Preliminary

Hyperbolic functions $\sinh' \alpha = \cosh \alpha$, $\cosh' \alpha = \sinh \alpha$, $\cosh^2 \alpha - \sinh^2 \alpha = 1$

Euler formula $e^{i\alpha} = \cos \alpha + i \sin \alpha$

Integration by parts

$$\int_0^T \dot{\phi}\psi \, dt = -\int_0^T \phi\dot{\psi} \, dt + (\phi\psi)|_0^T, \qquad \int_0^L \phi'\psi \, dx = -\int_0^L \phi\psi' \, dx + (\phi\psi)|_0^L$$

Huygens-Steiner parallel axis theorem, the moment of inertia of a rigid body of mass m about an axis z

$$I_z = I_{\text{cm}} + md^2$$

I_{cm} is the moment of inertia about the parallel axis through the center of mass with perpendicular distance d.

SDOF - Free vibrations

Spring-mass-damper, IVP

$$\begin{aligned} m\ddot{x} + c\dot{x} + kx &= 0 \\ x(0) &= x_0 \\ \dot{x}(0) &= v_0 \end{aligned}$$

Equation of motion, $\omega_{\text{n}} = \sqrt{k/m}$ is natural frequency, $\xi = \dfrac{c}{2\sqrt{km}}$ is damping ratio

$$\ddot{x} + 2\xi\omega_{\text{n}}\dot{x} + \omega_{\text{n}}^2 x = 0$$

Solutions:

1. Underdamped $0 < \xi < 1$

$$\begin{aligned} x &= e^{-\xi\omega_{\text{n}}t} X \cos(\omega_{\text{d}}t - \phi) \\ X &= \sqrt{x_0^2 + \left(\frac{\xi\omega_{\text{n}}x_0 + v_0}{\omega_{\text{d}}}\right)^2} \\ \phi &= \arctan\left(\frac{\xi\omega_{\text{n}}x_0 + v_0}{x_0\omega_{\text{d}}}\right) \end{aligned}$$

$\omega_{\text{d}} = \sqrt{1-\xi^2}\,\omega_{\text{n}}$ is damped frequency, X is amplitude, ϕ is phase angle.

Logarithmic decrement, displacements at two adjacent cycles give damping ratio

$$\delta = \ln \frac{x(t)}{x(t + 2\pi/\omega_{\text{d}})} = \text{const.}, \qquad \xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

2. Critically damped $\xi = 1$

$$x(t) = (x_0 + (\omega_{\text{n}}x_0 + v_0)t\,)\, e^{-\omega_{\text{n}}t}$$

3. Overdamped $\xi > 1$

$$x = e^{-\xi\omega_{\text{n}}t}\left(\frac{\xi\omega_{\text{n}}x_0 + v_0}{\sqrt{\xi^2-1}\,\omega_{\text{n}}}\sinh(\sqrt{\xi^2-1}\,\omega_{\text{n}}t) + x_0\cosh(\sqrt{\xi^2-1}\,\omega_{\text{n}}t)\right)$$

SDOF - Periodic forced vibrations

Equation of motion, $F(t)$ is excitation force

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

Harmonic excitation, ω is driving frequency

$$F(t) = F_0 \cos\omega t = F_0 \text{ Re } e^{i\omega t}$$

Equation of motion, $x_{\text{st}} = F_0/k$ is static deflection

$$\ddot{x} + 2\xi\omega_{\text{n}}\dot{x} + \omega_{\text{n}}^2 x = \omega_{\text{n}}^2 x_{\text{st}} \cos\omega t$$

Particular solution (harmonic, steady-state)

$$\begin{aligned} x_{\text{p}}(t) &= X' \cos(\omega t - \phi') \\ X'(\omega) &= \frac{x_{\text{st}}}{\sqrt{(1-(\omega/\omega_{\text{n}})^2)^2 + (2\xi\omega/\omega_{\text{n}})^2}} \\ \phi'(\omega) &= \arctan\left(\frac{2\xi\omega/\omega_{\text{n}}}{1-(\omega/\omega_{\text{n}})^2}\right) \end{aligned}$$

Frequency response ($X' = x_{\text{st}}|G|$)

$$G(\omega) = \frac{X(\omega)}{x_{\text{st}}} = \frac{1}{(1 - (\omega/\omega_{\text{n}})^2) + 2i\xi\omega/\omega_{\text{n}}}$$

Resonance ($\xi < 1/\sqrt{2}$)

$$X_{\text{res}} = \max_{\omega} X' = X'|_{\omega=\sqrt{1-2\xi^2}\omega_{\text{n}}} = \frac{x_{\text{st}}}{2\xi\sqrt{1-\xi^2}}$$

Total undamped solution (transient + harmonic) at resonance

$$x = \frac{x_{\text{st}}}{2}\,\omega_{\text{n}}t\,\sin\omega_{\text{n}}t$$

Unbalanced mass m with eccentricity e, rotating with velocity ω within larger mass $M - m$

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin\omega t$$

x is position of big mass, with amplitude (phase unchanged)

$$|X| = e\,\frac{m}{M}\,\frac{(\omega/\omega_{\text{n}})^2}{\sqrt{(1-(\omega/\omega_{\text{n}})^2)^2 + (2\xi\omega/\omega_{\text{n}})^2}}$$

Vibration isolation, transmissibility > 1 for $0 < \omega/\omega_{\text{n}} < \sqrt{2}$, design isolation mount such that $\omega_{\text{n}} < \omega/\sqrt{2}$.
Harmonic motion of support, amplitude ratio as forces in vibration isolation.
Vibration measurement, relative motion as in unbalanced mass.
Periodic excitation $F(t) = F(t+T)$ with period $T = 2\pi/\omega_0$.
Expansion of periodic excitation as (complex) Fourier series, $\omega_p = p\omega_0$.

$$F(t) = \frac{1}{2}A_0 + \text{Re}\left(\sum_{p=1}^{\infty} A_p e^{i\omega_p t}\right)$$

Complex Fourier coefficients

$$A_p = \frac{2}{T}\int_{-T/2}^{T/2} F(t)e^{-i\omega_p t}\,dt, \qquad p = 0, 1, \ldots$$

Particular solution is superposition of harmonics

$$x = \frac{A_0}{2k} + \text{Re}\left(\sum_{p=1}^{\infty}\frac{A_p}{k}\,\frac{e^{i(\omega_p t - \phi_p)}}{\sqrt{(1-(\omega_p/\omega_{\text{n}})^2)^2 + (2\xi\omega_p/\omega_{\text{n}})^2}}\right)$$

$$\phi_p = \arctan\left(\frac{2\xi\omega_p/\omega_{\text{n}}}{1-(\omega_p/\omega_{\text{n}})^2}\right)$$

Resonances, for ω_0 such that $\omega_0 = \omega_{\text{n}}/p$ (when $\xi = 0$).

SDOF - General forced vibrations

Response to general excitation with homogeneous initial conditions.
Impulse load, Dirac delta defined by its action on continuous functions

$$\int_{-\infty}^{\infty} F(t)\delta(t-\bar{t})\,dt = F(\bar{t})$$

Equivalent to initial velocity $= 1/m$. Impulse response

$$g(t) = \frac{1}{m\omega_{\text{d}}}e^{-\xi\omega_{\text{n}}t}\sin\omega_{\text{d}}t, \quad t > 0$$

Step load, Heaviside function

$$H(t-\bar{t}) = \int_{-\infty}^{t}\delta(\tau-\bar{t})\,d\tau = \begin{cases} 0, & t < \bar{t} \\ 1, & t > \bar{t} \end{cases}$$

Step response

$$u(t) = \int_0^t g(\tau)\,d\tau = \frac{1}{k}\left(1 - e^{-\xi\omega_{\text{n}}t}\left(\cos\omega_{\text{d}}t + \frac{\xi\omega_{\text{n}}}{\omega_{\text{d}}}\sin\omega_{\text{d}}t\right)\right)H(t)$$

General response, convolution integrals

$$x(t) = \int_0^t F(\tau)g(t-\tau)\,d\tau = \int_0^t F(t-\tau)g(\tau)\,d\tau = F(0)u(t) + \int_0^t \frac{dF(\tau)}{d\tau}\,u(t-\tau)\,d\tau$$

Shock spectrum, dependence of $x_{\max}/x_{\text{st}}$ on T_0/T for loading characterized by T_0 ($T = 2\pi/\omega_{\text{n}}$).
Truncated ramp

$$\frac{x_{\max}}{x_{\text{st}}} = 1 + \frac{T}{T_0\pi}|\sin(\pi T_0/T)|$$

Multiple degrees of freedom
Undamped system, $N \times N$ coupled ODE's

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} &= \mathbf{F} \\ \mathbf{x}(0) &= \mathbf{x}_0 \\ \dot{\mathbf{x}}(0) &= \mathbf{v}_0 \end{aligned}$$

Modal analysis, generalized algebraic eigenvalue problem

$$\mathbf{K}\mathbf{u} = \omega^2 \mathbf{M}\mathbf{u}$$

Characteristic equation is polynomial of degree N

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0$$

Roots $0 \leq \omega_1 \leq \omega_2 \leq \ldots \leq \omega_N$ are natural frequencies.
Eigenvectors are modes of vibration

$$\left[\mathbf{K} - \omega_r^2 \mathbf{M}\right] \mathbf{u}^{(r)} = \mathbf{0}, \quad r = 1, \ldots, N$$

Modes are M-orthogonal, when $\omega_r \neq \omega_s$, $r, s = 1, \ldots, N$

$$\mathbf{u}^{(r)^T} \mathbf{M}\mathbf{u}^{(s)} = 0, \qquad \mathbf{u}^{(r)^T} \mathbf{K}\mathbf{u}^{(s)} = 0$$

Modes may be M-normalized

$$\mathbf{u}^{(r)^T} \mathbf{M}\mathbf{u}^{(r)} = 1, \qquad \mathbf{u}^{(r)^T} \mathbf{K}\mathbf{u}^{(r)} = \omega_r^2$$

Modal superposition

$$\mathbf{x}(t) = \sum_{r=1}^{N} \eta_r(t) \mathbf{u}^{(r)}$$

By orthonormality, system of ODE's decouples to scalar modal equations, modal force $N_r(t) = \mathbf{u}^{(r)^T} \mathbf{F}(t)$

$$\ddot{\eta}_r + \omega_r^2 \eta_r = N_r, \qquad r = 1, \ldots, N$$

Initial conditions

$$\begin{aligned} \eta_r(0) &= \mathbf{u}^{(r)^T} \mathbf{M}\mathbf{x}_0 \\ \dot{\eta}_r(0) &= \mathbf{u}^{(r)^T} \mathbf{M}\mathbf{v}_0 \end{aligned}$$

SDOF solution, general response by superposition

$$\eta_r = \mathbf{u}^{(r)^T} \mathbf{M}\mathbf{x}_0 \cos \omega_r t + \frac{1}{\omega_r} \mathbf{u}^{(r)^T} \mathbf{M}\mathbf{v}_0 \sin \omega_r t + \frac{1}{\omega_r} \int_0^t N_r(\tau) \sin \omega_r (t - \tau)\, d\tau$$

Semi-definite system, rigid body motion is possible, potential energy $U = \frac{1}{2} \mathbf{x}^T \mathbf{K}\mathbf{x} = 0$ for $\mathbf{x} \neq \mathbf{0}$.
At least one frequency is zero (orthogonality is unaffected). For $\omega_1 = 0$

$$\eta_1 = \mathbf{u}^{(1)^T} \mathbf{M}\mathbf{x}_0 + \mathbf{u}^{(1)^T} \mathbf{M}\mathbf{v}_0 t + \int_0^t N_1(\tau)(t - \tau)\, d\tau$$

Repeated frequencies, corresponding modes are arbirtrary to degree of repetition, mutually orthogonalize.
Rayleigh quotient

$$R(\mathbf{u}) = \frac{\mathbf{u}^T \mathbf{K}\mathbf{u}}{\mathbf{u}^T \mathbf{M}\mathbf{u}}, \qquad R(\mathbf{u}^{(r)}) = \omega_r^2, \quad r = 1, \ldots, N$$

Frequency bounds

$$\min R(\mathbf{u}) = R(\mathbf{u}^{(1)}) = \omega_1^2 \leq R(\mathbf{u}) \leq \omega_N^2 = R(\mathbf{u}^{(N)}) = \max R(\mathbf{u})$$

Estimate of fundamental frequency, R of trial mode (static deflection under forces proportional to masses).
Rayleigh damping $\mathbf{C} = a\mathbf{M} + b\mathbf{K}$, modal equation with damping ratio $\xi_r = \frac{1}{2}(a/\omega_r + b\omega_r)$

$$\ddot{\eta}_r + 2\xi_r \omega_r \dot{\eta}_r + \omega_r^2 \eta_r = N_r, \qquad r = 1, \ldots, N$$

Continuous systems

Axial vibrations of an elastic rod, $m = \rho A$

$$\begin{aligned} m\ddot{u} - \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) &= f, \qquad 0 < x < L, \quad t > 0 \\ u(x,0) &= u_0(x) \\ \dot{u}(x,0) &= v_0(x) \end{aligned}$$

Boundary conditions: clamped $u|_{0,L} = 0$, free $-EA\frac{\partial u}{\partial x}|_0 = 0$, $EA\frac{\partial u}{\partial x}|_L = 0$,

spring $-EA\frac{\partial u}{\partial x}|_0 = -ku|_0$, $EA\frac{\partial u}{\partial x}|_L = -ku|_L$, mass $-EA\frac{\partial u}{\partial x}|_0 = -M\ddot{u}|_0$, $EA\frac{\partial u}{\partial x}|_L = -M\ddot{u}|_L$.

Free vibrations ($f = 0$), separation of variables, substitute and collect terms

$$u(x,t) = X(x)T(t), \qquad \frac{\ddot{T}}{T} = \frac{(EAX')'}{mX} = -\omega^2$$

Time-dependence same as SDOF

$$\ddot{T} + \omega^2 T = 0, \qquad T = C\cos(\omega t - \phi)$$

Continuous eigenvalue problem in x

$$(EAX')' + \omega^2 mX = 0, \qquad 0 < x < L$$

Const. properties, $\beta^2 = \rho\omega^2/E$, form of mode

$$X = A\sin\beta x + B\cos\beta x$$

Natural frequencies ω_r (from characteristic equation), modes X_r, $r = 1, 2, \ldots$, depend on boundary conditions.

Orthonormal modes

$$\int_0^L mX_rX_s\,dx + \underbrace{MX_r(L)X_s(L)}_{\text{mass at } L} = \delta_{rs}, \qquad \int_0^L EAX'_rX'_s\,dx + \underbrace{kX_r(L)X_s(L)}_{\text{spring at } L} = \omega_r^2\delta_{rs}$$

Similar applications: transverse vibrations of a taut spring, torsional vibrations of a circular shaft.

Bending vibrations of a thin beam

$$m\ddot{u} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right) = f, \quad 0 < x < L, \quad t > 0$$

Time-dependence as before.

Boundary conditions: free $EI\frac{\partial^2 u}{\partial x^2}|_{0,L} = 0$ ($M = 0$), $-\frac{\partial}{\partial x}\left(EI\frac{\partial^2 u}{\partial x^2}\right)|_{0,L} = 0$ ($Q = 0$),

clamped $u|_{0,L} = 0$, $\frac{\partial u}{\partial x}|_{0,L} = 0$, pinned $u|_{0,L} = 0$, $EI\frac{\partial^2 u}{\partial x^2}|_{0,L} = 0$,

spring $-\frac{\partial}{\partial x}\left(EI\frac{\partial^2 u}{\partial x^2}\right)|_0 = ku|_0$ and $EI\frac{\partial^2 u}{\partial x^2}|_0 = 0$, $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 u}{\partial x^2}\right)|_L = ku|_L$ and $EI\frac{\partial^2 u}{\partial x^2}|_L = 0$.

mass $-\frac{\partial}{\partial x}\left(EI\frac{\partial^2 u}{\partial x^2}\right)|_0 = M\ddot{u}|_0$ and $EI\frac{\partial^2 u}{\partial x^2}|_0 = 0$, $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 u}{\partial x^2}\right)|_L = M\ddot{u}|_L$ and $EI\frac{\partial^2 u}{\partial x^2}|_L = 0$.

Free vibrations ($f = 0$), separation of variables, continuous eigenvalue problem in x

$$(EIX'')'' - \omega^2 mX = 0, \qquad 0 < x < L$$

Const. properties, $\beta^4 = m\omega^2/(EI)$, form of mode

$$X = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$$

Orthonormal modes

$$\int_0^L mX_rX_s\,dx + \underbrace{MX_r(L)X_s(L)}_{\text{mass at } L} = \delta_{rs}, \qquad \int_0^L EIX''_rX''_s\,dx + \underbrace{kX_r(0)X_s(0)}_{\text{spring at } 0} = \omega_r^2\delta_{rs}$$

Modal superpositon, modal equations, modal force $N_r(t) = \int_0^L X_r(x)f(x,t)\,dx$

$$u(x,t) = \sum_{r=1}^{\infty} X_r(x)\eta_r(t), \qquad \ddot{\eta}_r + \omega_r^2\eta_r = N_r, \quad r = 1, 2, \ldots$$

Modal solution, $\eta_r(0) = \int_0^L mX_ru_0\,dx$, $\dot{\eta}_r(0) = \int_0^L mX_rv_0\,dx$

$$\eta_r = \eta_r(0)\cos\omega_r t + \frac{\dot{\eta}_r(0)}{\omega_r}\sin\omega_r t + \frac{1}{\omega_r}\int_0^t N_r(\tau)\sin\omega_r(t-\tau)\,d\tau, \qquad r = 1, 2, \ldots$$