in order to determine their components in plane polar coordinates.

Let r,θ denote, see Fig. 2D.1, plane polar coordinates such that

$$r = (x_1^2+x_2^2)^{1/2}$$

$$\theta = \tan^{-1}\frac{x_2}{x_1}$$

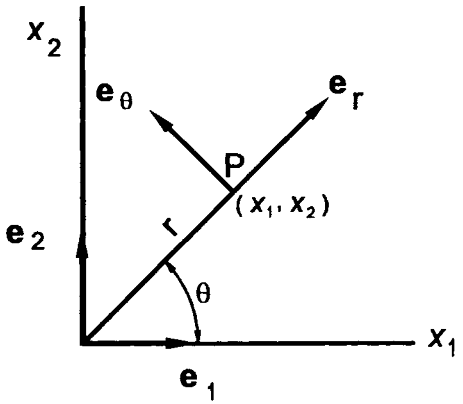

Fig. 2D.1

The unit base vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ can be expressed in terms of the Cartesian base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ as:

$$\mathbf{e}_r = \cos\theta\mathbf{e}_1+\sin\theta\mathbf{e}_2 \tag{2D1.1a}$$

$$\mathbf{e}_\theta = -\sin\theta\mathbf{e}_1+\cos\theta\mathbf{e}_2 \tag{2D1.1b}$$

These unit base vectors vary in direction as θ changes. In fact, from Eqs. (2D1.1a) and (2D1.1b), it is easily derived that

$$d\mathbf{e}_r = d\theta\mathbf{e}_\theta \tag{2D1.2a}$$

$$d\mathbf{e}_\theta = -d\theta\mathbf{e}_r \tag{2D1.2b}$$

The geometrical representation of $d\mathbf{e}_r$ and $d\mathbf{e}_\theta$ are shown in the following figure where one notes that $\mathbf{e}_r(P)$ has rotated an infinitesimal angle $d\theta$ to become $\mathbf{e}_r(Q)=\mathbf{e}_r(P)+d\mathbf{e}_r$ where $d\mathbf{e}_r$ is perpendicular to $\mathbf{e}_r(P)$ with a magnitude $|d\mathbf{e}_r|=(1)(d\theta)$. Similarly $d\mathbf{e}_\theta$ is perpendicular to $\mathbf{e}_\theta(P)$ but is pointing in the negative $\mathbf{e}_r$ direction and its magnitude is also $(1)d\theta$.

From the position vector $\mathbf{r}=r\mathbf{e}_r$, we have

$$d\mathbf{r} = dr\mathbf{e}_r+rd\mathbf{e}_r$$

Using Eq. (2D1.2a), we get

$$d\mathbf{r} = dr\mathbf{e}_r+rd\theta\mathbf{e}_\theta \tag{2D1.3}$$

The geometrical representation of this equation is also easily seen if one notes that $d\mathbf{r}$ is the vector PQ in Fig. 2D.2. The components of ∇f, $\nabla\mathbf{v}$ etc. in polar coordinates will now be obtained.

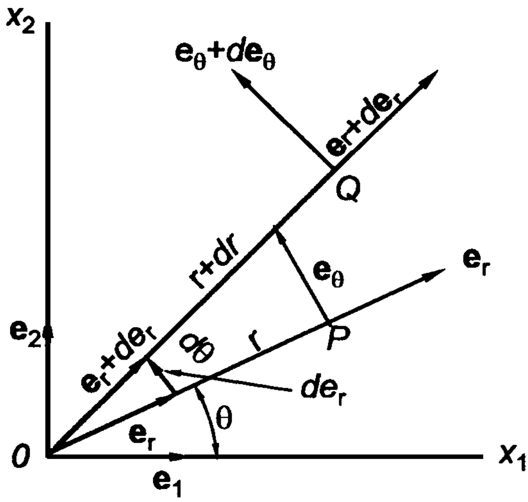

Fig. 2D.2

(i) *Components of* ∇f

Let $f(r,\theta)$ be a scalar field. By definition of the gradient of f, we have

$$df = \nabla f\cdot d\mathbf{r} = [a_r\mathbf{e}_r+a_\theta\mathbf{e}_\theta]].[dr\mathbf{e}_r+rd\theta\mathbf{e}_\theta]$$

where a_r and a_θ are components of ∇f in the $\mathbf{e}_r$ and $\mathbf{e}_\theta$ direction respectively.

Thus,

$$df = a_r dr + a_\theta r d\theta \tag{2D1.4}$$

But from Calculus,

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta \tag{2D1.5}$$

Since Eqs. (2D1.4) and (2D1.5) must yield the same result for all increments dr, $d\theta$, we have

$$a_r = \frac{\partial f}{\partial r} \text{ and } ra_\theta = \frac{\partial f}{\partial \theta}$$

Thus,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta \tag{2D1.6}$$

(ii) *Components of* $\nabla \mathbf{v}$

Let

$$\mathbf{v}(r,\theta) = v_r(r,\theta)\mathbf{e}_r + v_\theta(r,\theta)\mathbf{e}_\theta \tag{2D1.7}$$

By definition of $\nabla\mathbf{v}$, we have

$$d\mathbf{v} \equiv (\nabla\mathbf{v})d\mathbf{r}$$

Let $\mathbf{T} \equiv \nabla\mathbf{v}$, then

$$d\mathbf{v} = \mathbf{T}d\mathbf{r} = \mathbf{T}(dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta) = dr\mathbf{T}\mathbf{e}_r + rd\theta\mathbf{T}\mathbf{e}_\theta$$

Now,

$$\mathbf{T}\mathbf{e}_r = T_{rr}\mathbf{e}_r + T_{\theta r}\mathbf{e}_\theta \text{ and } \mathbf{T}\mathbf{e}_\theta = T_{r\theta}\mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta$$

Therefore,

$$d\mathbf{v} = (T_{rr}dr + T_{r\theta}rd\theta)\mathbf{e}_r + (T_{\theta r}dr + T_{\theta\theta}rd\theta)\mathbf{e}_\theta \tag{2D1.8}$$

But from Eq. (2D1.7)

$$d\mathbf{v} = dv_r\mathbf{e}_r + v_r d\mathbf{e}_r + dv_\theta\mathbf{e}_\theta + v_\theta d\mathbf{e}_\theta$$

and from calculus, we have,

$$dv_r = \frac{\partial v_r}{\partial r}dr + \frac{\partial v_r}{\partial \theta}d\theta \text{ and } dv_\theta = \frac{\partial v_\theta}{\partial r}dr + \frac{\partial v_\theta}{\partial \theta}d\theta$$

From the above three equations and Eqs. (2D1.2), we have

$$d\mathbf{v} = \left[\frac{\partial v_r}{\partial r}dr + \left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)d\theta\right]\mathbf{e}_r + \left[\frac{\partial v_\theta}{\partial r}dr + \left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)d\theta\right]\mathbf{e}_\theta \tag{2D1.9}$$

In order that Eqs. (2D1.8) and (2D1.9) agree for all increments $dr, d\theta$, we have

$$T_{rr}=\frac{\partial v_r}{\partial r},\quad T_{r\theta}=\frac{1}{r}\left(\frac{\partial v_r}{\partial \theta}-v_\theta\right),\quad T_{\theta r}=\frac{\partial v_\theta}{\partial r},\quad T_{\theta\theta}=\frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta}+v_r\right)$$

In matrix form,

$$[\nabla \mathbf{v}]=\begin{bmatrix}\dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta}-v_\theta\right)\\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta}+v_r\right)\end{bmatrix} \tag{2D1.10}$$

(iii) divv

Using the components of $\nabla\mathbf{v}$ obtained in (ii), we have

$$\mathrm{div}\mathbf{v}=\mathrm{tr}(\nabla\mathbf{v})=T_{rr}+T_{\theta\theta}=\frac{\partial v_r}{\partial r}+\frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta}+v_r\right) \tag{2D1.11}$$

(iv) curl v

From the definition that curlv≡ twice the dual vector of $(\nabla\mathbf{v})^A$, we have

$$\mathrm{curl}\mathbf{v}=\left(\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}-\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_3 \tag{2D1.12}$$

(v) *Components of* div **T**

The definition of the divergence of a second-order tensor is

$$(\mathrm{div}\mathbf{T})\cdot\mathbf{a}=\mathrm{div}(\mathbf{T}^T\mathbf{a})-\mathrm{tr}((\nabla\mathbf{a})\mathbf{T}^T)$$

for an arbitrary vector **a**.

Take $\mathbf{a}=\mathbf{e}_r$, then, the above equation gives

$$(\mathrm{div}\mathbf{T})_r=\mathrm{div}(\mathbf{T}^T\mathbf{e}_r)-\mathrm{tr}((\nabla\mathbf{e}_r)\mathbf{T}^T) \tag{2D1.13}$$

To evaluate the first term on the right hand side, we note that

$$\mathbf{T}^T\mathbf{e}_r=T_{rr}\mathbf{e}_r+T_{r\theta}\mathbf{e}_\theta$$

so that according to Eq. (2D1.11), with $v_r=T_{rr}$ and $v_\theta=T_{r\theta}$

$$\mathrm{div}(\mathbf{T}^T\mathbf{e}_r)=\mathrm{div}(T_{rr}\mathbf{e}_r+T_{r\theta}\mathbf{e}_\theta)=\frac{\partial T_{rr}}{\partial r}+\frac{1}{r}\left(\frac{\partial T_{r\theta}}{\partial \theta}+T_{rr}\right)$$

To evaluate the second term, we first use Eq. (2D1.10) to obtain $\nabla\mathbf{e}_r$. In fact, since $\mathbf{e}_r=(1)\mathbf{e}_r+0\mathbf{e}_\theta$, we have, with $v_r=1$ and $v_\theta=0$ in Eq. (2D1.10),

$$[\nabla\mathbf{e}_r]=\begin{bmatrix}0 & 0\\ 0 & \dfrac{1}{r}\end{bmatrix}\quad\text{and}\quad[\nabla\mathbf{e}_r][\mathbf{T}^T]=\begin{bmatrix}0 & 0\\ \dfrac{T_{r\theta}}{r} & \dfrac{T_{\theta\theta}}{r}\end{bmatrix}$$

so that $\text{tr}(\nabla \mathbf{e}_r \mathbf{T}^T) = \dfrac{T_{\theta\theta}}{r}$. Thus, from Eq. (2D1.13), we have

$$(\text{div}\mathbf{T})_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r} \tag{2D1.14}$$

In a similar manner, (see Prob. 2D1), one can derive

$$(\text{div}\mathbf{T})_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{T_{\theta\theta}}{\partial \theta} + \frac{T_{r\theta} + T_{\theta r}}{r} \tag{2D1.15}$$

2D2 Cylindrical Coordinates

In cylindrical coordinates, see Fig. 2D.3, the position of a point P is determined by (r,θ,z) where r and θ determine the position of the vertical projection of the point P on the xy plane (the point P' in the figure) and the coordinate z determines the height of the point P from the xy plane. In other words, the cylindrical coordinates is comprised of polar coordinates (r,θ) in the xy plane plus a coordinate z perpendicular to the xy plane.

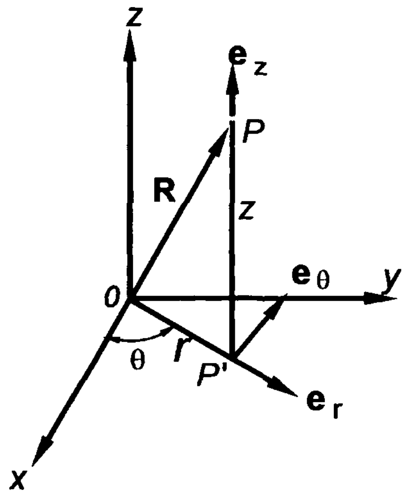

Fig.2D.3

We shall denote the position vector of P by $\mathbf{R}$, rather than $\mathbf{r}$, to avoid the possible confusion between the position vector $\mathbf{R}$ and the coordinate r (which is a radial distance in the xy plane). The unit vector $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are on the xy plane and it is clear from the above figure that

$$\mathbf{R} = r\mathbf{e}_r + z\mathbf{e}_z \tag{2D2.1}$$

and

$$d\mathbf{R} = dr\mathbf{e}_r + rd\mathbf{e}_r + dz\mathbf{e}_z + zd\mathbf{e}_z$$

In the above equation, $d\mathbf{e}_r$ is given by exactly the same equation given earlier for the polar coordinates, i.e., Eq. (2D1.2a). We note also that $\mathbf{e}_z$ never change its direction or magnitude regardless where the point P is, thus $d\mathbf{e}_z=0$. Thus,

$$d\mathbf{R} = dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + dz\mathbf{e}_z \tag{2D2.2}$$

By retracing all the step used in the section on polar coordinates, we can easily obtain the following results:

(i)*Components of* ∇f

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z \tag{2D2.3}$$

(ii) *Components of* $\nabla\mathbf{v}$

$$[\nabla\mathbf{v}] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - v_\theta\right) & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix} \tag{2D2.4}$$

(iii) div **v**

$$\mathrm{div}\mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{\partial v_z}{\partial z} \tag{2D2.5}$$

(iv) curl **v**

$$\mathrm{curl}\mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta + \left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_z \tag{2D2.6}$$

(v) *Components of* div **T**

$$(\mathrm{div}\mathbf{T})_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{\partial T_{rz}}{\partial z} \tag{2D2.7a}$$

$$(\mathrm{div}\mathbf{T})_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{T_{r\theta} + T_{\theta r}}{r} + \frac{\partial T_{\theta z}}{\partial z} \tag{2D2.7b}$$

$$(\mathrm{div}\mathbf{T})_z = \frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r} \tag{2D2.7c}$$

We note that in dyadic notation, $\text{div}\mathbf{T}^T$ is written as $\nabla\cdot\mathbf{T}$, so that $(\text{div}\,\mathbf{T})_{r\theta} = (\nabla\cdot\mathbf{T})_{\theta r}$, etc.

2D3 Spherical Coordinates

In Fig. 2D.4a, we show the spherical coordinates (r,θ,ϕ) of a general point P. In this figure, $\mathbf{e}_r,\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are unit vectors in the direction of increasing r,θ,ϕ respectively.

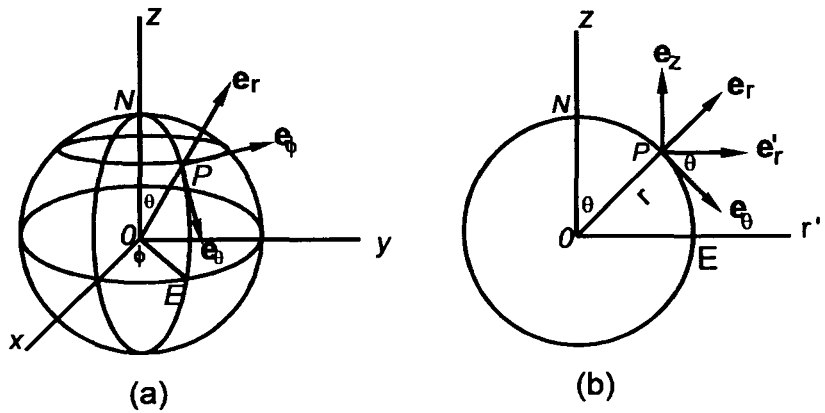

Fig. 2D.4

The position vector for the point P can be written as

$$\mathbf{r} = r\mathbf{e}_r \tag{2D3.1}$$

where r is the magnitude of the vector $\mathbf{r}$. Thus,

$$d\mathbf{r} = dr\mathbf{e}_r + rd\mathbf{e}_r \tag{2D3.2}$$

To evaluate $d\mathbf{e}_r$, we note from Fig. 2D.4b that

$$\mathbf{e}_r = \cos\theta\mathbf{e}_z + \sin\theta\mathbf{e}_r{}' \;, \qquad \mathbf{e}_\theta = \cos\theta\mathbf{e}_r{}' - \sin\theta\mathbf{e}_z \tag{2D3.3}$$

where $\mathbf{e}_r{}'$ is the unit vector in the r' (OE) direction (r' is in the xy plane). Thus,

$$d\mathbf{e}_r = -\sin\theta d\theta\mathbf{e}_z + \cos\theta d\theta\mathbf{e}_r{}' + \sin\theta d\mathbf{e}_r{}' = d\theta(-\sin\theta\mathbf{e}_z + \cos\theta\mathbf{e}_r{}') + \sin\theta d\mathbf{e}_r{}'$$
$$= d\theta\mathbf{e}_\theta + \sin\theta d\mathbf{e}_r{}' \tag{i}$$

But, just like in polar coordinates, due to $d\phi$, $d\mathbf{e}_r{}' = (1)d\phi\mathbf{e}_\phi$, therefore,

$$d\mathbf{e}_r = (d\theta)\mathbf{e}_\theta + (\sin\theta d\phi)\mathbf{e}_\phi \tag{2D3.4a}$$

Again, from Fig. 2D.4b, we have

$$\mathbf{e}_r' = \cos\theta\mathbf{e}_\theta + \sin\theta\mathbf{e}_r \tag{ii}$$

therefore,

$$d\mathbf{e}_\theta = \cos\theta d\mathbf{e}_r' - \sin\theta d\theta\mathbf{e}_r' - \cos\theta d\theta\mathbf{e}_z = -d\theta(\sin\theta\mathbf{e}_r' + \cos\theta\mathbf{e}_z) + \cos\theta d\mathbf{e}_r' \tag{iii}$$

that is,

$$d\mathbf{e}_\theta = -(d\theta)\mathbf{e}_r + (cos\theta d\phi)\mathbf{e}_\phi \tag{2D3.4b}$$

From Fig. 2D.4a, it is clear that $d\mathbf{e}_\phi = d\phi(-\mathbf{e}_r')$, therefore,

$$d\mathbf{e}_\phi = -(\sin\theta d\phi)\mathbf{e}_r - (\cos\theta d\phi)\mathbf{e}_\theta \tag{2D3.4c}$$

Substituting Eq.(2D3.4a) into Eq.(2D3.2), we have

$$d\mathbf{r} = dr\mathbf{e}_r + r(d\theta)\mathbf{e}_\theta + r(\sin\theta d\phi)\mathbf{e}_\phi \tag{2D3.5}$$

We are now in a position to obtain the components of ∇f, $\nabla\mathbf{v}$, div $\mathbf{v}$, curl $\mathbf{v}$ and div $\mathbf{T}$ in spherical coordinates.

(i)*Components of* ∇f

Let (r,θ,ϕ) be a scalar field. By the definition of the gradient of f, we have,

$$df = (\nabla f)\cdot d\mathbf{r} = [(\nabla f)_r\mathbf{e}_r + (\nabla f)_\theta\mathbf{e}_\theta + (\nabla f)_\phi\mathbf{e}_\phi]\cdot[(dr)\mathbf{e}_r + (rd\theta)\mathbf{e}_\theta + (r\sin\theta d\phi)\mathbf{e}_\phi] \tag{2D3.6}$$

i.e.,

$$df = (\nabla f)_r dr + (\nabla f)_\theta r d\theta + (\nabla f)_\phi r\sin\theta d\phi \tag{2D3.7}$$

From calculus, the total derivative of f is

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\theta}d\theta + \frac{\partial f}{\partial\phi}d\phi \tag{2D3.8}$$

Comparing Eq. (2D3.7) with Eq. (2D3.8), we obtain

$$(\nabla f)_r = \frac{\partial f}{\partial r} \qquad (\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial\theta} \qquad (\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi} \tag{2D3.9}$$

(ii) *Components of* $\nabla\mathbf{v}$

Let the vector field $\mathbf{v}$ be represented as:

$$\mathbf{v}(r,\theta,\phi) = v_r(r,\theta,\phi)\mathbf{e}_r + v_\theta(r,\theta,\phi)\mathbf{e}_\theta + v_\phi(r,\theta,\phi)\mathbf{e}_\phi \tag{2D3.10}$$

Letting $\mathbf{T} \equiv \nabla\mathbf{v}$, we have

$$d\mathbf{v} = \mathbf{T}d\mathbf{r} = \mathbf{T}(dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + r\sin\theta d\phi\mathbf{e}_\phi) = dr\mathbf{T}\mathbf{e}_r + rd\theta\mathbf{T}\mathbf{e}_\theta + r\sin\theta d\phi\mathbf{T}\mathbf{e}_\phi \tag{2D3.11}$$

Now by definition of the components of tensor $\mathbf{T}$ in spherical coordinates

$$\begin{aligned}\mathbf{T}\mathbf{e}_r &= T_{rr}\mathbf{e}_r+T_{\theta r}\mathbf{e}_\theta+T_{\phi r}\mathbf{e}_\phi\\ \mathbf{T}\mathbf{e}_\theta &= T_{r\theta}\mathbf{e}_r+T_{\theta\theta}\mathbf{e}_\theta+T_{\phi\theta}\mathbf{e}_\phi\\ \mathbf{T}\mathbf{e}_\phi &= T_{r\phi}\mathbf{e}_r+T_{\theta\phi}\mathbf{e}_\theta+T_{\phi\phi}\mathbf{e}_\phi\end{aligned}\tag{2D3.12}$$

Substituting this equation into Eq. (2D3.11) and rearranging terms we have

$$\begin{aligned}d\mathbf{v} = {}&(T_{rr}dr+rT_{r\theta}d\theta+r\sin\theta T_{r\phi}d\phi)\mathbf{e}_r\\ &+(T_{\theta r}dr+rT_{\theta\theta}d\theta+r\sin\theta T_{\theta\phi}d\phi)\mathbf{e}_\theta\\ &+(T_{\phi r}dr+rT_{\phi\theta}d\theta+r\sin\theta T_{\phi\phi}d\phi)\mathbf{e}_\phi\end{aligned}\tag{2D3.13}$$

But from Eq. (2D3.10) we have,

$$d\mathbf{v} = dv_r\mathbf{e}_r+v_rd\mathbf{e}_r+dv_\theta\mathbf{e}_\theta+v_\theta d\mathbf{e}_\theta+dv_\phi\mathbf{e}_\phi+v_\phi d\mathbf{e}_\phi\tag{2D3.14}$$

and from calculus we have

$$\begin{aligned}dv_r &= \frac{\partial v_r}{\partial r}dr+\frac{\partial v_r}{\partial\theta}d\theta+\frac{\partial v_r}{\partial\phi}d\phi\\ dv_\theta &= \frac{\partial v_\theta}{\partial r}dr+\frac{\partial v_\theta}{\partial\theta}d\theta+\frac{\partial v_\theta}{\partial\phi}d\phi\\ dv_\phi &= \frac{\partial v_\phi}{\partial r}dr+\frac{\partial v_\phi}{\partial\theta}d\theta+\frac{\partial v_\phi}{\partial\phi}d\phi\end{aligned}\tag{2D3.15}$$

Thus, using Eqs. (2D3.15) and Eqs. (2D3.4) , Eq. (2D3.14) becomes

$$\begin{aligned}d\mathbf{v} = {}&\left[\frac{\partial v_r}{\partial r}dr+\left(\frac{\partial v_r}{\partial\theta}-v_\theta\right)d\theta+\left(\frac{\partial v_r}{\partial\phi}-v_\phi\sin\theta\right)d\phi\right]\mathbf{e}_r+\\ &\left[\frac{\partial v_\theta}{\partial r}dr+\left(\frac{\partial v_\theta}{\partial\theta}+v_r\right)d\theta+\left(\frac{\partial v_\theta}{\partial\phi}-v_\phi\cos\theta\right)d\phi\right]\mathbf{e}_\theta+\\ &\left[\frac{\partial v_\phi}{\partial r}dr+\frac{\partial v_\phi}{\partial\theta}d\theta+\left(\frac{\partial v_\phi}{\partial\phi}+v_r\sin\theta+v_\theta\cos\theta\right)d\phi\right]\mathbf{e}_\phi\end{aligned}\tag{2D3.16}$$

In order that Eqs. (2D3.13) and (2D3.16) agree for all increments dr, $d\theta$, $d\phi$, we have

$$T_{rr} = \frac{\partial v}{\partial r},\quad T_{r\theta} = \frac{1}{r}\left(\frac{\partial v_r}{\partial\theta}-v_\theta\right),\quad .....$$

which we display in matrix form as

$$\nabla \mathbf{v} = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - v_\theta\right) & \dfrac{1}{r\sin\theta}\left(\dfrac{\partial v_r}{\partial \phi} - v_\phi \sin\theta\right) \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) & \dfrac{1}{r\sin\theta}\left(\dfrac{\partial v_\theta}{\partial \phi} - v_\phi \cos\theta\right) \\ \dfrac{\partial v_\phi}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_\phi}{\partial \theta} & \dfrac{1}{r\sin\theta}\dfrac{\partial v_\phi}{\partial \phi} + \dfrac{v_r}{r} + \dfrac{v_\theta \cot\theta}{r} \end{bmatrix} \tag{2D3.17}$$

(iii)div v

Using the components of $\nabla\mathbf{v}$ obtained in (ii), we have

$$\mathrm{div}\mathbf{v} = \mathrm{tr}(\nabla\mathbf{v}) = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + 2\frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}$$

$$= \frac{1}{r^2}\frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{2D3.18}$$

(iv)curl v

From the definition of the curl and Eq. (2D3.17) we have

$$\mathrm{curl}\mathbf{v} = \left[\frac{1}{r\sin\theta}\frac{\partial (v_\phi \sin\theta)}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]\mathbf{e}_r + \left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{1}{r}\frac{\partial (r v_\phi)}{\partial r}\right]\mathbf{e}_\theta + \left[\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]\mathbf{e}_\phi \tag{2D3.19}$$

(v)*Components of* div **T**

Using the definition of the divergence of a tensor, Eq. (2C4.3), with the vector **a** equal to the unit base vector $\mathbf{e}_r$ gives

$$(\mathrm{div}\mathbf{T})_r = \mathrm{div}(\mathbf{T}^T\mathbf{e}_r) - \mathrm{tr}((\nabla\mathbf{e}_r)\mathbf{T}^T) \tag{2D3.20}$$

To evaluate the first term on the right-hand side, we note that

$$\mathbf{T}^T\mathbf{e}_r = T_{rr}\mathbf{e}_r + T_{r\theta}\mathbf{e}_\theta + T_{r\phi}\mathbf{e}_\phi$$

so that according to Eq. (2D3.18), with $v_r = T_{rr}$, $v_\theta = T_{r\theta}$, $T_\phi = T_{r\phi}$

$$\mathrm{div}(\mathbf{T}^T\mathbf{e}_r) = \frac{1}{r^2}\frac{\partial (r^2 T_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (T_{r\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial (T_{r\phi})}{\partial \phi} \tag{2D3.21}$$

To evaluate the second term on the right-hand side of Eq. (2D3.20) we first use Eq. (2D3.17) with $\mathbf{v} = \mathbf{e}_r$ to obtain

$$[\nabla \mathbf{e}_r] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{r} \end{bmatrix} \quad \text{and} \quad [\nabla \mathbf{e}_r \, \mathbf{T}^T] = \begin{bmatrix} 0 & 0 & 0 \\ \frac{T_{r\theta}}{r} & \frac{T_{\theta\theta}}{r} & \frac{T_{\phi\theta}}{r} \\ \frac{T_{r\phi}}{r} & \frac{T_{\theta\phi}}{r} & \frac{T_{\phi\phi}}{r} \end{bmatrix} \tag{2D3.22}$$

so that

$$\mathrm{tr}(\nabla \mathbf{e}_r \mathbf{T}^T) = \frac{T_{\theta\theta} + T_{\phi\phi}}{r} \tag{2D3.23}$$

From Eq. (2D3.20), we obtain

$$(\mathrm{div}\mathbf{T})_r = \frac{1}{r^2}\frac{\partial(r^2 T_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{r\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi} - \frac{T_{\theta\theta}+T_{\phi\phi}}{r} \tag{2D3.24a}$$

In a similar manner, we can obtain (see Prob. 2D9)

$$(\mathrm{div}\mathbf{T})_\theta = \frac{1}{r^3}\frac{\partial(r^3 T_{\theta r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{\theta\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi} + \frac{T_{r\theta}-T_{\theta r}-T_{\phi\phi}\cot\theta}{r} \tag{2D3.24b}$$

$$(\mathrm{div}\mathbf{T})_\phi = \frac{1}{r^3}\frac{\partial(r^3 T_{\phi r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{\phi\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi} + \frac{T_{r\phi}-T_{\phi r}+T_{\theta\phi}\cot\theta}{r} \tag{2D3.24c}$$

PROBLEMS

2A1. Given

$$[S_{ij}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix} \quad \text{and} \quad [a_i] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

evaluate (a) S_{ii}, (b) $S_{ij}S_{ij}$, (c) $S_{jk}S_{kj}$, (d) $a_m a_m$, (e) $S_{mn}a_m a_n$.

2A2. Determine which of these equations have an identical meaning with $a_i = Q_{ij}a_j'$

(a) $a_p = Q_{pm}a_m'$,

(b) $a_p = Q_{qp}a_q'$,

(c) $a_m = a_n' Q_{mn}$.

2A3. Given the following matrices

$$[a_i] = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad [B_{ij}] = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad [C_{ij}] = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

Demonstrate the equivalence of the following subscripted equations and the corresponding matrix equations.

(a) $D_{ji} = B_{ij} \quad [D] = [B]^T$,

(b) $b_i = B_{ij}a_j \quad [b] = [B][a]$,

(c) $c_j = B_{ji}a_i \quad [c] = [B][a]$,

(d) $s = B_{ij}a_i a_j \quad s = [a]^T[B][a]$,

(e) $D_{ik} = B_{ij}C_{jk} \quad [D] = [B][C]$,

(f) $D_{ik} = B_{ij}C_{kj} \quad [D] = [B][C]^T$.

2A4. Given that $T_{ij} = 2\mu E_{ij} + \lambda(E_{kk})\delta_{ij}$, show that

(a)

$$W = \frac{1}{2}T_{ij}E_{ij} = \mu E_{ij}E_{ij} + \frac{\lambda}{2}(E_{kk})^2$$

(b)

$$P = T_{ij}T_{ij} = 4\mu^2 E_{ij}E_{ij} + (E_{kk})^2(4\mu\lambda + 3\lambda^2)$$

2A5. Given

$$[a_i] = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad [b_i] = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \quad [S_{ij}] = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix}$$

(a) Evaluate $[T_{ij}]$ if $T_{ij} = \varepsilon_{ijk}a_k$

(b) Evaluate $[c_i]$ if $c_i = \varepsilon_{ijk}S_{jk}$

(c) Evaluate $[d_i]$ if $d_k = \varepsilon_{ijk}a_ib_j$ and show that this result is the same as $d_k = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{e}_k$

2A6.

(a) If $\varepsilon_{ijk}T_{jk} = 0$,show that $T_{ij} = T_{ji}$

(b) Show that $\delta_{ij}\varepsilon_{ijk} = 0$

2A7. (a)Verify that

$$\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$$

By contracting the result of part (a) show that

(b)$\varepsilon_{ilm}\varepsilon_{jlm} = 2\delta_{ij}$

(c)$\varepsilon_{ijk}\varepsilon_{ijk} = 6$

2A8. Using the relation of Problem 2A7a, show that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

2A9. (a) If $T_{ij} = -T_{ji}$ show that $T_{ij}a_ia_j = 0$

(b) If $T_{ij} = -T_{ji}$ and $S_{ij} = S_{ji}$, show that $T_{kl}S_{kl} = 0$

2A10. Let $T_{ij} = \frac{1}{2}(S_{ij}+S_{ji})$ and $R_{ij} = \frac{1}{2}(S_{ij}-S_{ji})$, show that

$$S_{ij} = T_{ij}+R_{ij},\ T_{ij} = T_{ji},\ \text{and}\ R_{ij} = -R_{ji}$$

2A11. Let $f(x_1,x_2,x_3)$ be a function of x_i and $v_i(x_1,x_2,x_3)$ represent three functions of x_i. By expanding the following equations, show that they correspond to the usual formulas of differential calculus.

(a) $df = \dfrac{\partial f}{\partial x_i}dx_i$

(b) $dv_i = \dfrac{\partial v_i}{\partial x_j}dx_j$

2A12. Let $|A_{ij}|$ denote the determinant of the matrix $[A_{ij}]$. Show that $|A_{ij}| = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$.

2B1. A transformation **T** operates on a vector **a** to give $\mathbf{Ta} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$, where $|\mathbf{a}|$ is the magnitude of **a**. Show that **T** is not a linear transformation.

2B2. (a) A tensor **T** transforms every vector **a** into a vector $\mathbf{Ta} = \mathbf{m}\times\mathbf{a}$, where **m** is a specified vector. Prove that **T** is a linear transformation.

(b) If $\mathbf{m} = \mathbf{e}_1 + \mathbf{e}_2$, find the matrix of the tensor **T**

2B3. A tensor $\mathbf{T}$ transforms the base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ so that

$$\mathbf{T}\mathbf{e}_1 = \mathbf{e}_1+\mathbf{e}_2$$
$$\mathbf{T}\mathbf{e}_2 = \mathbf{e}_1-\mathbf{e}_2$$

If $\mathbf{a} = 2\mathbf{e}_1+3\mathbf{e}_2$ and $\mathbf{b} = 3\mathbf{e}_1+2\mathbf{e}_2$, use the linear property of $\mathbf{T}$ to find

(a) $\mathbf{Ta}$ (b) $\mathbf{Tb}$ and (c) $\mathbf{T}(\mathbf{a}+\mathbf{b})$.

2B4. Obtain the matrix for the tensor $\mathbf{T}$ which transforms the base vectors as follows:

$$\mathbf{T}\mathbf{e}_1 = 2\mathbf{e}_1+\mathbf{e}_3$$
$$\mathbf{T}\mathbf{e}_2 = \mathbf{e}_2+3\mathbf{e}_3$$
$$\mathbf{T}\mathbf{e}_3 = -\mathbf{e}_1+3\mathbf{e}_2$$

2B5. Find the matrix of the tensor $\mathbf{T}$ which transforms any vector $\mathbf{a}$ into a vector $\mathbf{b} = \mathbf{m}(\mathbf{a}\cdot\mathbf{n})$ where

$$\mathbf{m} = \frac{\sqrt{2}}{2}(\mathbf{e}_1+\mathbf{e}_2) \quad \text{and} \quad \mathbf{n} = \frac{\sqrt{2}}{2}(-\mathbf{e}_1+\mathbf{e}_3)$$

2B6. (a) A tensor $\mathbf{T}$ transforms every vector into its mirror image with respect to the plane whose normal is $\mathbf{e}_2$. Find the matrix of $\mathbf{T}$.

b) Do part (a) if the plane has a normal in the $\mathbf{e}_3$ direction instead.

2B7. a) Let $\mathbf{R}$ correspond to a right-hand rotation of angle θ about the x_1-axis. Find the matrix of $\mathbf{R}$.

b) Do part (a) if the rotation is about the x_2-axis.

2B8. Consider a plane of reflection which passes through the origin. Let $\mathbf{n}$ be a unit normal vector to the plane and let $\mathbf{r}$ be the position vector for a point in space

(a) Show that the reflected vector for $\mathbf{r}$ is given by $\mathbf{Tr} = \mathbf{r}-2(\mathbf{r}\cdot\mathbf{n})\mathbf{n}$, where $\mathbf{T}$ is the transformation that corresponds to the reflection.

(b) Let $\mathbf{n}=\frac{1}{\sqrt{3}}(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)$, find the matrix of the linear transformation $\mathbf{T}$ that corresponds to this reflection.

(c) Use this linear transformation to find the mirror image of a vector $\mathbf{a} = \mathbf{e}_1+2\mathbf{e}_2+3\mathbf{e}_3$.

2B9. A rigid body undergoes a right hand rotation of angle θ about an axis which is in the direction of the unit vector $\mathbf{m}$. Let the origin of the coordinates be on the axis of rotation and $\mathbf{r}$ be the position vector for a typical point in the body.

(a) Show that the rotated vector of $\mathbf{r}$ is given by $\mathbf{Rr} = (1-\cos\theta)(\mathbf{m}\cdot\mathbf{r})\mathbf{m}+\cos\theta\mathbf{r}+\sin\theta\mathbf{m}\times\mathbf{r}$, where $\mathbf{R}$ is the transformation that corresponds to the rotation.

(b) Let $\mathbf{m}=\frac{1}{\sqrt{3}}(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)$, find the matrix of the linear transformation that corresponds to this rotation.

(c) Use this linear transformation to find the rotated vector of $\mathbf{a} = \mathbf{e}_1+2\mathbf{e}_2+3\mathbf{e}_3$.

2B10. (a) Find the matrix of the tensor **S** that transforms every vector into its mirror image in a plane whose normal is $\mathbf{e}_2$ and then by a 45^0 right-hand rotation about the $\mathbf{e}_1$-axis.

(b) Find the matrix of the tensor **T** that transforms every vector by the combination of first the rotation and then the reflection of part (a).

(c) Consider the vector $\mathbf{e}_1+2\mathbf{e}_2+3\mathbf{e}_3$, find the transformed vector by using the transformations **S**. Also, find the transformed vector by using the transformation **T**.

2B11. a) Let **R** correspond to a right-hand rotation of angle θ about the x_3-axis.

(a)Find the matrix of $\mathbf{R}^2$.

(b)Show that $\mathbf{R}^2$ corresponds to a rotation of angle 2θ about the same axis.

(c)Find the matrix of $\mathbf{R}^n$ for any integer n.

2B12. Rigid body rotations that are small can be described by an orthogonal transformation $\mathbf{R} = \mathbf{I}+\varepsilon\mathbf{R}^*$, where $\varepsilon \to 0$ as the rotation angle approaches zero. Considering two successive rotations $\mathbf{R}_1$ and $\mathbf{R}_2$, show that for small rotations (so that terms containing ε^2 can be neglected) the final result does not depend on the order of the rotations.

2B13. Let **T** and **S** be any two tensors. Show that

(a) $\mathbf{T}^T$is a tensor.

(b) $\mathbf{T}^T+\mathbf{S}^T = (\mathbf{T}+\mathbf{S})^T$

(c) $(\mathbf{TS})^T = \mathbf{S}^T\mathbf{T}^T$.

2B14. Using the form for the reflection in an arbitrary plane of Prob. 2B8, write the reflection tensor in terms of dyadic products.

2B15. For arbitrary tensors **T** and **S**, without relying on the component form, prove that

(a) $(\mathbf{T}^{-1})^T = (\mathbf{T}^T)^{-1}$.

(b) $(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\mathbf{T}^{-1}$.

2B16. Let **Q** define an orthogonal transformation of coordinates, so that $\mathbf{e}_i' = Q_{mi}\mathbf{e}_m$. Consider $\mathbf{e}_i'\cdot\mathbf{e}_j'$ and verify that $Q_{mi}Q_{mj} = \delta_{ij}$.

2B17. The basis $\mathbf{e}_i'$ is obtained by a 30° counterclockwise rotation of the $\mathbf{e}_i$ basis about $\mathbf{e}_3$.

(a) Find the orthogonal transformation **Q** that defines this change of basis, i.e., $\mathbf{e}_i' = Q_{mi}\mathbf{e}_m$

(b) By using the vector transformation law, find the components of $\mathbf{a} = \sqrt{3}\mathbf{e}_1 + \mathbf{e}_2$ in the primed basis (i.e., find a_i')

(c) Do part (b) geometrically.

2B18. Do the previous problem with $\mathbf{e}_i'$ obtained by a 30° clockwise rotation of the $\mathbf{e}_i$-basis about $\mathbf{e}_3$.

2B19. The matrix of a tensor $\mathbf{T}$ in respect to the basis $\{\mathbf{e}_i\}$ is

$$[\mathbf{T}] = \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

Find T_{11}', T_{12}' and T_{31}' in respect to a right-hand basis $\mathbf{e}_i'$ where $\mathbf{e}_1'$ is in the direction of $-\mathbf{e}_2+2\mathbf{e}_3$ and $\mathbf{e}_2'$ is in the direction of $\mathbf{e}_1$

2B20. (a) For the tensor of the previous problem, find $[T_{ij}']$ if $\mathbf{e}_i'$ is obtained by a 90° right-hand rotation about the $\mathbf{e}_3$-axis.

(b) Compare both the sum of the diagonal elements and the determinants of $[\mathbf{T}]$ and $[\mathbf{T}]'$.

2B21. The dot product of two vectors $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b}_i = b_i\mathbf{e}_i$ is equal to a_ib_i. Show that the dot product is a scalar invariant with respect to an orthogonal transformation of coordinates.

2B22. (a) If T_{ij} are the components of a tensor, show that $T_{ij}T_{ij}$ is a scalar invariant with respect to an orthogonal transformation of coordinates.

(b) Evaluate $T_{ij}T_{ij}$ if in respect to the basis $\mathbf{e}_i$

$$[\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}_{\mathbf{e}_i}$$

(c) Find $[\mathbf{T}]'$ if $\mathbf{e}_i' = \mathbf{Q}\mathbf{e}_i$ and

$$[\mathbf{Q}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{\mathbf{e}_i}$$

(d) Show for this specific $[\mathbf{T}]$ and $[\mathbf{T}]'$ that

$$T_{mn}'T_{mn}' = T_{ij}T_{ij}.$$

2B23. Let $[\mathbf{T}]$ and $[\mathbf{T}]'$ be two matrices of the same tensor $\mathbf{T}$, show that

$$\det[\mathbf{T}] = \det[\mathbf{T}]'.$$

2B24. (a) The components of a third-order tensor are R_{ijk}. Show that R_{iik} are components of a vector.

(b) Generalize the result of part (a) by considering the components of a tensor of n^{th} order $R_{ijk\ldots}$ Show that $R_{iik\ldots}$ are components of an $(n\text{-}2)^{th}$ order tensor.

2B25. The components of an arbitrary vector **a** and an arbitrary second-order tensor **T** are related by a triply subscripted quantity R_{ijk} in the manner $a_i = R_{ijk}T_{jk}$ for any rectangular Cartesian basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Prove that R_{ijk} are the components of a third-order tensor.

2B26. For any vector **a** and any tensor **T**, show that

(a) $\mathbf{a}\cdot\mathbf{T}^A\mathbf{a} = 0$,

(b) $\mathbf{a}\cdot\mathbf{T}\mathbf{a} = \mathbf{a}\cdot\mathbf{T}^S\mathbf{a}$.

2B27. Any tensor may be decomposed into a symmetric and antisymmetric part. Prove that the decomposition is unique. (Hint: Assume that it is not unique.)

2B28. Given that a tensor **T** has a matrix

$$[\mathbf{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(a) find the symmetric and antisymmetric part of **T**.

(b) find the dual vector of the antisymmetric part of **T**.

2B29 From the result of part (a) of Prob. 2B9, for the rotation about an arbitrary axis **m** by an angle θ,

(a) Show that the rotation tensor is given by $\mathbf{R} = (1-cos\theta)(\mathbf{mm})+\sin\theta\mathbf{E}$, where **E** is the antisymmetric tensor whose dual vector is **m**. [note **mm** denotes the dyadic product of **m** with **m**].

(b) Find $\mathbf{R}^A$, the antisymmetric part of **R**.

(c) Show that the dual vector for $\mathbf{R}^A$ is given by $\sin\theta\mathbf{m}$

2B30. Prove that the only possible real eigenvalues of an orthogonal tensor are $\lambda = \pm 1$.

2B31. Tensors **T**, **R**, and **S** are related by **T** = **RS**. Tensors **R** and **S** have the same eigenvector **n** and corresponding eigenvalues r_1 and s_1. Find an eigenvalue and the corresponding eigenvector of **T**.

2B32. If **n** is a real eigenvector of an antisymmetric tensor **T**, then show that the corresponding eigenvalue vanishes.

2B33. Let **F** be an arbitrary tensor. It can be shown (Polar Decomposition Theorem) that any invertible tensor **F** can be expressed as **F** = **VQ** = **QU**, where **Q** is an orthogonal tensor and **U** and **V** are symmetric tensors.

(b) Show that $\mathbf{VV} = \mathbf{FF}^T$ and $\mathbf{UU} = \mathbf{F}^T\mathbf{F}$.

(c) If λ_i and $\mathbf{n}_i$ are the eigenvalues and eigenvectors of $\mathbf{U}$, find the eigenvectors and eigenvectors of $\mathbf{V}$.

2B34. (a) By inspection find an eigenvector of the dyadic product $\mathbf{ab}$

(b) What vector operation does the first scalar invariant of $\mathbf{ab}$ correspond to?

(c) Show that the second and the third scalar invariants of $\mathbf{ab}$ vanish. Show that this indicates that zero is a double eigenvalue of $\mathbf{ab}$. What are the corresponding eigenvectors?

2B35. A rotation tensor $\mathbf{R}$ is defined by the relations

$$\mathbf{Re}_1 = \mathbf{e}_2, \quad \mathbf{Re}_2 = \mathbf{e}_3, \quad \mathbf{Re}_3 = \mathbf{e}_1$$

(a) Find the matrix of $\mathbf{R}$ and verify that $\mathbf{RR}^T = \mathbf{I}$ and $\det|\mathbf{R}| = 1$.

(b) Find the angle of rotation that could have been used to effect this particular rotation.

2B36. For any rotation transformation a basis $\mathbf{e}_i'$ may be chosen so that $\mathbf{e}_3'$ is along the axis of rotation.

(a) Verify that for a right-hand rotation angle θ, the rotation matrix in respect to the $\mathbf{e}_i'$ basis is

$$[\mathbf{R}]' = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}_i'}$$

(b) Find the symmetric and antisymmetric parts of $[\mathbf{R}]'$.

(c) Find the eigenvalues and eigenvectors of $\mathbf{R}^S$.

(d) Find the first scalar invariant of $\mathbf{R}$.

(e) Find the dual vector of $\mathbf{R}^A$.

(f) Use the result of (d) and (e) to find the angle of rotation and the axis of rotation for the previous problem.

2B37. (a) If $\mathbf{Q}$ is an improper orthogonal transformation (corresponding to a reflection), what are the eigenvalues and corresponding eigenvectors of $\mathbf{Q}$?

(b) If the matrix $\mathbf{Q}$ is

$$[\mathbf{Q}] = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

find the normal to the plane of reflection.

2B38. Show that the second scalar invariant of $\mathbf{T}$ is

$$I_2 = \frac{T_{ii}T_{jj}}{2} - \frac{T_{ij}T_{ji}}{2}$$

by expanding this equation.

2B39. Using the matrix transformation law for second-order tensors, show that the third scalar invariant is indeed independent of the particular basis.

2B40. A tensor **T** has a matrix

$$[\mathbf{T}] = \begin{bmatrix} 5 & 4 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Find the scalar invariants, the principle values and corresponding principal directions of the tensor **T**.

(b) If $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are the principal directions, write $[\mathbf{T}]_{\mathbf{n}_i}$.

(c) Could the following matrix represent the tensor **T** in respect to some basis?

$$\begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

2B41. Do the previous Problem for the matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

2B42. A tensor **T** has a matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Find the principal values and three mutually orthogonal principal directions.

2B43. The inertia tensor $\bar{\mathbf{I}}_o$ of a rigid body with respect to a point o, is defined by

$$\bar{\mathbf{I}}_o = \int (r^2\mathbf{I} - \mathbf{rr})\rho dV$$

where **r** is the position vector, $r = |\mathbf{r}|, \rho =$ mass density, **I** is the identity tensor, and dV is a differential volume. The moment of inertia, with respect to an axis pass through o, is given by $\bar{\mathbf{I}}_{nn} = \mathbf{n}\cdot\bar{\mathbf{I}}_o\mathbf{n}$, (no sum on n), where **n** is a unit vector in the direction of the axis of interest.

(a) Show that $\bar{\mathbf{I}}_o$ is symmetric.

(b) Letting $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$, write out all components of the inertia tensor $\bar{\mathbf{I}}_o$.

(c) The diagonal terms of the inertia matrix are the moments of inertia and the off-diagonal terms the products of inertia. For what axes will the products of inertia be zero? For which axis will the moments of inertia be greatest (or least)?

Let a coordinate frame $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ be attached to a rigid body which is spinning with an angular velocity $\boldsymbol{\omega}$. Then, the angular momentum vector $\mathbf{H}_c$, in respect to the mass center, is given by

$$\mathbf{H}_c = \bar{\mathbf{I}}_c\boldsymbol{\omega}$$

and

$$\frac{d\mathbf{e}_i}{dt} = \boldsymbol{\omega}\times\mathbf{e}_i.$$

(d) Let $\boldsymbol{\omega} = \omega_i\mathbf{e}_i$ and demonstrate that

$$\dot{\boldsymbol{\omega}} = \frac{d\boldsymbol{\omega}}{dt} = \frac{d\omega_i}{dt}\mathbf{e}_i$$

and that

$$\dot{\mathbf{H}}_c = \frac{d}{dt}\mathbf{H}_c = \bar{\mathbf{I}}_c\dot{\boldsymbol{\omega}}+\boldsymbol{\omega}\times(\bar{\mathbf{I}}_c\boldsymbol{\omega})$$

2C1. Prove the identities (2C1.2a-e) of Section 2C1.

2C2. Consider the scalar field defined by $\phi = x^2+3xy+2z$.

(a) Find a unit normal to the surface of constant ϕ at the origin (0,0,0).

(b) What is the maximum value of the directional derivative of ϕ at the origin?

(c) Evaluate $d\phi/dr$ at the origin if $d\mathbf{r} = ds(\mathbf{e}_1+\mathbf{e}_3)$.

2C3. Consider the ellipsoid defined by the equation $x^2/a^2+y^2/b^2+z^2/c^2=1$.

Find the unit normal vector at a given position (x,y,z).

2C4. Consider a temperature field given by $\theta = 3xy$.

(a) Find the heat flux at the point $A(1,1,1)$ if $\mathbf{q} = -k\nabla\theta$.

(b) Find the heat flux at the same point as part (a) if $\mathbf{q} = -\mathbf{K}\nabla\theta$, where

$$[\mathbf{K}] = \begin{bmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 3k \end{bmatrix}$$

2C5. Consider an electrostatic potential given by $\phi = \alpha[x\cos\theta+y\sin\theta]$, where α and θ are constants.

(a) Find the electric field $\mathbf{E}$ if $\mathbf{E} = -\nabla\phi$.

(b) Find the electric displacement $\mathbf{D}$ if $\mathbf{D} = \varepsilon\mathbf{E}$, where the matrix of ε is

$$[\varepsilon] = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

(c) Find the angle θ for which the magnitude of $\mathbf{D}$ is a maximum.

2C6. Let $\phi(x,y,z)$ and $\psi(x,y,z)$ be scalar fields, and let $\mathbf{v}(x,y,z)$ and $\mathbf{w}(x,y,z)$ be vector fields. By writing the subscripted component form, verify the following identities:

(a) $\nabla(\phi+\psi) = \nabla\phi+\nabla\psi$

Sample solution:

$$[\nabla(\phi+\psi)]_i = \frac{\partial}{\partial x_i}(\phi+\psi) = \frac{\partial\phi}{\partial x_i}+\frac{\partial\psi}{\partial x_i} = (\nabla\phi)_i+(\nabla\psi)_i$$

(b) $\operatorname{div}(\mathbf{v}+\mathbf{w}) = \operatorname{div}\mathbf{v}+\operatorname{div}\mathbf{w}$,

(c) $\operatorname{div}(\phi\mathbf{v}) = (\nabla\phi)\cdot\mathbf{v}+\phi(\operatorname{div}\mathbf{v})$,

(d) $\operatorname{curl}(\nabla\phi) = 0$,

(e) $\operatorname{div}(\operatorname{curl}\mathbf{v}) = 0$.

2C7. Consider the vector field $\mathbf{v} = x^2\mathbf{e}_1 + z^2\mathbf{e}_2 + y^2\mathbf{e}_3$. For the point (1, 1, 0):

(a) Find the matrix of $\nabla\mathbf{v}$.

(b) Find the vector $(\nabla\mathbf{v})\mathbf{v}$.

(c) Find div $\mathbf{v}$ and curl $\mathbf{v}$.

(d) if $d\mathbf{r} = ds(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$, find the differential $d\mathbf{v}$.

2D1. Obtain Eq. (2D1.15)

2D2. Calculate div $\mathbf{u}$ for the following vector field in cylindrical coordinates:

(a)$u_r = u_\theta = 0$, $u_z = A + Br^2$,

(b)$u_r = \dfrac{\sin\theta}{r}$, $u_\theta = 0$, $u_z = 0$,

(c)$u_r = \dfrac{1}{2}\sin\theta r^2$, $u_\theta = \dfrac{1}{2}\cos\theta r^2$, ,$u_z = 0$,

(d)$u_r = \dfrac{\sin\theta}{r^2}$, $u_\theta = -\dfrac{\cos\theta}{r^2}$, $u_z = 0$.

2D3. Calculate div $\mathbf{u}$ for the following vector field in spherical coordinates:

$$u_r = Ar + \frac{B}{r^2}, \quad u_\theta = u_\phi = 0$$

2D4. Calculate $\nabla\mathbf{u}$ for the following vector field in cylindrical coordinate

$$u_r = \frac{A}{r}, \quad u_\theta = Br, \quad v_z = 0$$

2D5. Calculate $\nabla\mathbf{u}$ for the following vector field in spherical coordinate

$$u_r = Ar + \frac{B}{r^2}, \quad u_\theta = u_\phi = 0$$

2D6. Calculate div **T** for the following tensor field in cylindrical coordinates:

$$T_{rr} = \frac{Az}{R^3} - \frac{3r^2z}{R^5},\quad T_{\theta\theta} = \frac{Az}{R^3},\quad T_{zz} = -\left[\frac{Az}{R^3} + \frac{3z^3}{R^5}\right],\quad T_{rz} = -\left[\frac{Ar}{R^3} + \frac{3rz^2}{R^5}\right]$$

$$T_{z\theta} = T_{r\theta} = 0, \text{ where } R^2 = r^2 + z^2$$

2D7. Calculate div **T** for the following tensor field in cylindrical coordinates:

$$T_{rr} = A + \frac{B}{r^2},\quad T_{\theta\theta} = A - \frac{B}{r^2},\quad T_{zz} = \text{constant},\ T_{r\theta} = T_{rz} = T_{\theta z} = 0$$

2D8. Calculate div **T** for the following tensor field in spherical coordinates:

$$T_{rr} = A - \frac{2B}{r^3},\quad T_{\theta\theta} = T_{\phi\phi} = A + \frac{B}{r^3}$$

$$T_{\theta r} = T_{\phi r} = T_{\phi\phi} = 0$$

2D9. Derive Eq. (2D3.24b) and Eq. (2D3.24c).