Solitary Waves in Flexible, Arbitrary Elastic Helix

By Leonid Slepyan,[1] Viacheslav Krylov,[2] and Philip Rosenau[3]

Abstract: The traveling solitary wave was recently discovered to be a very stable object existing in an inextensible, flexible helical fiber. In the present work, a flexible helical fiber with an arbitrary stress-strain relation is considered, and a general, analytical, steady-state solution of the 3D nonlinear vector equation is found. This solution describes a new class of spatial motion of an elastic string, which is shown to be a waveguide for subsonic solitary waves. The results confirm that the known solution for an inextensible fiber is a low-velocity or low-energy asymptote of the solution presented here. Another type of asymptotic solution derived here corresponds to a high-energy wave. In this case, when the amplitude of the wave increases, the wave speed and energy tend to infinity, and the effective wavelength tends to zero.

INTRODUCTION AND BACKGROUND

An elastic, flexible fiber of mass density ρ per unit length is considered whose equation of motion is

$$\left[F(\Lambda)\frac{\mathbf{R}'}{\Lambda}\right]' = \rho\ddot{\mathbf{R}}; \quad \Lambda = |\mathbf{R}'| \tag{1}$$

Here, F is a nonnegative tensile force and $\mathbf{R}$ is the position vector [see, for example, Weinberger (1965)]. The primes and dots appearing in this equation denote derivatives with respect to the coordinate along the fiber S and time t, respectively. Lagrange's variables are used, so the modulus $|\mathbf{R}'|$ is the stretch of the string. The function $F(\Lambda)$ is assumed to be invertible. In particular, a positive constant, c_*, is assumed to exist such that

$$\frac{dF}{d\Lambda} > \rho c_*^2 \tag{2}$$

This is the only restriction on the elasticity law of the fiber. Also, this inequality signifies stability of the fiber under extension. Thus, a general, geometrically and physically nonlinear, stable-under-extension fiber (or string) is considered.

The fibers treated here have no bending stiffness, so any curve can be its natural state. However, the solitary-wave solution derived in this paper corresponds to a helical natural state (the helical shape of the fiber in front of the wave is restored after the wave has passed, but axial and angular shifts arise). Note that the model of the string that does not possess bending stiffness is a long-wave approximation for a flexible rod (cable, pipe, etc.).

It may be noted that helical systems are relevant to a wide variety of fields in which coiled structures are important, from the modeling of macromolecules such as DNA [Austin et al. (1997) and references therein] to deployable structures for satellite applications (Beletsky and Levin 1990; Penzo and Ammann 1996). Various mechanical models of helical cables, ropes, etc., are considered in a comprehensive review by Cardou and Jolicoeur (1997). The dynamics of helical fiber is related to the motion of vortex filament in fluids (Hasimoto 1971; Fukumoto and Miyazaki 1991) and to the textile yarn manufacturing processes (Stump and Fraser 1996). Helical fibers are usable as reinforcements of composite materials (Kagawa et al. 1982). Such a fiber (or a helical-fiber composite) can also be used as an energy absorber under dynamic action (Cherkaev and Slepyan 1995).

Eq. (1) describes the nonlinear coupling between axial and transverse motions of the string, and only in the simplest case of a constant tensile force, $F = F_0$, is it satisfied by the well-known Dalambert solution in the form $\mathbf{R} = \mathbf{R}(s - vt)$ or $\mathbf{R} = \mathbf{R}(s + vt)$, under the condition $|\mathbf{R}'| = \text{const}(F(|\mathbf{R}'|) = F_0)$ in which $v = \sqrt{F_0/\rho}$. Note that even in this simplest case the problem remains nonlinear; the superposition principle is not valid.

There is a large body of works devoted to the nonlinear dynamics of elastic strings. Because the solution of (1) presents serious mathematical difficulties, various approximate models were developed to take into account the coupling between the axial tension in the string and the transverse displacements. Probably the first approximate model of such a coupling was introduced by Kirchhoff (1883). Among other models, note the approximation by Carrier (1945). The latter was then studied analytically by Oplinger (1960) and Narasimha (1968) and numerically by Leissa and Saad (1994). For an extended reference list and historical notes, see, for example, Antman (1995). A different formulation of the equation of string dynamics can be found in Antman (1980); group properties of the equation were studied by Peters and Ames (1990).

There are a few works that present exact solutions of (1) for special conditions. Keller (1959) found an exact solution of (1) for cases in which the tension is assumed to be a linear function of a stretch $F = a\Lambda$. It was shown that only in this case can the string perform purely transverse motion. Lee and Ames (1973) have described a class of exact solutions under the assumption of a specific connection between the transverse velocity and the inclination of the string. In this case, the equation for the tension is independent of all the other variables. The drawing and whirling of a string were described by Antman and Reeken (1987). Eq. (1) was considered by Rosenau and Rubin (1986), where the specific cases of time-independent and coordinate-independent tensions were studied. The possibility of launching periodical waves in the plane coiled string (i.e., for a helix with a zero pitch) was noted by Rosenau (1987). Kinematical conditions of steady motion of a string were recently formulated by Nordenholz and O'Reilly (1995).

The first exact, nonlinear solution describing a wave in a helical fiber (assumed to be inextensible) is presented in Slepyan et al. (1995a), with some numerical results given in Slepyan et al. (1995b). It was shown that a solitary wave can propagate along the helical fiber as a very stable object. The complete description of various types of periodic and solitary waves for a more general case of a helix rotating around its axis was derived by Krylov et al. (1998). An extraordinary nonstationary binary wave that arises in a helical thread under

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axial dynamic tension was described in Krylov and Slepyan (1997).

In the present paper, it is shown that extensibility does not prevent the existence of a solitary wave. However, it does bound the wave velocity; only subsonic solitary waves can exist. The results confirm that the solution for an inextensible fiber is a low-velocity or low-energy asymptote of the solution presented here.

SOLUTION

Let us note that a general relation between the force and the position vector follows immediately from (1), as its projection onto the tangent line yields

$$\Lambda(F)F' = \rho\ddot{\mathbf{R}}\cdot\mathbf{R}' \tag{3}$$

This relation is valid for a fiber of arbitrary shape. It is shown below that in the case of a steady-state solitary wave in a helix, (3) can be reduced to a relation between the force and radius of the deformed helix.

Consider a helix of an initial radius R_0, and let γ denote the initial angle between the fiber and the axis of the helix, x. Using the same representation of the steady-state wave as in Slepyan et al. (1995a), one notes that, to an observer moving along the helical fiber with a speed v and an associated angular velocity about the x-axis, $v \sin \gamma/R_0$ (with an orthogonal triad natural to the helix), the initial geometry appears invariant; hence, a solitary wave is expected to exist as a steady-state solution in a coordinate system attached to the moving observer.

Letting $\mathbf{R}(S, t)$ be represented as the sum of the longitudinal vector, $\mathbf{R}_x(S, t)$, and a vector $\mathbf{R}_1(S, t)$ perpendicular to the x-axis, the following nondimensional quantities are introduced:

$$\mathbf{p} = \mathbf{R}_x/R_0; \quad \mathbf{r} = \mathbf{R}_1/R_0; \quad s = S/R_0; \quad f = F/\rho v^2; \quad \tau = vt/R_0 \tag{4}$$

and vectors $\mathbf{p}$ and $\mathbf{r}$ are expressed as follows:

$$\mathbf{p}(s, \xi) = [s \cos \gamma + u(\xi)]\mathbf{k}_x; \quad \lambda = \sin \gamma; \quad \xi = s - \tau \tag{5a}$$

$$\mathbf{r}(s, \xi) = \mathrm{Re}\Omega\mathbf{k}_y + \mathrm{Im}\Omega\mathbf{k}_z; \quad \Omega = A(\xi)e^{i\lambda s} = B(\xi)e^{i\lambda\tau} \quad (B = Ae^{i\lambda\xi}) \tag{5b}$$

where x is the axis coordinate; y and z are the orthogonal coordinates in the normal plane; $\mathbf{k}_x$, $\mathbf{k}_y$, and $\mathbf{k}_z$ are the unit vectors; and $u(\xi)$ is the axial displacement.

In the case $A = 1$, representation (5) describes an unmoving helix, whereas the case $A = A(\xi)$ corresponds to a wave propagating along the helical fiber. A generalization of the problem for a rotating helix was considered for both an inextensible helix (Krylov et al. 1998) and an extensible helix (Krylov and Rosenau 1996). Note that the present paper was completed before the paper by Krylov and Rosenau (1996) and is referred to therein.

For the conversions below, it is convenient to use the complex representation, Ω, of vector $\mathbf{r}(s, \xi)$ defined by (5). Note that the scalar product of two coplanar vectors can be expressed in terms of their complex representations, Ω_1 and Ω_2, as $\mathrm{Re}(\Omega_1\bar{\Omega}_2)$ as follows immediately from (5).

For $v > 0$, the conditions at infinity in front of the wave are imposed to correspond to the initial shape of the unmoving helix, i.e.

$$(u, u', A', f) \to 0; \quad A \to 1 \quad (\xi \to +\infty, \tau \geq 0) \tag{6}$$

Here and below, primes and dots denote derivatives with respect to the nondimensional coordinate s and time τ, respectively.

Based on the representations given in (5), the components of the acceleration vector can now be expressed as follows:

$$\ddot{\mathbf{p}} = \mathbf{p}''; \quad \ddot{\mathbf{r}} = \mathbf{r}'' - 2i\lambda\mathbf{r}' - \lambda^2\mathbf{r} \tag{7}$$

This allows extension of the fiber to be expressed in terms of the deformed helix radius. In fact, substitution in (3) leads to

$$\Phi' - \frac{1}{2}(\Lambda^2)' = -\frac{\lambda^2}{2}(r^2)'; \quad \Phi = \int_0^f \Lambda\, df \tag{8}$$

where $r^2 = |\mathbf{r}|^2$. Eq. (8) and condition (6) yield the relation with $\Lambda = 1$ at infinity, where $r^2 = 1$

$$U(f) \equiv \Phi - \frac{1}{2}(\Lambda^2 - 1) = \frac{\lambda^2}{2}(1 - r^2) \tag{9}$$

As follows from (2) for the range $0 < v^2 < c_*^2$:

$$\frac{df}{d\Lambda} > 1 \tag{10}$$

and hence, the derivative of U is positive

$$\frac{dU(f)}{df} = \Lambda\left(1 - \frac{d\Lambda}{df}\right) > 0 \quad (\Lambda \geq 1) \tag{11}$$

At the same time, local sound velocity in the fiber

$$c = \sqrt{\frac{1}{\rho}\frac{dF}{d\Lambda}} = v\sqrt{\frac{df}{d\Lambda}} \tag{12}$$

It can now be seen that condition (10) means that $v < c$. Thus, in the case of a subsonic wave, the derivative, dU/df, is positive; hence, there is a one-to-one correspondence between the force and the radius of the deformed helix [see (9)].

Let us represent relation (9) in the form

$$f = \frac{1}{2}(\Lambda^2 - 1) - \int_1^\Lambda (\Lambda - 1)\frac{df}{d\Lambda}\, d\Lambda + \frac{\lambda^2}{2}(1 - r^2) \tag{13}$$

Inequality (10) allows us to make the estimation

$$f \leq \Lambda - 1 + \frac{\lambda^2}{2} < \Lambda \tag{14}$$

Furthermore, under condition (10), $U \geq 0$; hence

$$r^2 \leq 1 \tag{15}$$

In the particular case of inextensibility when $\Lambda \equiv 1$ and $\Phi \equiv f$, (9) yields

$$f = \frac{\lambda^2}{2}(1 - r^2) \tag{16}$$

In the case of linear extensibility when

$$\Lambda = 1 + M^2 f; \quad \Phi = f + \frac{M^2}{2}f^2 \quad \left(M = \frac{v}{c}\right) \tag{17}$$

the nondimensional internal force is

$$f = M^{-2}\left[\sqrt{1 + \frac{M^2\lambda^2(1 - r^2)}{1 - M^2}} - 1\right] \tag{18}$$

Note that (16) follows from (18) as a limit of $M \to 0$. In an inextensible helix, the wave velocity is unbounded. If extensibility is taken into account, the wave velocity is bounded. The upper bound is independent of the helix parameter λ in the case of the linear extensibility, and it depends on λ for a nonlinear elastic law. Consider, for example, the constitutive relation $F = F_0\sqrt{\Lambda - 1}$. Then

$$f = \frac{\sqrt{\Lambda - 1}}{\omega}; \quad \Phi = f + \frac{1}{3}\omega^2 f^3; \quad c^2 = \frac{F_0}{2\rho\sqrt{\Lambda - 1}}; \quad \omega = \frac{\rho v^2}{F_0} \tag{19}$$

and the condition $v^2 < c^2$ leads to the inequality

$$\omega < \frac{1}{2\sqrt{\Lambda_{max} - 1}} \tag{20}$$

It now follows from (11) that

$$\Lambda^2 + 4\Lambda - 5 - 3\lambda^2 < 0; \quad \Lambda_{max} = \sqrt{9 + 3\lambda^2} - 2 \tag{21}$$

Let us return to the general case. For a given λ, the internal force f, and therefore the stretch Λ, depend on r^2 only, and the ratio f/Λ can be denoted as a function of r^2

$$\frac{f}{\Lambda} = G(r^2); \quad G \to 0 \quad \text{when} \quad r \to 1 \quad (s \to \infty) \tag{22}$$

This function is defined by a given elastic law, $\Lambda(F)$, and the derived equation, (9). It can be seen from (14) that $G < 1$.

Substitution of (7) in (1) [see (22)] leads to

$$[G(u' + \cos\gamma)]' = u'' \tag{23}$$

and

$$(G\mathbf{r}')' = \mathbf{r}'' - 2i\lambda\mathbf{r}' - \lambda^2\mathbf{r} \tag{24}$$

From (23) and (6), one now has

$$u' = \frac{G\cos\gamma}{1 - G} \tag{25}$$

In solving the 2D vector equation, (24), let us represent the vector $\mathbf{r}$ by means of complex representation, (5), with

$$\Omega = re^{i\phi} \tag{26}$$

where $r(\xi)$ and $\phi(\xi, s)$ are real functions.

As shown below, the fiber crosses the axis of the helix at the point of maximum tensile force; we define this point to be $\xi = 0$. Furthermore, we find it convenient to define $r = |\mathbf{r}|$ for $\xi \geq 0$ and $r = -|\mathbf{r}|$ for $\xi \leq 0$. In particular, we note that in the initial geometry, ϕ is given by the linear relation $\phi = s\lambda$, and under condition (6)

$$r \to 1; \quad \phi \sim \lambda s \ (\xi \to +\infty) \tag{27}$$

Substituting representation (26) into (24), multiplying by $e^{-i\phi}$ and separating the real and imaginary parts, one obtains

$$(1 - G)r\phi'' + 2\left[(1 - G)r' - \frac{dG}{d(r^2)}r^2r'\right]\phi' = 2\lambda r' \tag{28}$$

and

$$[(1 - G)r']' - (1 - G)r(\phi')^2 + 2\lambda r\phi' - \lambda^2 r = 0 \tag{29}$$

Eqs. (28) and (29) are satisfied by the initial shape of the helix ($r = 1$, $\phi' = \lambda$). However, as we now show, these equations also lead to a solitary wave solution, just for inextensible strings, as was shown by Slepyan et al. (1995a). To this end, we first consider (28), which is linear and of the first order in $\psi \equiv \phi'$. Under the condition at infinity, $\phi' = \lambda$, (28) possesses a solution

$$\psi \equiv \phi' = \frac{\lambda}{1 - G} \tag{30}$$

Substitution of this last result into (29) yields a second-order equation with respect to r

$$[(1 - G)r']' + \frac{\lambda^2 rG}{1 - G} = 0 \tag{31}$$

Multiplying this equation by $(1 - G)r'$ and integrating the resulting expression, one obtains

$$(r')^2 = \frac{\lambda^2}{[1 - G(r^2)]^2}\int_{r^2}^{1} G(r^2)\,dr^2 \quad (r' = 0 \quad \text{when} \quad r = 1) \tag{32}$$

It can be seen that the derivative r' is nonnegative at the point $r = 0$ ($r > 0$ if $\xi > 0$)

$$r'(0) = \frac{\lambda}{[1 - G(0)]}\left(\int_0^1 G(a)\,da\right)^{1/2} > 0 \tag{33}$$

The phase portrait associated with (31) is shown in Fig. 1. The upper part corresponds to the wave propagating to the right (this wave is considered here), and the bottom part corresponds to the wave propagating to the left.

Thus, the radius of the deformed helix as a function of ξ is defined by the relation

$$\int_0^r \left\{\frac{\lambda^2}{[1 - G(r^2)]^2}\int_{r^2}^1 G(a)\,da\right\}^{-1/2} dr = \xi \tag{34}$$

The axial displacement, u, and the angle, ϕ, can be expressed as follows [see (25) and (30)]:

$$u(\xi) = -\int_\xi^\infty \frac{G}{1 - G}\,d\xi\,\cos\gamma;$$

$$\phi(\xi) = -\int_\xi^\infty \frac{G}{1 - G}\,d\xi\,\sin\gamma + s\sin\gamma \tag{35}$$

Note that, as follows from the derived relations, the variables $r(\xi)$, $u(\xi) - u(0)$, and $\theta(\xi) - \theta(0)$ ($\theta = \phi - \lambda s$) are antisymmetrical, monotonically increasing functions, and the internal force, $f(\xi)$, is a symmetrical function that monotonously decreases and tends to zero with an increasing $|\xi|$.

We now have to show that the wave is truly solitary and that the integrals are convergent. With this in mind, we consider the asymptotic expressions for the helix radius and the tensile force. Taking into account that

$$f \to 0; \quad \varepsilon \to 0; \quad \Phi = O(f\varepsilon) \quad (r^2 \to 1) \tag{36}$$

the following asymptotic expression of (9) is obtained:

$$f \sim G \sim \frac{\lambda^2(1 - r^2)}{2} + \varepsilon \sim \frac{\lambda^2(1 - r^2)}{2(1 - v^2/c_0^2)}; \quad c_0 = \lim_{\varepsilon \to 0} c \tag{37}$$

Now, as follows from (34)

$$1 - r^2 \sim C\exp\left(-\frac{\lambda^2|\xi|}{\sqrt{1 - v^2/c_0^2}}\right) \tag{38}$$

where C is a constant. Thus, the intensity of the wave decreases exponentially when $|\xi| \to \infty$, which is the required result.

We now consider the case of linear extensibility in more detail. In this case, the use of expressions (17) and (18) in (22) and then in (34) leads to the following relation between the

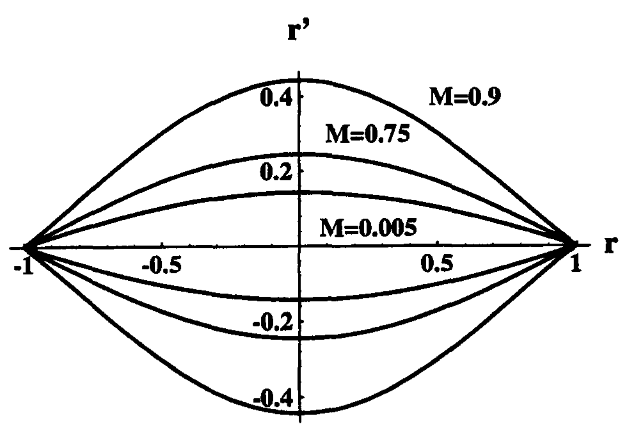

FIG. 1. Phase Portrait

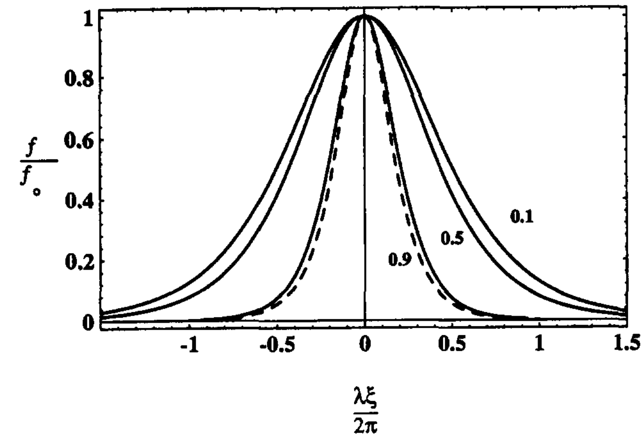

FIG. 2. Nondimensional Tensile Force

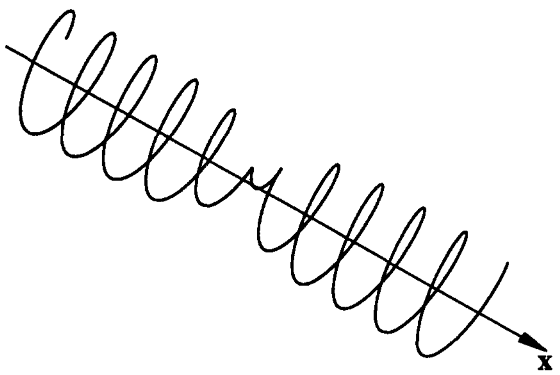

FIG. 3. Shape of Deformed Fiber

radius of the helix and the coordinate ξ (recall that $r = |\mathbf{r}|\xi/|\xi|$):

$$\ln\frac{(1+r)(\Lambda+r)}{(1-r)(\Lambda-r)} - \frac{2\lambda}{M}\sqrt{1-M^2}\arcsin\left(\frac{\lambda r M}{\sqrt{1-M^2+M^2\lambda^2}}\right)$$

$$= \frac{2\lambda^2\xi}{\sqrt{1-M^2}};\quad \Lambda = \sqrt{1+\frac{M^2\lambda^2(1-r^2)}{1-M^2}};\quad M = \frac{v}{c} \tag{39}$$

Asymptotic representations [$v \rightarrow c$ ($v < c$)] follow from (39) and (18):

$$r \sim \tanh\frac{\lambda^2\xi}{\sqrt{1-M^2}};\quad f \sim f_0\left(\cosh\frac{\lambda^2\xi}{\sqrt{1-M^2}}\right)^{-1};$$

$$f_0 \sim \frac{\lambda}{\sqrt{1-M^2}} \tag{40}$$

where $f_0 = f$ at the peak of the wave ($\xi = r = 0$). As a consequence of (18), the asymptotic expression (40) for the force is valid under the condition $\lambda^2(1-r^2)/(1-M^2) >> 1$. The strain and kinetic energies of the wave are asymptotically the same, and the total energy is

$$W = \frac{\rho v^2}{2}R_0\int_{-\infty}^{\infty}(M^2f^2 + \dot{u}^2 + \dot{r}^2 + r^2\dot{\phi}^2)ds \sim \frac{2\rho c^2 R_0}{\sqrt{1-M^2}} \tag{41}$$

Thus, with an increase in wave speed, the effective wavelength tends to zero, and, at the same time, the energy of the wave increases unboundedly. The ratio f/f_0 for $\lambda = 1/2$ is presented in Fig. 2 for several values of v/c. The dotted curve corresponds to the asymptotic solution (40). The helical fiber deformed by the propagating solitary wave is shown in Fig. 3.

The derived results confirm that the solution for an inextensible fiber, Slepyan et al. (1995a), is a low-velocity or low-energy asymptote of the solution presented here.

In conclusion, let us consider whether a supersonic steady-state wave can exist. In a general case, the local sound velocity $c = c(\Lambda)$, and if such a wave does exist, the inequality $v > c$ can be valid within a range of the stretch, $\Lambda_1 < \Lambda < \Lambda_2$, where $\Lambda_1 \geq 1$, $\Lambda_2 \leq \infty$. Within this range, as follows from relations (9)–(12), $d\Lambda/df > 0$, $dU/df < 0$, and $f'r' \geq 0$. In the case $\Lambda_1 = 1$, at any point within the interval $1 < \Lambda < \Lambda_2$, the tension $f > 0$, $U < 0$, and hence, $r > 1$ ($r = 1$ at $s = +\infty$ where $\Lambda = 1$). However, this conflicts with expression (32), the right-hand part of which becomes negative. In the other case, when $\Lambda_1 > 1$, the derivative r' changes its sign when Λ crosses the point Λ_1. But this, again, is inconsistent with (32), because r' must be continuous, as can be seen in (31). Thus, in the considered system, supersonic steady-state waves cannot exist.

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