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THE AXIAL-VELOCITY DISTRIBUTION FUNCTION AND THE LONGITUDINAL SUSCEPTIBILITY OF AN e-BEAM IN A PLANAR WIGGLER FEL

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This paper introduces a development of a normalized axial velocity distribution function and a susceptibility term for an electron beam in a planar wiggler FEL. The model includes the independent contributions of the energy spread and the angular spread, the emittance and the betatron motion, to the axial velocity distribution function.

In the case of an emittance dominated spread, the resulting distribution function is a skewed, asymmetrical function. The e-beam susceptibility is described in this case by the first derivative of the complex error function (the plasma dispersion function), convolved with a decaying exponent.

The "exact" distribution function and the susceptibility integral that are represented in this paper may be used in any linear, kinetic FEL model to improve its accuracy in cases where the angular spread, the emittance or the betatron motion, are dominant spread sources.

1. Introduction

Kinetic, linear models of the free electron laser interaction [1–4] are usually involved with a susceptibility integral that has a typical form as follows

$$\chi_z(s+\mathrm{i}nk_\mathrm{W},\ \omega) = -\frac{\mathrm{i}e^2}{\omega}\int_{\bar{p}_z=-\infty}^{\infty} \frac{\partial f(\bar{p}_z)/\partial \bar{p}_z}{s+\mathrm{i}nk_\mathrm{W}-\mathrm{i}\omega/\bar{v}_z}\,\mathrm{d}\bar{p}_z, \tag{1}$$

where s and ω are the complex wave number and the angular frequency of the electromagnetic wave, $\bar{p}_z$ and $\bar{v}_z$ are the axial momentum and velocity, averaged over z, of a single electron. The wiggler wave number is k_W and n is the harmonic order. $f(\bar{p}_z)$ is the e-beam distribution function of the averaged axial momentum component of the electrons. In the cold beam limit, $f(\bar{p}_z)=\delta(\bar{p}_z-\bar{P}_{0z})$, the pole $s_0+\mathrm{i}k_\mathrm{W}-\mathrm{i}\omega/\bar{v}_{0z}=0$ represents the full synchronism condition of the FEL interaction, i.e. the phase velocity of the ponderomotive wave $v_p=\mathrm{i}\omega/(s_0+\mathrm{i}k_\mathrm{W})$ equals the e-beam velocity $\bar{v}_{0z}$. In a realistic case, the distribution function $f(\bar{p}_z)$ has a finite width, hence the susceptibility integral represents a *continuous spectra* of synchronism conditions between the EM wave and the various components of the e-beam. The "spreading" of the susceptibility poles due to the finite e-beam distribution function leads to an inhomogeneous broadening of the FEL spontaneous emission and to a reduction of its stimulated amplification [7–13].

Various functions have been chosen for the e-beam longitudinal momentum distribution function $f(\bar{p}_z)$. These are the "window" function [7] and the Lorentzian function [9], both provide easy analytical solutions for the susceptibility integral, and the Gaussian function [3,10,11,13] that leads to the complex error functions representation (the plasma dispersion function) of the susceptibility integral $\chi_z(s,\ \omega)$. These distribution functions are symmetrical and can be useful to describe the energy spread effect, or any other symmetrical spread effect. However, the emittance (the angular spread multiplied by the e-beam width) and the betatron motion [5] are e-beam spread causes that are substantially asymmetrical, i.e. the electron that propagates on-axis determines the maximum value of $\bar{p}_z$ for the longitudinal momentum $f(\bar{p}_z)$ while the other electrons, that are subjected to an angular spread, contribute a tail of lower $\bar{p}_z$ values of the distribution function. This effect of the emittance was demonstrated numerically by Goldstein et al. [12] and has been recently discussed by Colson et al. [13]. The present paper describes in detail a development of expressions for the normalized axial velocity distribution function and the longitudinal susceptibility of an electron beam in a planar wiggler FEL.

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2. The normalized axial-velocity distribution function

A normalized axial-velocity distribution function for an e-beam, propagating in a planar wiggler field, is developed in this section. The axial velocity is considered as a random variable that is a function of two other random variables, the electron energy γmc^2 and the generalized electron off-axis angle ϕ. This axis integrates the spread contributions of the transverse phase-space components φ_{x0}, φ_{y0}, x_0, y_0 and of the betatron motion. The fundamental random variables γ, φ_{x0}, φ_{y0}, x_0, y_0 are considered as independent Gaussian distributed random variables. Using the known formulae of the probability theory, the distribution function of the generalized random variables ϕ and $\bar{v}_z$ are developed.

The average axial momentum $\bar{p}_z$ of an electron can be related to its randomal initial values γ, $\boldsymbol{p}_{0\perp}$ and $\boldsymbol{p}_\beta$, by [5]

$$\bar{p}_z = \sqrt{(\gamma^2-1)m^2c^2 - \overline{|\boldsymbol{p}_{W\perp}|^2} - |\boldsymbol{p}_{0\perp}|^2 - |\boldsymbol{p}_\beta|^2}, \tag{2a}$$

where γmc^2 is the initial energy of the electron, $\boldsymbol{p}_{0\perp}$ is its initial canonical momentum and $\boldsymbol{p}_\beta$ is the transverse momentum component of its betatron motion, all are considered as random variables. The transverse momentum of the wiggling motion, $\boldsymbol{p}_{W\perp}$, is related to the wiggler strength $\bar{a}_W$, by $\bar{a}_W = \sqrt{\overline{|\boldsymbol{p}_{W\perp}|^2}}/mc$, and $\bar{p}_z$ is conserved along z. The canonical angles of the electron trajectory are defined as

$$\varphi_{x0} = \frac{p_{0x}}{p_0}, \quad \varphi_{y0} = \frac{p_{0y}}{p_0}, \tag{2b,c}$$

and eq. (2a) becomes

$$\bar{p}_z = \left[(\gamma^2-1)m^2c^2\left(1-\varphi_{x0}^2-\varphi_{y0}^2-k_{\beta x}^2x_0^2-k_{\beta y}^2y_0^2\right)-\bar{a}_W^2m^2c^2\right]^{1/2}, \tag{2d}$$

where $k_{\beta x}$ and $k_{\beta y}$ are the betatron wave numbers in the $\hat{x}$ and $\hat{y}$ directions, respectively. The variables x_0 and y_0 are the initial off-axis coordinates of the electron. The relation $|\boldsymbol{p}_\beta|^2 = (\gamma^2-1)m^2c^2(k_{\beta x}^2x_0^2 + k_{\beta y}^2y_0^2)$ that is given in ref. [5] was used to obtain eq. (2d). The variables γ, φ_{x0}, φ_{y0}, x_0, y_0 are assumed to be independent random variables and a new generalized random variable that integrates the angular deviation of the electron, is defined as

$$\phi = +\sqrt{\varphi_{x0}^2+\varphi_{y0}^2+k_{\beta x}^2x_0^2+k_{\beta y}^2y_0^2}\,. \tag{3}$$

In general, a distribution function of a random variable z that is a function of two other independent random variables x and y (i.e. $z=g(x, y)$) is related to the distribution functions $f_x(x)$ and $f_y(y)$ as

$$f_z(z) = \sum_{n=1}^{N}\int_x f_x(x)f_y(y_n(x, z))\left|\frac{\partial y_n(x, z)}{\partial z}\right| dx, \tag{4a}$$

where the inverse function, $y(x, z)$, can be in general a multivalued function that satisfies

$$z = g(x, y_1) = g(x, y_2) = \cdots = g(x, y_N). \tag{4b}$$

In practice, the contribution of the energy spread to the axial-momentum spread is much greater than the contribution of the angular spread, since

$$\Delta\phi^2 \ll \Delta\gamma/\gamma, \tag{5a}$$

while for a practical electron beam, the angular spread and the energy spread contributions to the axial-velocity spread are typically of the same order,

$$\Delta\phi^2 \sim \frac{\Delta\gamma}{\gamma_z^2\gamma}. \tag{5b}$$

Hence, in order to properly evaluate the effect of the angular spread, the axial-velocity distribution should be examined. Using eq. (4a), a relation between the axial-velocity distribution and the axial-momentum distribution is obtained as follows

$$f_{\bar{p}_z}(\bar{p}_z) \cong \frac{1+\bar{a}_{\mathrm{W}}^2}{\langle\gamma^3\rangle} f_{\bar{v}_z}(\bar{v}_z), \tag{6a}$$

where the average axial velocity $\bar{v}_z$ is given by

$$\bar{v}_z = \sqrt{(1-\gamma^{-2})c^2(1-\phi^2) - \gamma^{-2}\bar{a}_{\mathrm{W}}^2 c^2}\,. \tag{6b}$$

The axial-velocity distribution is written, using eq. (4a), as

$$f_{\bar{v}_z}(\bar{v}_z) = \int_{\phi=0}^{\pi/2} f_\phi(\phi) f_\gamma(\gamma(\phi, \bar{v}_z)) \left| \frac{\partial\gamma(\phi, \bar{v}_z)}{\partial\bar{v}_z} \right| \mathrm{d}\phi, \tag{7a}$$

and an assumption is taken that the energy spread is described by a Gaussian distribution function

$$f_\gamma(\gamma) = \frac{1}{\sqrt{\pi}\,\Delta\gamma} \exp\left[-\left(\frac{\gamma-\gamma_0}{\Delta\gamma}\right)^2\right]. \tag{7b}$$

Eq. (6b) yields

$$\gamma(\phi, \bar{v}_z) = +\sqrt{\frac{1+\bar{a}_{\mathrm{W}}^2-\phi^2}{1-\phi^2-\bar{v}_z^2/c^2}}\,, \tag{8a}$$

and consequently

$$\frac{\partial\gamma(\phi, \bar{v}_z)}{\partial\bar{v}_z} = \frac{\gamma^3\bar{v}_z}{c^2(1+\bar{a}_{\mathrm{W}}^2-\phi^2)}. \tag{8b}$$

In practice $\Delta\phi^2 \ll 1$, thus eq. (7a) becomes

$$f_{\bar{v}_z}(\bar{v}_z) = \frac{\gamma^3\bar{v}_z}{\sqrt{\pi}\,\Delta\gamma c^2(1+\bar{a}_{\mathrm{W}}^2)} \int_{\phi=0}^{\infty} f_\phi(\phi) \exp\left[-\left(\frac{\gamma(\phi, \bar{v}_z)-\gamma_0}{\Delta\gamma}\right)^2\right] \mathrm{d}\phi, \tag{8c}$$

and eq. (8a) is approximated by

$$\gamma(\phi, \bar{v}_z) = +\sqrt{\frac{1+\bar{a}_{\mathrm{W}}^2}{\bar{\gamma}_z^{-2}-\phi^2}}\,, \tag{9a}$$

where $\bar{\gamma}_z$ is defined as

$$\bar{\gamma}_z \equiv (1-\bar{v}_z^2/c^2)^{-1/2}. \tag{9b}$$

Eq. (9a), expanded to the second order in ϕ^2, yields

$$\gamma \cong \sqrt{1+\bar{a}_{\mathrm{W}}^2}\,(\bar{\gamma}_z + \tfrac{1}{2}\bar{\gamma}_z^3\phi^2 + \tfrac{3}{8}\bar{\gamma}_z^5\phi^4), \tag{10a}$$

and similarly

$$\gamma^2 \cong (1+\bar{a}_{\mathrm{W}}^2)(\bar{\gamma}_z^2 + \bar{\gamma}_z^4\phi^2 + \bar{\gamma}_z^6\phi^4). \tag{10b}$$

The argument $(\gamma-\gamma_0)^2/\Delta\gamma^2$ in eq. (8c) becomes

$$\left(\frac{\gamma-\gamma_0}{\Delta\gamma}\right)^2 \cong \left(\frac{\gamma_0\bar{\gamma}_z^2}{2\,\Delta\gamma}\right)^2\phi^4 + \frac{\gamma_0\bar{\gamma}_z^2}{\Delta\gamma}\phi^2 u + u^2, \tag{10c}$$

where the normalized random variable, u, is defined as

$$u = \left[\sqrt{\frac{1 + \bar{a}_W^2}{1 - \bar{v}_z^2/c^2}} - \gamma_0 \right] / \Delta\gamma. \tag{11a}$$

The energy spread contribution to the axial-velocity spread, i.e. the standard deviation of the axial-velocity distribution due only to the energy spread, is deduced from eq. (6a) and is defined as

$$\delta\gamma = \frac{\Delta\gamma}{\gamma_{0z}^2 \gamma_0}, \tag{11b}$$

where γ_{0z} is given by

$$\gamma_{0z} = \frac{\gamma_0}{\sqrt{1 + \bar{a}_W^2}} \cong \bar{\gamma}_z, \tag{11c}$$

and eq. (8c) is written in the form

$$f_{\bar{v}_z}(\bar{v}_z) = \frac{\bar{v}_z}{\sqrt{\pi}\, c^2 \delta\gamma} \int_{\phi=0}^{\infty} f_\phi(\phi) \exp\left[-\frac{\phi^4}{4\delta\gamma^2} - \frac{\phi^2}{\delta\gamma} u - u^2 \right] \mathrm{d}\phi. \tag{12}$$

The normalized distribution function $f_u(u)$ is related to $f_{\bar{v}_z}(\bar{v}_z)$ by

$$f_u(u) = f_{\bar{v}_z}(\bar{v}_z) \frac{d\bar{v}_z}{du}, \tag{13a}$$

and the derivative $\mathrm{d}\bar{v}_z/\mathrm{d}u$ is obtained from eq. (11a) as

$$\frac{\mathrm{d}\bar{v}_z}{\mathrm{d}u} = \frac{c^2}{\bar{v}_z} \delta\gamma. \tag{13b}$$

The normalized distribution function gets the form

$$f_u(u) = \frac{1}{\sqrt{\pi}} \int_{\phi=0}^{\infty} f_\phi(\phi) \exp\left[-\frac{\phi^4}{4\delta\gamma^2} - \frac{\phi^2}{\delta\gamma} u - u^2 \right] \mathrm{d}\phi. \tag{14}$$

It can be easily verified that in a case of zero emittance, i.e. when

$$f_\phi(\phi) = \delta(\phi), \tag{15a}$$

eq. (14) is reduced to a normalized Gaussian distribution function

$$f_u(u) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-u^2}, \tag{15b}$$

as expected.

An ideal planar wiggler field is described by

$$\boldsymbol{B}_W = B_W [\hat{y} \cosh k_w y \cos k_w z - \hat{z} \sinh k_w y \sin k_w z]. \tag{16a}$$

In a case where no means are taken to focus the e-beam in the x–z plane, one may assume that

$$\Delta\varphi_y \gg \Delta\varphi_x, \quad k_{\beta x} = 0. \tag{16b,c}$$

and therefore eq. (3) becomes

$$\phi \cong \phi_y = +\sqrt{\varphi_{y0}^2 + k_{\beta y}^2 y_0^2}\,. \tag{16d}$$

The distribution function for ϕ_y is given by eq. (9a) as

$$f_{\phi_y}(\phi_y) = 2\int_{-\phi_y}^{\phi_y} f_{\varphi_{y0}}(\varphi_{y0}) f_{y0}\big(y_0(\varphi_{y0}, \phi_y)\big)\left|\frac{\partial y_0}{\partial \phi_y}\right| \mathrm{d}\varphi_{y0}, \tag{17a}$$

where eq. (16d) yields

$$y_0(\varphi_{y0}, \phi_y) = \sqrt{\frac{\phi_y^2 - \varphi_{y0}^2}{k_{\beta y}^2}}, \tag{17b}$$

and consequently

$$\frac{\mathrm{d} y_0}{\mathrm{d}\phi_y} = \pm \frac{\phi_y}{k_{\beta y}\sqrt{\phi_y^2 - \varphi_{y0}^2}}. \tag{17c}$$

It should be noted that the singular point $\phi_y = \varphi_{y_0}$ (due to the electrons with $y_0 = 0$) is considered as a removable singularity for a particle e-beam. Assuming that φ_{y0} and y_0 are both Gaussian-distributed with standard deviations of $\Delta\varphi_{y0}$ and Δy_0, respectively, and both have zero means, eq. (17a) becomes

$$f_{\phi_y}(\phi_y) = \frac{2\phi_y}{\Delta\varphi_{y0} k_{\beta y}\, \Delta y_0} \exp\left(\frac{-\phi_y^2}{k_{\beta y}^2\, \Delta y_0^2}\right) \int_{-\phi_y}^{\phi_y} \left(\phi_y^2 - \varphi_{y0}^2\right)^{-1/2} \exp\left[\varphi_{y0}^2\left(\frac{1}{k_{\beta y}^2 \Delta y^2} - \frac{1}{\Delta\varphi_{y0}^2}\right)\right] \mathrm{d}\varphi_{y0}. \tag{18a}$$

Changing variables, $\varphi_{y0} = \phi_y \sin\alpha$, the integral is represented by

$$f_{\phi_y}(\phi_y) = \frac{2\phi_y}{\Delta\varphi_{y0} k_{\beta y} \Delta y_0} \exp\left[-\left(\frac{1}{\Delta\varphi_{y0}^2} + \frac{1}{k_{\beta y}^2 \Delta y^2}\right)\phi_y^2/2\right] I_0\left(\left(\frac{1}{\Delta\varphi_{y0}^2} - \frac{1}{k_{\beta y}^2 \Delta y^2}\right)\phi_y^2/2\right), \tag{18b}$$

where $I_0(x)$ is the modified Bessel function of order zero, generated by

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(x \sin\theta)\, \mathrm{d}\theta. \tag{18c}$$

It is previously known that the e-beam width, Δy_0, and the angular spread, $\Delta\phi_y$, are optimized with the betatron oscillation wave number, $k_{\beta y}$, when the electron beam is represented on the phase space, y–ϕ_y, by a circle that its radius is conserved along the wiggler axis [5]. The width of the e-beam is given in this case by

$$\Delta y_0 = \frac{\Delta\varphi_{y0}}{k_{\beta y}}. \tag{19a}$$

Consequently the argument of the Bessel function in eq. (18b) becomes zero in this case. Substituting $I_0(0) = 1$, and $k_{\beta y}\Delta y_0 = \Delta\varphi_{y0}$, into eq. (18b), it becomes a Rayleigh density function as follows

$$f_{\phi_y}(\phi_y) = \frac{2\phi_y}{\Delta\varphi_{y0}^2} \exp\left[-\frac{\phi_y^2}{\Delta\varphi_{y0}^2}\right], \quad \phi_y \geq 0. \tag{19b}$$

Substituting eq. (19b) into eq. (14), the normalized distribution function becomes

$$f_u(u) = \frac{2}{\sqrt{\pi}\, \Delta\varphi_{y0}^2} \int_{\phi=0}^{\infty} \phi_y \exp\left[-\frac{\phi_y^4}{4\delta\gamma^2} - \left(\frac{u}{\delta\gamma} + \frac{1}{\Delta\varphi_{y0}^2}\right)\phi_y^2 - u^2\right] \mathrm{d}\phi_y. \tag{19c}$$

Changing variables, $\phi_y^2 = \theta$, results in the integral

$$f_u(u) = \frac{2}{\sqrt{\pi}\,\Delta\varphi_{y0}^2} \int_{\theta=0}^{\infty} \exp\left[-\frac{\theta^2}{4\delta\gamma^2} - \left(\frac{u}{\delta\gamma} + \frac{1}{\Delta\varphi_{y0}^2} \right)\theta - u^2 \right] \mathrm{d}\theta, \tag{19d}$$

of which the standard solution is given by

$$f_u(u) = \frac{\delta\gamma}{\Delta\varphi_{y0}^2} \exp\left[\frac{\delta\gamma}{\Delta\varphi_{y0}^2} \left(\frac{\delta\gamma}{\Delta\varphi_{y0}^2} + 2u \right) \right] \mathrm{erfc}\left(\frac{\delta\gamma}{\Delta\varphi_{y0}^2} + u \right), \tag{20a}$$

where the complementary error function, erfc(p), is defined as

$$\mathrm{erfc}(p) = \frac{2}{\sqrt{\pi}} \int_p^{\infty} \mathrm{e}^{-x^2} dx. \tag{20b}$$

The ratio $U \equiv \delta\gamma/\Delta\varphi_{y0}^2$ gives a measure for the relation of the contributions of the energy spread and the angular spread to the axial-velocity spread. It can be related to the e-beam emittance

$$\epsilon_y = \pi\Delta y_0 \Delta\varphi_{y0} \tag{21a}$$

by the expression

$$U = \frac{\delta\gamma}{\Delta\varphi_{y0}^2} = \frac{\pi}{k_{\beta y}} \frac{\delta\gamma}{\epsilon_y}. \tag{21b}$$

Finally, the normalized distribution function for the axial velocity in a planar wiggler is given simply by

$$f_u(u) = U \exp[U(U+2u)]\ \mathrm{erfc}(U+u). \tag{22a}$$

The normalized variable u, given in eq. (11a), can be approximated to the first order in $\bar{v}_z$ by

$$u \cong \frac{\bar{v}_z - \bar{v}_{0z}}{\bar{v}_{0z}\delta\gamma}; \tag{22b}$$

thus it relates the axial-velocity deviation of an electron to the standard deviation of the axial velocity due only to the energy spread.

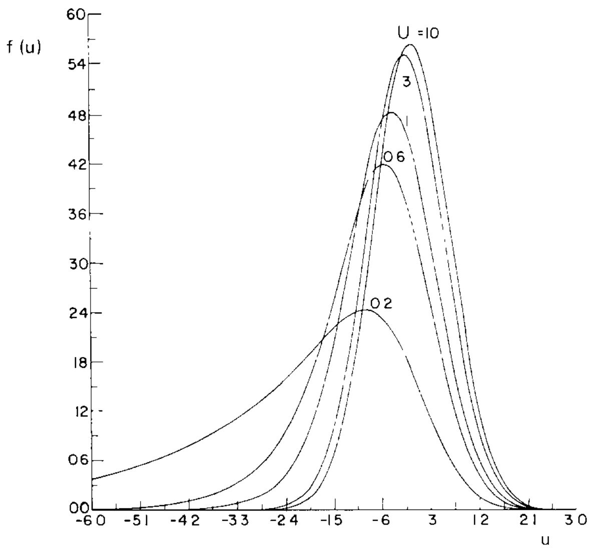

Fig. 1. Curves of the normalized axial velocity distribution function $f_u(u) = U\exp[U(v+2U)]\,\mathrm{erfc}(U+u)$, eq. (22a).

Fig. 1 shows curves of the noralized distribution function $f_u(u)$ for various values of the ratio U. It is shown that for $U \gg 1$, when the energy spread is dominated over the angular spread, the distribution function tends to be a Gaussian, as expected. In the other extreme, for $U \ll 1$ when the angular spread is the dominant spread effect, the normalized distribution function becomes asymmetric. The positive slope of the curve is more moderate than the negative slope and this may further reduce the gain in the warm-gain regime.

3. The e-beam longitudinal susceptibility and the FEL parameters

The longitudinal susceptibility integral eq. (1) is solved in this section, using the normalized distribution function eq. (22a) that brings into account the energy spread, the emittance and the betatron motion effects. The solution of the susceptibility integral leads to the definition of the FEL parameters; the synchronism parameter θ, the space-charge parameter $\bar{\theta}_p$ and the thermal-spread parameters.

The longitudinal susceptibility $\chi_z(s, \omega)$, given in eq. (1), can be written in terms of $\bar{v}_z$, using eq. (6a), as

$$\chi_z(s, \omega) = -\frac{e^2 n_0 (1 + \bar{a}_W^2)}{m\langle\gamma^3\rangle s^2} \int_{\bar{v}_z=-\infty}^{\infty} \frac{\partial f(\bar{v}_z)/\partial \bar{v}_z}{\bar{v}_z - i\omega/s} d\bar{v}_z, \tag{23a}$$

or in a normalized form, using eqs. (13a,b), as

$$\chi_z(s, \omega) = -\frac{e^2 n_0 (1 + \bar{a}_W^2)}{m\langle\gamma^3\rangle s^2 (\bar{v}_{0z}\delta\gamma)^2} \int_u \frac{df(u)/du}{u - \zeta} du, \tag{23b}$$

where the complex variable ζ is defined as

$$\zeta^{(n)} = \frac{i\omega/\bar{v}_{0z} - s - ink_W}{(s + ink_W)\delta\gamma}. \tag{23c}$$

The normalized integral in eq. (23b) is sometimes defined as $G'(\zeta)$ [3] and maintains

$$G'(\zeta) = \int_u \frac{df(u)/du}{u - \zeta} du = \frac{d}{d\zeta} \int_u \frac{f(u)}{u - \zeta} du. \tag{23d}$$

The first derivative of the normalized distribution function $f_u(u)$ (eq. (22a)) is:

$$\frac{d}{du} f_u(u) = 2U^2 \exp[U(U+u)] \operatorname{erfc}(U+u) - \frac{2}{\sqrt{\pi}} U \exp(-u^2). \tag{24a}$$

Curves of $f_u'(u)$ for various values of U are shown in fig. 2. These curves can be related to the FEL gain in the warm-beam limit [3]. Substituting eq. (24a) into eq. (23b) results in

$$\chi_z(s, \omega)/\epsilon_0 = \frac{\omega_p'^2}{s^2(\bar{v}_{0z}\delta\gamma)^2} \left[2U^2 e^{U^2} \int_{x=0}^{\infty} e^{-2Uu} \operatorname{erfc}(U+u)\, du - 2UZ(\zeta) \right], \tag{24b}$$

where ω_p', the relativistic plasma frequency, is defined as

$$\omega_p' = e\sqrt{\frac{n_0(1 + \bar{a}_W^2)}{m\epsilon_0\langle\gamma_0^3\rangle}} \tag{24c}$$

and $Z(\zeta)$ is the complex error function that is also known as the "plasma dispersion function" [6]. It is defined as

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{z=-\infty}^{\infty} \frac{e^{-z^2}}{z - \zeta} dz. \tag{24d}$$

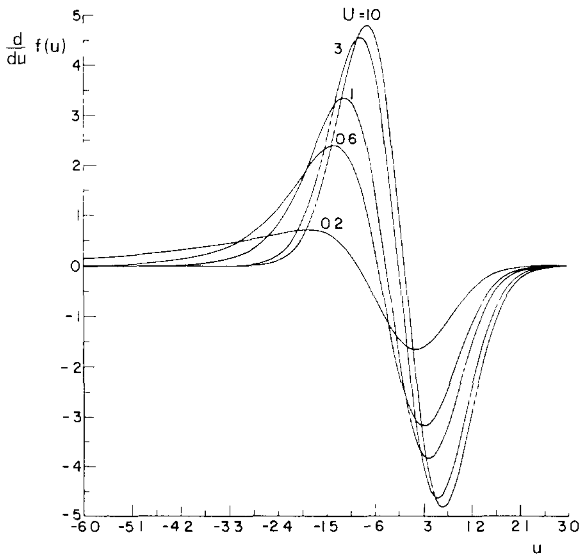

Fig. 2. Curves of $\mathrm{d}f_u(u)/\mathrm{d}u$, eq. (24a), for various U values.

Using the definition for erfc(p) (eq. (20b)) and changing variables, eq. (24b) becomes

$$\chi_z(s,\,\omega)/\epsilon_0=\frac{\omega_\mathrm{p}'^2}{s^2(\bar{v}_{0z}\delta\gamma)^2}\left[2U^2\,\mathrm{e}^{U^2}\int_{u=-\infty}^{\infty}\frac{1}{u-\zeta}\mathrm{e}^{-2Uu}\cdot\frac{2}{\sqrt{\pi}}\int_{y=0}^{\infty}\mathrm{e}^{-(y+U+u)^2}\,\mathrm{d}y\,\mathrm{d}u-2UZ(\zeta)\right] \tag{25a}$$

Changing the order between the integrals over u and y leads to

$$\chi_z(s,\,\omega)=\frac{\epsilon_0\omega_\mathrm{p}'^2}{s^2(\bar{v}_{0z}\delta\gamma)^2}\left[\frac{4}{\sqrt{\pi}}U\int_{y=0}^{\infty}\mathrm{e}^{-2Uy}\int_{u=-\infty}^{\infty}\frac{1}{u-\zeta}\mathrm{e}^{-(y+u)^2}\,\mathrm{d}u\,\mathrm{d}y-2UZ(\zeta)\right]. \tag{25b}$$

The inner integral (over u) is replaced now by $Z(\zeta+y)$ (eq. (24d)), and eq. (25b) becomes

$$\chi_z(s,\,\omega)/\epsilon_0=\frac{\omega_\mathrm{p}'^2}{s^2(\bar{v}_{0z}\delta\gamma)^2}2U\left[2U\int_{y=0}^{\infty}\mathrm{e}^{-2Uy}Z(\zeta+y)\,\mathrm{d}y-Z(\zeta)\right]. \tag{25c}$$

Using the identity

$$Z'(\zeta+y)=\frac{\mathrm{d}}{\mathrm{d}\zeta}Z(\zeta+y)=\frac{\mathrm{d}}{\mathrm{d}y}Z(\zeta+y), \tag{25d}$$

and integrating eq. (25c) by parts, one obtains an expression for the e-beam susceptibility as follows

$$\chi_z(s,\,\omega)/\epsilon_0=\frac{\omega_\mathrm{p}'^2}{s^2(\bar{v}_{0z}\delta\gamma)^2}2U\int_{y=0}^{\infty}\mathrm{e}^{-2Uy}Z'(\zeta+y)\,\mathrm{d}y. \tag{26}$$

In practice, the FEL interaction of an electromagnetic wave with a wiggling e-beam is not strong enough to change the wave number of the EM wave substantially. Thus the s variable in eq. (23c) can be written as [4]

$$s=\mathrm{i}k_{0z}+\mathrm{i}\delta k \tag{27a}$$

where k_{0z} is the longitudinal wave number of the central component of the angular spectrum of the initial EM plane-wave group. It is assumed that

$$|\delta k| \ll k_{0z}, \tag{27b}$$

and therefore the shifted $\zeta^{(n)}$ parameter can be written in terms of the FEL parameters as

$$\zeta^{(n)} = \frac{\theta_n - \delta k}{\theta_{\text{th}}^{\text{e.s}}}, \tag{28a}$$

where θ_n, the detuning parameter, is defined as

$$\theta_n = \omega/\bar{v}_{0z} - k_{0z} - nk_{\text{W}}, \tag{28b}$$

The detuning spread parameter due to the energy spread, $\theta_{\text{th}}^{\text{e.s}}$, is defined as

$$\theta_{\text{th}}^{\text{e.s}} = (k_{0z} + nk_{\text{W}})\delta\gamma, \tag{28c}$$

or in a condition of near synchronism operation, where $\theta_n \sim 0$, as

$$\theta_{\text{th}}^{\text{e.s}} = \frac{\omega}{\bar{v}_{0z}}\delta\gamma. \tag{28d}$$

The detuning parameter, θ_n, measures the synchronism between an electron of which the axial velocity equals the average, $\bar{v}_z$, and the nth harmonic of the ponderomotive wave. The detuning spread parameter, $\theta_{\text{th}}^{\text{e.s}}$, shows the standard deviation of the electrons (due only to the energy spread) from the average detuning value θ_n. Another constitutive parameter of the FEL interaction is the space-charge parameter θ_{p} that is defined as

$$\theta_{\text{p}} \equiv \frac{\omega_{\text{p}}'}{\bar{v}_{0z}} = \sqrt{\frac{e}{m\epsilon_0 c^3}\frac{\left(1+\bar{a}_{\text{W}}^2\right)}{\langle\gamma_0^3\rangle\langle\beta_z^3\rangle}J_0}, \tag{28e}$$

where J_0 is the on-axis e-beam current density.

The FEL parameters, θ_n, $\theta_{\text{th}}^{\text{e.s.}}$ and θ_{p}, multiplied by the interaction length L_{w}, can be interpreted as phase variables with respect to the ponderomotive wave.

The shifted e-beam susceptibility, $\chi_z^{(n)} = \chi_z(s + \text{i}nk_{\text{W}})/\epsilon_0$, that is modulated in the *s-plane* by $+\text{i}nk_{\text{W}}$, is written in terms of the FEL parameters θ_{p}, $\theta_{\text{th}}^{\text{e.s.}}$, U and $\zeta^{(n)}(\theta_n, \theta_{\text{th}}^{\text{e.s}})$ as

$$\chi_z^{(n)} = \frac{\theta_p^2}{(\theta_{\text{th}}^{\text{e.s}})^2} 2U\int_{y=0}^{\infty} \text{e}^{-2Uy} Z'(\zeta^{(n)} + y)\,\text{d}y. \tag{29}$$

One can easily verify that in a case that $U \gg 1$, i.e. in a case that the angular spread is small with respect to the symmetrical energy spread, eq. (29) is reduced to the known expression [3]

$$\chi_z^{(n)} = \frac{\theta_{\text{p}}^2}{(\theta_{\text{th}}^{\text{e.s}})^2} Z'(\zeta^{(n)}) \tag{30a}$$

In the cold beam limit, $\delta\gamma \to 0$, $\theta_{\text{th}}^{\text{e.s}} \to 0$, and $\zeta \to \infty$ (for any $s \neq \text{i}\omega/\bar{v}_{0z}$). The infinite limit of the complex error function (24d) is $Z(\zeta \to \infty) = -1/\zeta$ and the susceptibility term is further simplified to

$$\chi_z^{(n)} = -\frac{\theta_{\text{p}}^2}{(\theta - \delta k)^2}. \tag{30b}$$

The effect of the asymmetrical contribution of the angular spread is seen in the susceptibility integral (29). This can be regarded as a convolution of the known $Z'(\zeta)$ solution, with a decaying exponent e^{-2Uy} that represents the angular spread effect, as a tail spreading of the e-beam susceptibility.

The ratio $\theta_p^2/(\theta_{th}^{es})^2$ can be related to be Debay wave number which is defined as

$$k_D = \sqrt{2}\,\frac{\omega_p'}{(\bar{v}_{0z}\delta\gamma)}; \tag{31}$$

thus the coefficient $\theta_p^2/(\theta_{th}^{es})^2 = (k_D/(k_{0z}+k_W))^2$, appearing in eqs. (29) and (30a), can be regarded as the ratio between the ponderomotive wave wavelength and the e-beam plasma shielding radius.

The susceptibility integral eq. (30) can be easily computed by any standard integration method. The convergence of the decaying exponent e^{-2Uv} is determined by U and is typically fast. The computation of $Z'(\zeta)$ is based on its asymptotic expansions and continued fraction representations [6] and is usually provided by computer mathematical libraries. The evaluation of the susceptibility term $\chi_z^{(n)}$ enables obtaining solutions for the gain dispersion equation [11] and for its higher-harmonics and three-dimensional versions [4].

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