Preliminary
Euler formula

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

Integration by parts

$$\int_0^T \dot\phi\psi\,dt = -\int_0^T \phi\dot\psi\,dt + (\phi\psi)|_0^T, \qquad \int_0^L \phi'\psi\,dx = -\int_0^L \phi\psi'\,dx + (\phi\psi)|_0^L$$

Huygens-Steiner parallel axis theorem, the moment of inertia of a rigid body of mass m about an axis z

$$I_z = I_{\rm cm} + md^2$$

$I_{\rm cm}$ is the moment of inertia about the parallel axis through the center of mass with perpendicular distance d.

SDOF - Free vibrations (initial excitation)
Spring-mass-damper, IVP

$$\begin{aligned} m\ddot x + c\dot x + kx &= 0 \\ x(0) &= x_0 \\ \dot x(0) &= v_0 \end{aligned}$$

Equation of motion, $\omega_{\rm n} = \sqrt{k/m}$ is natural frequency, $\xi = \dfrac{c}{2\sqrt{km}}$ is damping ratio

$$\ddot x + 2\xi\omega_{\rm n}\dot x + \omega_{\rm n}^2 x = 0$$

Solutions:
1. Underdamped $0 < \xi < 1$

$$\begin{aligned} x &= e^{-\xi\omega_{\rm n}t} X\cos(\omega_{\rm d}t - \phi) \\ X &= \sqrt{x_0^2 + \left(\frac{\xi\omega_{\rm n}x_0 + v_0}{\omega_{\rm d}}\right)^2} \\ \phi &= \arctan\left(\frac{\xi\omega_{\rm n}x_0 + v_0}{x_0\omega_{\rm d}}\right) \end{aligned}$$

$\omega_{\rm d} = \sqrt{1-\xi^2}\,\omega_{\rm n}$ is damped frequency, X is amplitude, ϕ is phase angle.
Logarithmic decrement, displacements at two adjacent cycles give damping ratio

$$\delta = \ln\frac{x(t)}{x(t+2\pi/\omega_{\rm d})} = \text{const.}, \qquad \xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

2. Critically damped $\xi = 1$

$$x(t) = (x_0 + (\omega_{\rm n}x_0 + v_0)t\,)\,e^{-\omega_{\rm n}t}$$

3. Overdamped $\xi > 1$

$$x = e^{-\xi\omega_{\rm n}t}\left(\frac{\xi\omega_{\rm n}x_0 + v_0}{\sqrt{\xi^2-1}\,\omega_{\rm n}}\sinh(\sqrt{\xi^2-1}\,\omega_{\rm n}t) + x_0\cosh(\sqrt{\xi^2-1}\,\omega_{\rm n}t)\right)$$

SDOF - Periodic forced vibrations
Equation of motion, $F(t)$ is excitation force

$$m\ddot x + c\dot x + kx = F(t)$$

Harmonic excitation, ω is driving frequency

$$F(t) = F_0\cos\omega t = F_0\ \text{Re}\ e^{i\omega t}$$

Equation of motion, $x_{\rm st} = F_0/k$ is static deflection

$$\ddot x + 2\xi\omega_{\rm n}\dot x + \omega_{\rm n}^2 x = \omega_{\rm n}^2 x_{\rm st}\cos\omega t$$

Particular solution (harmonic, steady-state)

$$\begin{aligned} x_{\rm p}(t) &= X'\cos(\omega t - \phi') \\ X'(\omega) &= \frac{x_{\rm st}}{\sqrt{(1-(\omega/\omega_{\rm n})^2)^2 + (2\xi\omega/\omega_{\rm n})^2}} \\ \phi'(\omega) &= \arctan\left(\frac{2\xi\omega/\omega_{\rm n}}{1-(\omega/\omega_{\rm n})^2}\right) \end{aligned}$$

Frequency response ($X' = x_{\mathrm{st}}|G|$)

$$G(\omega) = \frac{X(\omega)}{x_{\mathrm{st}}} = \frac{1}{(1-(\omega/\omega_{\mathrm{n}})^2) + 2i\xi\omega/\omega_{\mathrm{n}}}$$

Resonance ($\xi < 1/\sqrt{2}$)

$$X_{\mathrm{res}} = \max_{\omega} X' = X'|_{\omega=\sqrt{1-2\xi^2}\omega_{\mathrm{n}}} = \frac{x_{\mathrm{st}}}{2\xi\sqrt{1-\xi^2}}$$

Total undamped solution (transient + harmonic) at resonance

$$x = \frac{x_{\mathrm{st}}}{2}\,\omega_{\mathrm{n}}t\,\sin\omega_{\mathrm{n}}t$$

Unbalanced mass m with eccentricity e, rotating with velocity ω within larger mass $M-m$

$$M\ddot{x} + c\dot{x} + kx = me\omega^2\sin\omega t$$

x is position of big mass, with amplitude (phase unchanged)

$$|X| = e\,\frac{m}{M}\,\frac{(\omega/\omega_{\mathrm{n}})^2}{\sqrt{(1-(\omega/\omega_{\mathrm{n}})^2)^2 + (2\xi\omega/\omega_{\mathrm{n}})^2}}$$

Vibration isolation, transmissibility > 1 for $0 < \omega/\omega_{\mathrm{n}} < \sqrt{2}$, design isolation mount such that $\omega_{\mathrm{n}} < \omega/\sqrt{2}$.
Harmonic motion of support, amplitude ratio as forces in vibration isolation.
Vibration measurement, relative motion as in unbalanced mass.
Periodic excitation $F(t) = F(t+T)$ with period $T = 2\pi/\omega_0$.
Expansion of periodic excitation as (complex) Fourier series, $\omega_p = p\omega_0$.

$$F(t) = \frac{1}{2}A_0 + \mathrm{Re}\left(\sum_{p=1}^{\infty} A_p e^{i\omega_p t}\right)$$

Complex Fourier coefficients

$$A_p = \frac{2}{T}\int_{-T/2}^{T/2} F(t)e^{-i\omega_p t}\,dt, \qquad p = 0, 1, \ldots$$

Particular solution is superposition of harmonics

$$\begin{aligned} x &= \frac{A_0}{2k} + \mathrm{Re}\left(\sum_{p=1}^{\infty}\frac{A_p}{k}\,\frac{e^{i(\omega_p t - \phi_p)}}{\sqrt{(1-(\omega_p/\omega_{\mathrm{n}})^2)^2 + (2\xi\omega_p/\omega_{\mathrm{n}})^2}}\right) \\ \phi_p &= \arctan\left(\frac{2\xi\omega_p/\omega_{\mathrm{n}}}{1-(\omega_p/\omega_{\mathrm{n}})^2}\right) \end{aligned}$$

Resonances, for ω_0 such that $\omega_0 = \omega_{\mathrm{n}}/p$ (when $\xi = 0$).

SDOF - General forced vibrations

Response to general excitation with homogeneous initial conditions.
Impulse load, Dirac delta defined by its action on continuous functions

$$\int_{-\infty}^{\infty} F(t)\delta(t-\bar{t})\,dt = F(\bar{t})$$

Equivalent to initial velocity $= 1/m$. Impulse response

$$g(t) = \frac{1}{m\omega_{\mathrm{d}}}e^{-\xi\omega_{\mathrm{n}}t}\sin\omega_{\mathrm{d}}t, \quad t > 0$$

Step load, Heaviside function

$$H(t-\bar{t}) = \int_{-\infty}^{t}\delta(\tau-\bar{t})\,d\tau = \begin{cases} 0, & t < \bar{t} \\ 1, & t > \bar{t} \end{cases}$$

Step response

$$u(t) = \int_0^t g(\tau)\,d\tau = \frac{1}{k}\left(1 - e^{-\xi\omega_{\mathrm{n}}t}\left(\cos\omega_{\mathrm{d}}t + \frac{\xi\omega_{\mathrm{n}}}{\omega_{\mathrm{d}}}\sin\omega_{\mathrm{d}}t\right)\right)H(t)$$

General response, convolution integrals

$$x(t) = \int_0^t F(\tau)g(t-\tau)\,d\tau = \int_0^t F(t-\tau)g(\tau)\,d\tau = F(0)u(t) + \int_0^t \frac{dF(\tau)}{d\tau}\,u(t-\tau)\,d\tau$$

Shock spectrum, dependence of $x_{\max}/x_{\mathrm{st}}$ on T_0/T for loading characterized by T_0 ($T = 2\pi/\omega_{\mathrm{n}}$).
Truncated ramp

$$\frac{x_{\max}}{x_{\mathrm{st}}} = 1 + \frac{T}{T_0\pi}|\sin(\pi T_0/T)|$$