

A LINEAR THREE-DIMENSIONAL MODEL FOR FREE ELECTRON LASER AMPLIFIERS

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This paper introduces a generalized, linear, three-dimensional model of the FEL amplifier. This 3-D model that is represented by a matrix gain–dispersion equation is valid in the various FEL gain regimes; the low and high gain regimes and the space-charge dominated regimes. It includes electron-beam longitudinal velocity spread effects that are caused by energy spread, transverse emittance and betatron motion. The model provides solutions for the EM-wave amplitude and phase profiles for any initial transverse profiles of the electron beam and of the EM wave, given at the entrance to the interaction region. The matrix gain–dispersion equation consists of angular spectrum components and therefore it is applicable for free-space FEL schemes and for rectangular waveguide schemes as well. Different linear three-dimensional effects, such as optical guiding, gain focusing, reduction of space-charge effects, bending of the radiation beam, off axis-gain, etc., are all inherently included in the presented model. Figurative results, derived according to the parameter sets of two representative high-gain FELs, are demonstrated.

1. Introduction

The FEL interaction is basically a longitudinal effect in which the electron beam bunching and the optical field growth evolve along a main axis, the longitudinal axis. In the standard FEL configuration this axis is common to the three elements that comprise the FEL interaction: the electron beam, the electromagnetic field and the wiggler field. The amplification process evolves along the longitudinal axis as a consequence of the interaction between the electron beam and a longitudinal ponderomotive force that is induced by the combined nonlinear response of the electrons to the wiggler force and the EM wave. The ponderomotive force causes the electron beam to become longitudinally bunched and to transfer a small portion of its energy to the EM wave, which is consequently amplified.

The basic features of the longitudinal FEL interaction process are well described in the linear regime by a 1-D gain–dispersion equation [1] that was analyzed in a previous publication [2]. However, an accurate analysis of the FEL interaction must bring into account transverse effects, that are consequences of the finite transverse dimensions of the e-beam and of the radiation beam. These include optical effects and space-charge effects.

The optical transverse effects result in a transverse modification of the EM wavefront which evolves along the interaction axis. This is caused by the partial filling of the radiation beam cross-section by the active medium, i.e. the bunched electron beam. In the high-gain strong coupling regime this may lead to guiding of the radiation beam inside the electron beam. The guiding effect that would otherwise take place in free-space propagation of a radiation beam, and facilitates high

gain operation of long wiggler FELs. The transverse modification of the EM wave-front can be described by expansion of the resultant radiation field in terms of the free-space transverse modes. This approach that was proposed by Tang and Sprangle [3,4] was adopted in the present work. The optical guiding effect is related to the transverse variation of the complex index of refraction of the interaction medium. This mechanism has recently been a subject for an intensive study [5–11].

Tang and Sprangle [3,4] formalized a 3-D nonlinear theory for the FEL interaction. This model was solved numerically by a simulation code, demonstrating the FEL self-focusing effect. An analytical treatment for the guiding effect in a uniformly distributed electron beam was presented by Moore [7], who defined an analytical condition for the guiding effect at the high-gain regime. The analogy between an optical fiber and the high-gain FEL has been used by Scharlemann, Sessler and Wurtele [6] to demonstrate the guiding phenomena, and to distinguish between *gain focusing* and *refractive guiding* effects. A WKB solution for the transverse mode equation of an FEL that employs an arbitrary shaped electron beam, was introduced by Lucini and Solimeno [9]. Simplified 3-D FEL models were proposed by Amir and Greenzweig [10] and by Xie and Deacon [11]. Both assume a Gaussian variable form for the optical beam, leading to an easily solved set of equations. An observation on optical guiding effects has been published recently [23].

The great interest in the optical guiding effect clearly stems from the desire to realize long FEL systems that will operate at short wavelength in the high-gain regime [20].

Another kind of transverse phenomenon in FELs is the reduction of the space-charge effects due to the

finite transverse dimensions of the e-beam [14]. This leads to transverse components of the electrostatic fields that are produced by the 3-D space-charge waves. Therefore, the space-charge effects that usually impede the FEL interaction, become weaker for a narrow e-beam. This effect is important in the space-charge dominated regimes in cases where the EM wavelength is of the same order, or greater than the e-beam radius. The reduction of space-charge effects may substantially modify the synchronism condition of the FEL interaction in the Raman regime and may increase the FEL gain. Three dimensional aspects of the Raman FEL were investigated by various workers. Freund et al. [15] studied the collective FEL interaction between a cylindrical waveguide mode and a realizable wiggler that consists of a helical field and a uniform axial guide field. Bekefi and Yin [16] studied the collective interaction in a linear waveguide, and recently Steinberg et al. [17] introduced a 3-D collective theory for an arbitrary waveguide shape. They developed a gain–dispersion equation for the interaction between a single waveguide mode and an infinite number of beam modes, and studied the dispersion of plasma waves in a bounded Raman FEL.

The linear 3-D model that is presented here, combines the optical and the space-charge FEL phenomena in a single generalized gain–dispersion equation. The concept of an angular spectrum of plane waves is used, and the transverse dependences of the EM wave and the e-beam profiles are represented by a free-space modal expansion that is easily computed by a discrete Fourier series. Each component represents a plane wave that propagates in a certain angle with respect to the FEL axis and interacts with the wiggling electron beam. In addition, it is coupled to other plane waves and exchanges energy with them. The relation between the two packets of plane waves, the exit and the entrance of the interaction region, is obtained by a gain–dispersion matrix equation.

2. Three-dimensional gain–dispersion equation

The basic linear regime features of the FEL are described by the known 1-D gain–dispersion equation [1,2]:

$$E_x^{(0)} = \left\{ (1 + r\chi_z^{(+1)})(s - ik_{0z}) - i\kappa F\chi_z^{(+1)} \right\}^{-1} \times (1 + r\chi_z^{(+1)})E_{x0} \quad (1)$$

where $E_x^{(0)} = E_x(s)$ is the Laplace transform of the \hat{x} -polarized electric-field component $E_x(z)$, and E_{x0} is its initial value at $z=0$. The function $\chi_z^{(+1)}$ is the

longitudinal susceptibility of the electron beam inside the wiggler and is given by

$$\chi_z^{(+1)} = \chi_z(s + ik_w, \omega)/\epsilon_0 = -\frac{ie^2}{\epsilon_0\omega} \int_{\bar{p}_z=-\infty}^{\infty} \frac{\partial f(\bar{p}_z)/\partial \bar{p}_z}{s + ik_w - i\omega/\bar{v}_z} d\bar{p}_z, \quad (2)$$

where k_{0z} and k_w are the wavenumbers of the single mode EM wave and of the wiggler field, respectively. $f(\bar{p}_z)$ is the e-beam distribution function and $\bar{p}_z = \gamma m \bar{v}_z$ is the longitudinal momentum of the electrons. The coupling parameter κ is defined as

$$\kappa = \frac{\omega V_{wx}^2}{8c^2 \langle \bar{v}_z \rangle}, \quad (3)$$

where V_{wx} is the amplitude of the wiggling motion velocity. The factors r and F are the (single-mode) space-charge reduction factor [17] and the FEL filling factor, respectively.

Using a multimode expansion in the transverse dimension, the three-dimensional version of the gain–dispersion equation (1) is written in a matrix form in the $\{s, \mathbf{k}_\perp, \omega\}$ four-dimensional space, as follows [18]:

$$E_X^{(0)} = \left\{ [\mathbf{I} + \mathbf{R}\chi_z^{(+1)}] \mathbf{K}^{(0)} - i\kappa \mathbf{G}\chi_z^{(+1)} \right\}^{-1} \times [\mathbf{I} + \mathbf{R}\chi_z^{(+1)}] \hat{E}_{X0}. \quad (4)$$

The EM field vectors, $E_X^{(0)}$ and E_{X0} , consists of vector components; each of them represents a transverse spatial Fourier component of the radiation wave at the exit of and at the entrance to the interaction region respectively. Assuming a finite transverse extent of the fields, the \mathbf{k}_\perp spectrum is discrete. Using either periodic or vanishing field boundary conditions in the transverse dimensions, the free-space transverse modes and the corresponding components of the vectors $E_X^{(0)}$ and \hat{E}_{X0} are characterized by mode indices (n, m) , and axial wave-numbers $k_{znm} = (k^2 - k_{\perp nm}^2)^{1/2}$. The unit matrix is denoted by \mathbf{I} , and \mathbf{R} is the space-charge reduction matrix. $\mathbf{K}^{(0)}$ is a diagonal matrix that represents the EM wave propagation constants for the various vacuum modes. It is a pure diagonal matrix and the diagonal terms are $(s^2 + k_{znm}^2)/2s$. The matrix \mathbf{G} is the e-beam coupling matrix. It consists of the transverse spatial Fourier components of the e-beam profile function $g(x, y)$ in an order that is determined from the convolution operation between the spatial Fourier spectra of the e-beam and the radiation beam profiles. The space-charge reduction matrix \mathbf{R} was found to be [18]

$$\mathbf{R} = -\frac{\omega^2}{2 \langle \gamma_z^2 \bar{v}_z^2 \rangle (s + ik_w)} \mathbf{G} \mathbf{K}^{(+1)-1}, \quad (5)$$

where $\mathbf{K}^{(+1)}$ is equal to $\mathbf{K}^{(0)}$, shifted by the transformation $s \rightarrow s + ik_w$ in the complex s -plane.

The matrix gain–dispersion equation (4) is formally similar to the scalar gain–dispersion equation (1). The coupling parameter κ and the susceptibility $\chi_z^{(+1)}$ are the same, and an analogy can be drawn between the adequate terms of both equations. Moreover, the matrix equation (4) is reduced, as expected, to the scalar gain–dispersion equation (1) in the single mode case. The matrix $[\mathbf{I} + \mathbf{R}\chi_z^{(+1)}]$, being the 3-D equivalent of the 1-D term $(1 + r\chi_z^{(+1)})$, represents the dielectric tensor of the electron beam in the wiggler and holds in it the space-charge effects. The gain matrix, $\kappa\mathbf{G}\chi_z^{(+1)}$, measures the strength of the FEL interaction. Its diagonal terms describe the coupling of each vacuum mode with itself. The off-diagonal terms measure the coupling between the different mode and provide the *FEL mode coupling mechanism*.

The order of the 3-D matrix equation, i.e. the number of angular spectrum components that are needed to properly describe the FEL interaction, can be determined by various practical considerations; The highest reasonable order can be simply taken as the number of transverse Fourier components that are needed to properly sample either the e-beam profile or the EM wave profile. Neglect of longitudinally evanescent radiation modes when appropriate and of modes which are out of resonance with the FEL interaction helps to limit the number of EM modes necessary to sample the radiation field, though it does not affect the order of the reduction matrix \mathbf{R} that represents the space-charge waves. In a quasi-free-space scheme, i.e. in a system that is bounded by the wiggler gap or in an over-moded waveguide, one may neglect modes that are out of resonance [13]. The longitudinal wavenumber components $k_{z_{nm}}$ of these modes correspond to detuning parameters $\bar{\theta}_{nm} = (k_{z_{nm}} + k_w - \omega/\bar{v}_z)L$ that are out of the FEL detuning-gain curve $G(\bar{\theta})$, the width of which is $\bar{\theta}^{ac}$, the acceptance detuning parameter [2]. This consideration leads to a simple estimation for the system order N , in terms of the system's transverse dimension a and length L

$$N \cong 2 \frac{a^2}{\lambda_0 L} \frac{\bar{\theta}^{ac}}{\pi}. \quad (8)$$

In a typical FEL system $a \leq \lambda_w$ and the system order can be evaluated by

$$N \leq 4 \frac{\bar{\theta}^{ac}}{\pi N_w} \gamma_z^2.$$

Thus, taking for example a case where $\bar{\theta}^{ac} = 2\pi$ (corresponding to the low gain regime [2]), $\gamma_z = 30$ and $N_w = 100$, only less than ten angular spectral components are needed to properly describe each transverse dimension. The matrix equation in this case is of an order that is smaller than 100 and it can be easily solved by standard library subroutines for complex matrices inversion.

The general kinetic definition of the susceptibility

function $\chi_z^{(+1)}$ (eq. (2)) permits to take into consideration all the quality degradation factors of the e-beam. The normalized distribution function that incorporated the effects of the emittance, the transverse magnetic field gradient and the energy spread on the axial velocity spread, was found to be [19]:

$$f_u(u) = U \exp[U(U+u)] \operatorname{erfc}(U+u), \quad (9)$$

where the normalized variable is defined as $u \cong (\bar{v}_z - \bar{v}_{0z})/(\bar{v}_{0z}\delta_\gamma)$ and the factor U is given by

$$U = \frac{\lambda_\beta}{2} \frac{\delta_\gamma}{\epsilon_b}. \quad (10)$$

The axial velocity spread due to the energy spread contribution is $\delta_\gamma = \Delta\gamma/(\gamma_0^2\gamma_0)$, the emittance is $\epsilon_b = \pi r_b \Delta\phi$, and the period of the betatron motion is λ_β . Therefore, the factor U can be regarded as a ratio between the contributions of the energy spread and the angular spread to the axial velocity spread. Substituting the distribution function (9) into the longitudinal susceptibility integral (2) results in

$$\chi_z^{(+1)} = \frac{\theta_p^2}{(\theta_{th}^{es})^2} 2U \int_{y=0}^{\infty} e^{-2Uy} Z'(\zeta^{(+1)} + y) dy, \quad (11)$$

where the FEL constitutive parameters are the space-charge parameter $\theta_p = \omega'_p/\bar{v}_{0z}$ (ω'_p is the longitudinal relativistic plasma frequency on axis) and the detuning spread parameter (due to energy spread) $\theta_{th}^{es} = (\omega/\bar{v}_{0z})\delta_\gamma$. The *complex error function* is $Z'(z)$ and its argument in eq. (11) includes a complex variable, given by

$$\zeta^{(+1)} = \frac{i\omega/\bar{v}_{0z} - s - ik_w}{(s + ik_w)\delta_\gamma}.$$

The susceptibility integral (11) is asymptotically reduced to the standard plasma susceptibility function for $U \gg 1$ [2], otherwise it can be solved numerically. Following these numerical steps, an inverse Laplace transform is applied to solve the matrix gain–dispersion equation (4) in a similar manner to the 1-D case [2]. This results in a vector that consists of spatial Fourier components of the field at a given distance z . The components can now be summed up as an inverse discrete Fourier transform to yield the EM field complex profile $E_x(x, y, z = L)$. Two examples of results that were obtained by the 3-D gain–dispersion equation (4) are presented in the next section.

3. Transverse profile modification effects

The FEL guiding effect is demonstrated here by solving the 3-D gain–dispersion equation (4) for the parameters sets of two representative FEL experiments, both of which are in the high gain regimes.

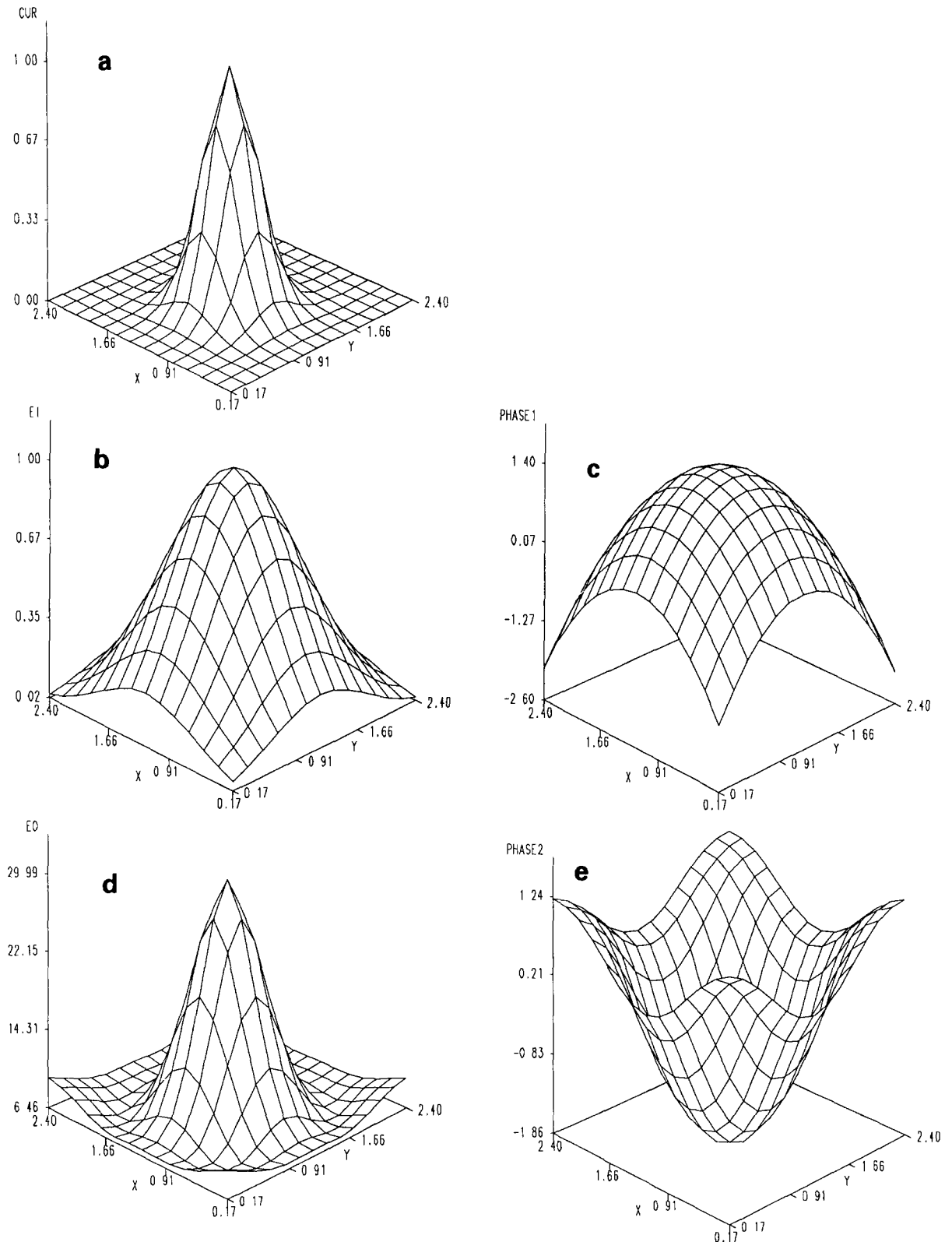


Fig. 1. The Stanford proposed FEL experiments [8,20]: $L_w = 27$ m, $\lambda_w = 0.114$ m, $\lambda = 0.43$ μ m, $\bar{\theta} = -6.0$. (a) Transverse profile of the initial e-beam current density. (b) Transverse profile of the initial EM wave amplitude, $z = 0$. (c) Transverse profile of the initial EM wave phase, $z = 0$. (d) Transverse profile of the initial EM wave amplitude, $z = 27$ m. (e) Transverse profile of the initial EM wave phase, $z = 27$ m.

The first FEL parameter set corresponds to the FEL experiment that has been recently proposed by the Stanford FEL group [8,20]. It employs a storage ring 1 GeV, 270 A e-beam. The wiggler length is 27 m and its period is 11.4 cm. The FEL parameters for this scheme are $\bar{\theta}_p = 1.2$, $\bar{\theta}_{th}^{cs} = 2.4$, $U = 2.4$ and $\bar{a}_w = \gamma V_{wx}/c = 5.4$. The radiation wavelength is $\lambda = 0.43 \mu\text{m}$ and the gain parameter is of the order of 10^3 , which clearly corresponds to operating at the strong-coupling high-gain regime. Figs. 1a–1c show the initial transverse profiles of the e-beam (fig. 1a), the EM wave amplitude (fig. 1b) and the phase (fig. 1c), at the entrance to the wiggler. The initial EM wave amplitude and phase profiles fit to a free-space Gaussian beam of a 0.6 mm waist size at a

2.6 m Rayleigh length. The amplitude and the phase profiles of the EM wave after a 27 m interaction length, which is equivalent to about ten Rayleigh lengths, are shown in figs. 1d and 1e, respectively. The amplitude profile after the interaction is very well confined to the electron beam and it is apparent that both the amplitude profile and the e-beam profile have a similar shape. The phase profile in fig. 1e shows a tendency of a positive diffraction, i.e. the beam is optically defocused, thus the guiding effect is associated in this case to the *gain guiding* mechanism [6]. The gain that was calculated by the semi-analytical model, based on eq. (4), is $G = 576$ and it fits well to the value that was obtained by the FRED simulation [8,20].

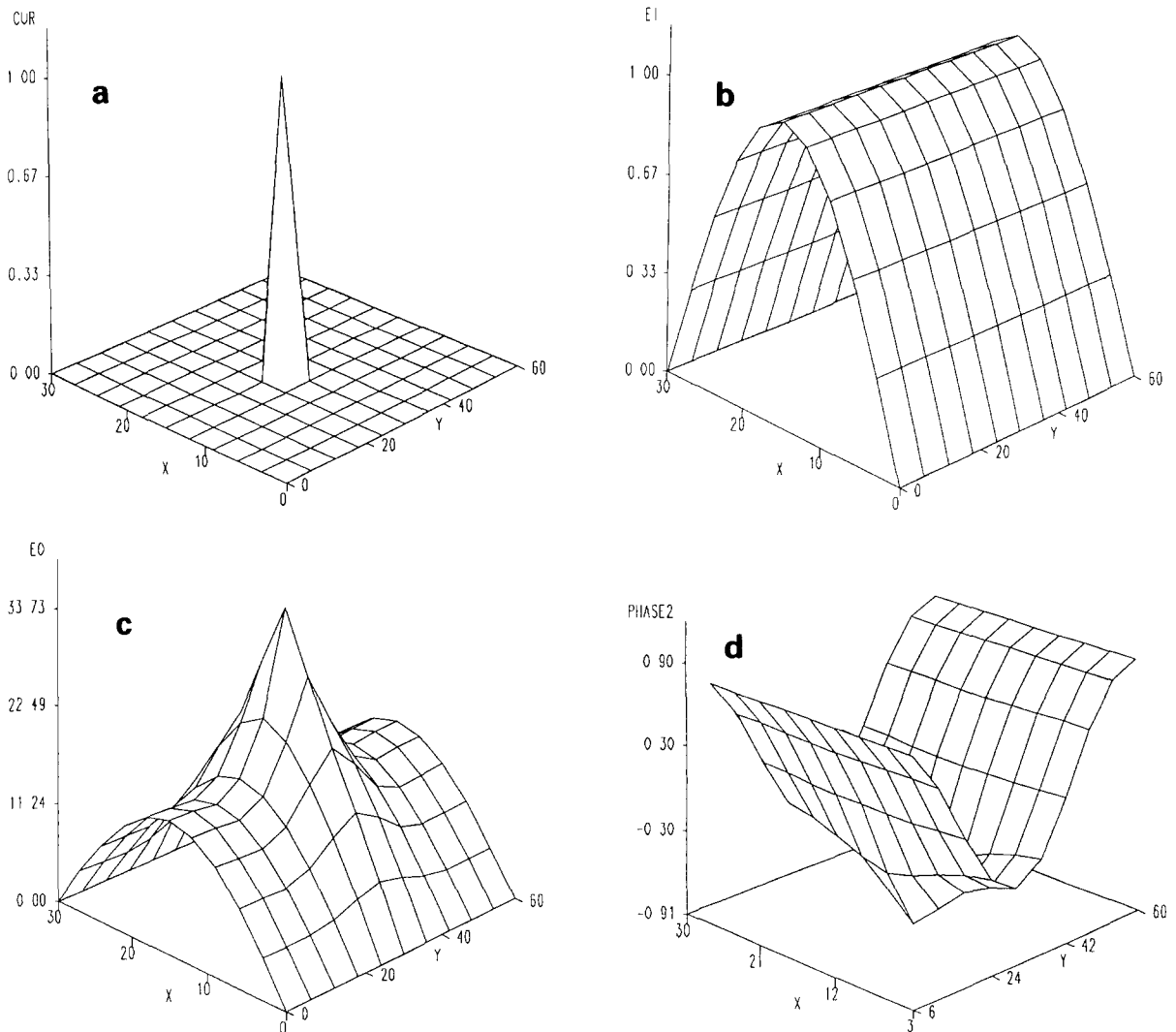


Fig. 2. The LLL ELF experiment [22]: $L_w = 3 \text{ m}$, $\lambda_w = 9.8 \text{ cm}$, $\lambda = 8.67 \text{ mm}$, $\bar{\theta} = -9.0$, $\bar{\theta}_p = 23$. (a) Transverse profile of the initial e-beam current density. (b) Transverse profile of the initial EM wave amplitude, $z = 0$ (c) transverse profile of the initial EM wave amplitude, $z = 1 \text{ m}$. (d) Transverse profile of the initial EM wave phase, $z = 1 \text{ m}$

The other FEL experiment for which we demonstrate the guiding effect by solving eq. (4), is the ELF based experiment, conducted at the Lawrence Livermore Laboratories [22]. The FEL interaction occurs inside a rectangular waveguide of a $6 \text{ cm} \times 3 \text{ cm}$ cross-section. It uses a 3 MeV, 850 A electron beam and a wiggler of 9.8 cm period. The radiation wavelength is 8.6 mm and the space-charge parameter is about $\bar{\theta}_p = 23$. A wiggler parameter value of $\bar{a}_w = 2.4$ was taken in this example.

The e-beam and the EM amplitude profiles are shown in figs. 2a and 2b, respectively. The e-beam has a $6 \text{ mm} \times 3 \text{ mm}$ cross-section and the EM wave is a TE_{01} mode of a rectangular waveguide. After 1 m interaction length, the EM wave profile is no longer the original TE_{01} mode but a packet of transverse modes that compose the profile shown in fig. 2c. It is clear that the growth rate of the EM wave is much stronger in the vicinity of the electron beam than in the rest of the waveguide cross-section. This effect of spatial concentration of the FEL amplification around the e-beam results from the gain guiding effect as in the previous example. The profile of the EM wave amplitude that is shown in fig. 2c can be regarded as a superposition of the original TE_{01} mode and additional, electron beam guided generated radiation. The phase profile after a 1 m interaction length is shown in fig. 2d. The same effect of positive diffraction, i.e. defocusing of the EM wave, occurs here as in the previous example.

The model presented in this article has been applied to other FEL schemes and other effects has been examined as well. A detailed description of the mathematical development of the model and a further discussion on its various aspects, including additional results derived for other FEL schemes, will be presented in a successive publication.

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