

1 One-dimensional Model

Strong form:

given $f(x)$ and consts. g & h , find $u(x)$ such that

$$(S) \begin{cases} -(EAu_{,x})_{,x} = f & 0 < x < L \\ -(EAu_{,x})|_{x=0} = h \\ u(L) = g \end{cases}$$

Weak form:

given f , g , & h , find the trial sol'n u ($u(L) = g$) such that for all weighting ft'ns w ($w(L) = 0$)

$$(W) \quad \int_0^L w_{,x} EAu_{,x} dx = \int_0^L wf dx + w(0)h$$

Integration by parts (Green's theorem): $\int_0^L \phi_{,x} \psi dx = - \int_0^L \phi \psi_{,x} dx + (\phi\psi)|_0^L$

$(S) \Rightarrow (W)$:

Assume u is a sol'n of $(S) \Rightarrow u(L) = g \Rightarrow u$ is a (admissible) trial sol'n.

u satisfies diff. eq'n in domain $\Rightarrow 0 = - \int_0^L w ((EAu_{,x})_{,x} + f) dx = \int_0^L w_{,x} EAu_{,x} dx - (wEAu_{,x})|_0^L - \int_0^L wf dx$
 $(wEAu_{,x})|_0^L = \underbrace{w(L)(EAu_{,x})|_{x=L}}_0 - \underbrace{w(0)(EAu_{,x})|_{x=0}}_{-h} \Rightarrow 0 = \int_0^L w_{,x} EAu_{,x} dx - w(0)h - \int_0^L wf dx$

Galerkin approximation:

given f , g , & h , find the approx. trial sol'n $u^h = v^h + g^h$ ($v^h(L) = 0, g^h(L) = g$) such that for all approx. weighting ft'ns w^h ($w^h(L) = 0$)

$$(G) \quad \int_0^L w^h_{,x} EA v^h_{,x} dx = \underbrace{\int_0^L w^h f dx + w^h(0)h}_{\text{given}} - \int_0^L w^h_{,x} EA g^h_{,x} dx$$

Mesh: nodes at x_A , $A = 1, \dots, n+1$. $x_1 = 0, x_{n+1} = L$. $n_{\text{el}} = n$ elements. typical element $x_A \leq x \leq x_{A+1}$.
Example, piecewise linear shape functions:

$$N_A = \begin{cases} \frac{x-x_{A-1}}{h_{A-1}}, & x_{A-1} \leq x \leq x_A \\ \frac{x_{A+1}-x}{h_A}, & x_A \leq x \leq x_{A+1} \\ 0, & \text{else} \end{cases} \quad A = 2, \dots, n$$

- Interpolation: $N_A(x_B) = \delta_{AB} = \begin{cases} 1, & A = B \\ 0, & A \neq B \end{cases}$
- Local support: $N_A \neq 0$ only in elem's that contain node A (banded or sparse matrices)

Matrix equations:

$$w^h(x) = \sum_{A=1}^n c_A N_A(x) \quad v^h(x) = \sum_{A=1}^n d_A N_A(x) \quad g^h(x) = g N_{n+1}(x)$$

$$(M) \quad \sum_{B=1}^n \underbrace{\left(\int_0^L N_{A,x} EAN_{B,x} dx \right)}_{K_{AB}} d_B = \underbrace{\int_0^L N_A f dx + N_A(0)h - \left(\int_0^L N_{A,x} EAN_{n+1,x} dx \right) g}_{F_A}, \quad A = 1, \dots, n$$

- Symmetry ($\mathbf{K} = \mathbf{K}^T$)
- Positive definite (bc's) – unique inverse, real positive eigv.
- \mathbf{K} is banded (tri-diagonal in 1-D, N_A have local support)
- Well conditioned (N_A “nearly orthogonal”)

Superconvergence
(only in 1-D, $EA = \text{const.}$):
 $u^h(x_A) = u(x_A), A = 1, \dots, n+1$

$$u_{,x}^h(x) - u_{,x}(x) = \begin{cases} O(h^2), & x = \frac{x_A+x_{A+1}}{2} \\ O(h), & \text{Otherwise} \end{cases}$$

Local description: Parent domain $-1 \leq \xi \leq 1$

$$\text{Linear element: } N_a(\xi) = \frac{1+\xi_a \xi}{2}, \quad \xi_a = (-1)^a, \quad a = 1, 2 \quad x^e(\xi) = \sum_{a=1}^2 N_a(\xi) x_a^e, \quad u^h(\xi) = \sum_{a=1}^2 N_a(\xi) d_a^e$$

$$\text{Derivatives: } \frac{\partial}{\partial \xi} \phi(x(\xi)) = \phi_{,x}(x(\xi)) x_{,\xi}(\xi) \quad N_{a,x} = N_{a,\xi}(x_{,\xi})^{-1}$$

$$\text{Integration: } \int_{x_1}^{x_2} \phi(x) dx = \int_{\xi_1}^{\xi_2} \phi(x(\xi)) x_{,\xi}(\xi) d\xi$$

$$k_{ab}^e = \int_{x_1^e}^{x_2^e} N_{a,x} E A N_{b,x} dx = \begin{cases} (-1)^{a+b} EA/h^e, & 1 \leq a, b \leq 2 \\ EA = \text{const.} & \end{cases}$$

$$f_a^e = \int_{x_1^e}^{x_2^e} N_a f dx + \begin{cases} h, & a = 1, e = 1 \\ -k_{a2}^e g, & e = n_{\text{el}} \\ 0, & \text{Otherwise} \end{cases} \quad \int_{x_1^e}^{x_2^e} N_a f dx \approx \frac{h^e}{6} \begin{cases} 2f(x_1^e) + f(x_2^e), & a = 1 \\ f(x_1^e) + 2f(x_2^e), & a = 2 \end{cases}$$

$$\text{Energy norm error: } \|u - u^h\|_E = \sqrt{\int_0^L EA (u_{,x} - u_{,x}^h)^2 dx}$$

2 Multi-dimensional Scalar Problems (membrane/heat conduction)

$$\text{Integration by parts: } \int_{\Omega} \phi_{,i} \psi d\Omega = - \int_{\Omega} \phi \psi_{,i} d\Omega + \int_{\Gamma} \phi \psi n_i d\Gamma$$

Strong form:

given $f(\mathbf{x})$, $g(\mathbf{x})$, and $h(\mathbf{x})$, find $u(\mathbf{x})$ such that

$$(S) \begin{cases} q_{i,i} = f & \text{in } \Omega \\ u = g & \text{on } \Gamma_g \\ -q_i n_i = h & \text{on } \Gamma_h \end{cases} \quad (\text{membrane: } n_{\text{sd}} = 2, \kappa_{ij} \leftarrow T \delta_{ij}, \mathbf{q} = -T \begin{Bmatrix} \theta \\ \psi \end{Bmatrix}, g = 0 \text{ on } \Gamma = \Gamma_g)$$

Weak form:

given f , g , & h , find the trial sol'n u ($u = g$ on Γ_g) such that for all weighting ft'ns w ($w = 0$ on Γ_g)

$$(W) \quad - \int_{\Omega} w_{,i} q_i d\Omega = \int_{\Omega} w f d\Omega + \int_{\Gamma_h} w h d\Gamma$$

$(S) \Rightarrow (W)$:

Assume u is a sol'n of $(S) \Rightarrow u = g$ on $\Gamma_g \Rightarrow u$ is a (admissible) trial sol'n.

u satisfies diff. eq'n in domain $\Rightarrow 0 = \int_{\Omega} w (-q_{i,i} + f) d\Omega = \int_{\Omega} w_{,i} q_i d\Omega - \int_{\Gamma_h} w q_i n_i d\Gamma + \int_{\Omega} w f d\Omega$

Galerkin approximation:

given f , g , & h , find the approx. trial sol'n $u^h = v^h + g^h$ ($v^h = 0, g^h \approx g$ on Γ_g) such that for all approx. weighting ft'ns w^h ($w^h = 0$ on Γ_g)

$$(G) \quad \int_{\Omega} (\nabla w^h)^T \boldsymbol{\kappa} (\nabla v^h) d\Omega = \int_{\Omega} w^h f d\Omega + \int_{\Gamma_h} w^h h d\Gamma - \int_{\Omega} (\nabla w^h)^T \boldsymbol{\kappa} (\nabla g^h) d\Omega$$

Matrix equations:

$$w^h(\mathbf{x}) = \sum_{A \text{ in active set}} N_A(\mathbf{x}) c_A \quad v^h(\mathbf{x}) = \sum_{A \text{ in active set}} N_A(\mathbf{x}) d_A \quad g^h(\mathbf{x}) = \sum_{g-\text{node set}} N_A(\mathbf{x}) g_A \quad (g_A = g(\mathbf{x}_A))$$

$$(M) \quad \sum_{Q=1}^{n_{\text{eq}}} K_{PQ} d_Q = F_P, \quad P = 1, \dots, n_{\text{eq}} \quad \mathbf{B}_A = \nabla N_A = \{N_{A,i}\}, \quad \mathbf{D} = \boldsymbol{\kappa}$$

$$\begin{aligned} K_{PQ} &= \int_{\Omega} \mathbf{B}_A^T \mathbf{D} \mathbf{B}_B d\Omega, & P = \text{ID}(A) && Q = \text{ID}(B) \\ F_P &= \int_{\Omega} N_A f d\Omega + \int_{\Gamma_h} N_A h d\Gamma - \sum_{\substack{B \text{ in} \\ g-\text{node set}}} \left(\int_{\Omega} \mathbf{B}_A^T \mathbf{D} \mathbf{B}_B d\Omega \right) g_B \end{aligned}$$

- Symmetry ($\mathbf{K} = \mathbf{K}^T$)
- Positive definite (bc's) – unique inverse, real positive eigenvalues
- Banded (N_A have local support)
- Well conditioned (N_A “nearly orthogonal”)

Parent domain & Local description:

$$\boldsymbol{x}(\xi) = \sum_{a=1}^{n_{\text{en}}} N_a(\xi) \boldsymbol{x}_a^e \quad u^h(\xi) = \sum_{a=1}^{n_{\text{en}}} N_a(\xi) d_a^e \quad d_a^e = u^h(\boldsymbol{x}_a^e) \quad \mathbf{B}_a = \nabla N_a = \{N_{a,i}\}$$

$$\begin{aligned} k_{ab}^e &= \int_{\Omega^e} \mathbf{B}_a^T \mathbf{D} \mathbf{B}_b d\Omega, \quad 1 \leq a, b \leq n_{\text{en}} \\ f_a^e &= \int_{\Omega^e} N_a f d\Omega + \int_{\Gamma_h^e} N_a h d\Gamma - \sum_{b=1}^{n_{\text{en}}} k_{ab}^e g_b^e, \quad g_b^e = \begin{cases} 0, & \text{if } \text{LM}(b, e) \neq 0 \\ g_B, & \text{if } \text{LM}(b, e) = 0, \text{ where } B = \text{IEN}(b, e) \end{cases} \end{aligned}$$

Data processing: $\text{ID}(A) = P$ $\text{IEN}(a, e) = A$ $\text{LM}(a, e) = \text{ID}(\underbrace{\text{IEN}(a, e)}_A) = P$

3 Multi-dimensional Vector Problems (linear elastostatics)

Strong form:

given $f_i(\boldsymbol{x})$, $g_i(\boldsymbol{x})$, and $h_i(\boldsymbol{x})$, find $u_i(\boldsymbol{x})$ such that

$$(S) \begin{cases} -\sigma_{ij,j} &= f_i \quad \text{in } \Omega \\ u_i &= g_i \quad \text{on } \Gamma_g \\ \sigma_{ij} n_j &= h_i \quad \text{on } \Gamma_h \end{cases}$$

$$u_{(i,j)} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \sigma_{ij} = \lambda \delta_{ij} u_{k,k} + 2\mu u_{(i,j)} \quad (\text{isotropic})$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)}$$

Weak form:

given f_i , g_i , & h_i , find the trial sol'n u ($u_i = g_i$ on Γ_g) such that for all weighting ft'ns w ($w_i = 0$ on Γ_g)

$$(W) \quad \int_{\Omega} w_{(i,j)} \sigma_{ij} d\Omega = \int_{\Omega} w_i f_i d\Omega + \int_{\Gamma_h} w_i h_i d\Gamma$$

$(S) \implies (W)$:

Assume u_i is a sol'n of $(S) \implies u_i = g_i$ on $\Gamma_g \implies u_i$ is a (admissible) trial sol'n.

u_i satisfies diff. eq'n in domain $\implies 0 = \int_{\Omega} w_i (\sigma_{ij,j} + f_i) d\Omega = -\int_{\Omega} w_{i,j} \sigma_{ij} d\Omega + \int_{\Gamma_h} w_i \sigma_{ij} n_j d\Gamma + \int_{\Omega} w_i f_i d\Omega$

$$w_{(i,j)} \sigma_{ij} = \boldsymbol{\epsilon}^T(\boldsymbol{w}) \mathbf{D} \boldsymbol{\epsilon}(\boldsymbol{u})$$

$n_{\text{sd}} = 3$:

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,3} + u_{3,2} \\ u_{1,3} + u_{3,1} \\ u_{1,2} + u_{2,1} \end{Bmatrix} \quad \mathbf{D} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

$n_{\text{sd}} = 2$ – plane strain (plane stress $\lambda \leftarrow \frac{2\lambda\mu}{\lambda+2\mu}$):

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{Bmatrix} \quad \mathbf{D} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

Galerkin approximation:

given f_i , g_i , & h_i , find the approx. trial sol'n $u_i^h = v_i^h + g_i^h$ ($v_i^h = 0$, $g_i^h \approx g_i$ on Γ_g) such that for all approx. weighting ft'ns w_i^h ($w_i^h = 0$ on Γ_g)

$$(G) \quad \int_{\Omega} \boldsymbol{\epsilon}^T(\boldsymbol{w}^h) \mathbf{D} \boldsymbol{\epsilon}(\boldsymbol{v}^h) d\Omega = \int_{\Omega} w_i^h f_i d\Omega + \int_{\Gamma_h} w_i^h h_i d\Gamma - \int_{\Omega} \boldsymbol{\epsilon}^T(\boldsymbol{w}^h) \mathbf{D} \boldsymbol{\epsilon}(\boldsymbol{g}^h) d\Omega$$

Matrix equations:

$$w_i^h(\mathbf{x}) = \sum_{\substack{A \text{ in} \\ \text{active set } i}} N_A(\mathbf{x}) c_{iA} \quad v_i^h(\mathbf{x}) = \sum_{\substack{A \text{ in} \\ \text{active set } i}} N_A(\mathbf{x}) d_{iA} \quad g_i^h(\mathbf{x}) = \sum_{\substack{A \text{ in} \\ g_i - \text{node set}}} N_A(\mathbf{x}) g_{iA} \quad (g_{iA} = g_i(\mathbf{x}_A))$$

$$(M) \quad \sum_{Q=1}^{n_{\text{eq}}} K_{PQ} d_Q = F_P, \quad P = 1, \dots, n_{\text{eq}}$$

$$(n_{\text{sd}} = 2) \quad \mathbf{B}_A = \begin{bmatrix} N_{A,1} & 0 \\ 0 & N_{A,2} \\ N_{A,2} & N_{A,1} \end{bmatrix} \quad (n_{\text{sd}} = 3) \quad \mathbf{B}_A = \begin{bmatrix} N_{A,1} & 0 & 0 \\ 0 & N_{A,2} & 0 \\ 0 & 0 & N_{A,3} \\ 0 & N_{A,3} & N_{A,2} \\ N_{A,3} & 0 & N_{A,1} \\ N_{A,2} & N_{A,1} & 0 \end{bmatrix}$$

$$\begin{aligned} K_{PQ} &= \mathbf{e}_i^T \int_{\Omega} \mathbf{B}_A^T \mathbf{D} \mathbf{B}_B d\Omega \mathbf{e}_j, & P = \text{ID}(i, A) && Q = \text{ID}(j, B) \\ F_P &= \int_{\Omega} N_A f_i d\Omega + \int_{\Gamma_{h_i}} N_A h_i d\Gamma - \sum_{j=1}^{n_{\text{dof}}} \left(\sum_{\substack{B \text{ in} \\ g_j - \text{node set}}} \mathbf{e}_i^T \int_{\Omega} \mathbf{B}_A^T \mathbf{D} \mathbf{B}_B d\Omega \mathbf{e}_j g_{jB} \right) \end{aligned}$$

Parent domain & Local description:

$$\begin{aligned} \mathbf{x}(\xi) &= \sum_{a=1}^{n_{\text{en}}} N_a(\xi) \mathbf{x}_a^e & \mathbf{u}^h(\xi) &= \sum_{a=1}^{n_{\text{en}}} N_a(\xi) \mathbf{d}_a^e & \mathbf{d}_a^e &= \{d_{ia}^e\} & d_{ia}^e &= u_i^h(\mathbf{x}_a^e) \\ (n_{\text{sd}} = 2) \quad \mathbf{B}_a &= \begin{bmatrix} N_{a,1} & 0 \\ 0 & N_{a,2} \\ N_{a,2} & N_{a,1} \end{bmatrix} & (n_{\text{sd}} = 3) \quad \mathbf{B}_a &= \begin{bmatrix} N_{a,1} & 0 & 0 \\ 0 & N_{a,2} & 0 \\ 0 & 0 & N_{a,3} \\ 0 & N_{a,3} & N_{a,2} \\ N_{a,3} & 0 & N_{a,1} \\ N_{a,2} & N_{a,1} & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} k_{pq}^e &= \mathbf{e}_i^T \int_{\Omega^e} \mathbf{B}_a^T \mathbf{D} \mathbf{B}_b d\Omega \mathbf{e}_j, & p = n_{\text{dof}}(a-1) + i && q = n_{\text{dof}}(b-1) + j \\ \mathbf{f}_p^e &= \int_{\Omega^e} N_a f_i d\Omega + \int_{\Gamma_{h_i}^e} N_a h_i d\Gamma - \sum_{q=1}^{n_{\text{ee}}} k_{pq}^e g_q^e, & g_q^e = g_{jB}^e &= \begin{cases} 0, & \text{if LM}(q, e) \neq 0 \\ g_{jB}, & \text{if LM}(q, e) = 0, \text{ where } B = \text{IEN}(b, e) \end{cases} \end{aligned}$$

Data processing: $\text{ID}(i, A) = P$ $\text{IEN}(a, e) = A$ $\text{LM}(p, e) = \text{ID}(i, \underbrace{\text{IEN}(a, e)}_A) = P$

Post-processing: $\boldsymbol{\epsilon}(\mathbf{u}^h) = \sum_{a=1}^{n_{\text{en}}} \mathbf{B}_a \mathbf{d}_a^e, \quad \boldsymbol{\sigma} = \mathbf{D} \boldsymbol{\epsilon}$

contracted notation for stress, for example $n_{\text{sd}} = 2$ $\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$

Energy perspective: potential energy $\pi(\mathbf{u}) = \underbrace{\frac{1}{2} \int_{\Omega} u_{i,j} \sigma_{ij} d\Omega}_{U} - \int_{\Omega} u_i f_i d\Omega - \int_{\Gamma_h} u_i h_i d\Gamma$

Solve (S) or (W) \iff min. potential energy $\implies \pi(\mathbf{u}^h) \geq \pi(\mathbf{u})$

Strain energy is underestimated $U(\mathbf{u}^h) \leq U(\mathbf{u})$

Best approximation, for any admissible trial solution $\bar{\mathbf{u}}^h$ $U(\mathbf{u}^h - \mathbf{u}) \leq U(\bar{\mathbf{u}}^h - \mathbf{u})$

4 Isoparametric elements

Convergence criteria:

- Compatibility: smoothness in element Ω^e (C^1) & continuity across elements Γ^e (C^0) \Rightarrow no gaps or overlaps
- Completeness: represent *arbitrary* lin. polynomial in \boldsymbol{x} ($c_0 + c_1x + c_2y$ in 2-D)

$$d_a^e = c_0 + c_1x_a^e + c_2y_a^e \implies u^h = \sum_{a=1}^{n_{\text{en}}} N_a(c_0 + c_1x_a^e + c_2y_a^e) = c_0 \sum_{a=1}^{n_{\text{en}}} N_a + c_1 \sum_{a=1}^{n_{\text{en}}} N_a x_a^e + c_2 \sum_{a=1}^{n_{\text{en}}} N_a y_a^e = c_0 + c_1x + c_2y$$

Bilinear quad.: $n_{\text{en}} = 4 \quad N_a(\xi, \eta) = \frac{1}{4}(1 + \xi_a \xi)(1 + \eta_a \eta)$

Degenerate quad. \rightarrow triangle: $\boldsymbol{x}_4^e = \boldsymbol{x}_3^e \rightarrow \hat{N}_3 = N_3 + N_4$

Isoparametric lin. triangle: $N_1(r, s) = r \quad N_2(r, s) = s \quad N_3(r, s) = t = 1 - r - s$

$$\text{Derivatives: } \boldsymbol{x}_{,\boldsymbol{\xi}} = \begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix} \quad j = \det(\boldsymbol{x}_{,\boldsymbol{\xi}}) \quad \boldsymbol{\xi}_{,\boldsymbol{x}} = \begin{bmatrix} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{bmatrix} = (\boldsymbol{x}_{,\boldsymbol{\xi}})^{-1} = \frac{1}{j} \begin{bmatrix} y_{,\eta} & -x_{,\eta} \\ -y_{,\xi} & x_{,\xi} \end{bmatrix}$$

$$\langle N_{a,x} \quad N_{a,y} \rangle = \frac{1}{j} \langle N_{a,\xi} \quad N_{a,\eta} \rangle \begin{bmatrix} y_{,\eta} & -x_{,\eta} \\ -y_{,\xi} & x_{,\xi} \end{bmatrix}$$

$$\text{Integration: } \int_{\Omega^e} \phi(x, y) d\Omega = \int_{-1}^1 \int_{-1}^1 \underbrace{\phi(x(\xi, \eta), y(\xi, \eta)) j(\xi, \eta)}_{g(\xi, \eta)} d\xi d\eta \approx \sum_{l=1}^{n_{\text{int}}} g(\tilde{\xi}_l, \tilde{\eta}_l) W_l$$

Gaussian rules:

n_{int} points with optimal locations $\tilde{\xi}_l$ and weights W_l , $1 \leq l \leq n_{\text{int}}$, such that order of accuracy = $2n_{\text{int}}$.

Triangle integration formula:

$$\int_{\Omega^e} r^\alpha s^\beta t^\gamma d\Omega = \frac{\alpha! \beta! \gamma!}{(\alpha + \beta + \gamma + 2)!} 2A \quad \text{here, } 2A = \det \begin{bmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{bmatrix}$$