Mechanics of Time-Dependent Materials **3:** 159–203, 1999.

A Lattice Model for Viscoelastic Fracture

L.I. SLEPYAN
Department of Solid Mechanics, Materials and Structures, Tel Aviv University, 69978 Israel; E-mail: leonid@eng.tau.ac.il; and Department of Civil and Environmental Engineering, Clarkson University, Potsdam, NY 13699-5710, U.S.A.

M.V. AYZENBERG-STEPANENKO
Institute for Industrial Mathematics, 4 Yehuda Hanachtom, Beer-Sheva, 84249 Israel

J.P. DEMPSEY
Department of Civil and Environmental Engineering, Clarkson University, Potsdam, NY 13699-5710, U.S.A.; E-mail: john.dempsey@clarkson.edu

(Received 17 August 1998; accepted in revised form 12 August 1999)

Abstract. A plane, periodic, square-cell lattice is considered, consisting of point particles connected by mass-less viscoelastic bonds. Homogeneous and inhomogeneous problems for steady-state semi-infinite crack propagation in an unbounded lattice and lattice strip are studied. Expressions for the local-to-global energy-release-rate ratios, stresses and strains of the breaking bonds as well as the crack opening displacement are derived. Comparative results are obtained for homogeneous viscoelastic materials, elastic lattices and homogeneous elastic materials. The influences of viscosity, the discrete structure, cell size, strip width and crack speed on the wave/viscous resistances to crack propagation are revealed. Some asymptotic results related to an important asymptotic case of large viscosity (on a scale relative to the lattice cell) are shown. Along with dynamic crack propagation, a theory for a slow crack in a viscoelastic lattice is derived.

Key words: asymptotics, clamped strip, cohesive-zone models, dynamic, fracture, Mode III, quasi-static, square-cell, steady-state, viscoelastic lattice

Nomenclature

a	= the bond length
A_e	= $a\mu\varepsilon^2(+0)/2 =$ elastic energy of a broken bond
A_v	= total energy of a broken bond
A_0	= $a\sigma(+0)\varepsilon(+0)/2 =$ effective elastic energy of a broken bond
c	= the long shear wave speed
C_α	= $\alpha c/a =$ the nondimensional parameter of viscosity
E	= $(1+ikV_\alpha)(1+ikV_\beta) =$ the complex modulus
G	= the global energy release rate as an energy flux from infinity
G_e	= A_e/a, $G_v = A_v/a$, $G_0 = A_0/a =$ local energy release rates
h	= $[2E(1-\cos k)+(0+ikV)^2]^{1/2}$
k	= the Fourier transform parameter
L	= $r/h = L_+L_-$
$L_+(k)$ $(L_-(k))$	= regular in the upper (lower) half-plane
K_{III}	= the Mode III stress intensity factor

$L_{+\alpha}$	=	$L_+(i/V_\alpha)$
M	=	the particle mass at each node in the lattice
m	=	node number of a particle on a plane parallel to the crack plane
n	=	node number of a particle on a plane perpendicular to the crack lying between the lines $n = 0$ and $n = 1$
N	=	$-(N+1) \leq n \leq N =$ a lattice strip of width $(2N+1)a$
q	=	external loading
r	=	$(h^2 + 4E)^{1/2}$
$\mathcal{R}_v$	=	G_v/G, $\mathcal{R}_e = G_e/G$, $\mathcal{R}_0 = G_0/G$
S	=	$\phi + (1-\phi)/L_{+\alpha}$
t	=	time
u	=	displacement
$u_{m,n}$	=	displacement of the particle marked by numbers m and n
$u^F(k)$	=	the Fourier transform of $u(\eta)$
$V = v/c$	=	the nondimensional crack speed
V_α	=	$\alpha v/a =$ a nondimensional creep time
V_β	=	$\beta v/a =$ a nondimensional relaxation time
x	=	$ma =$ the horizontal coordinate
y	=	$na =$ the vertical coordinate
α	=	the creep time
β	=	the relaxation time
η	=	$m - vt/a =$ the steady-state coordinate
μ	=	the bond stiffness
σ	=	tensile force
$\sigma_+(k)$ $(\sigma_-(k))$	=	the right (left) Fourier transform of $\sigma(\eta)$
$\sigma_{+\alpha}$	=	$\sigma_+(i/V_\alpha)$
ϕ	=	β/α
Ψ	=	$\sqrt{2N+1}\,\Omega$
ε	=	strain
$\varepsilon_+(k)$ $(\varepsilon_-(k))$	=	the right (left) Fourier transform of $\varepsilon(\eta)$
Ω	=	$\exp\left[\int_0^\infty \mathrm{Arg}L(\xi)\,\mathrm{d}\xi/\pi\xi\right]$

1. Introduction

In the case of a viscoelastic material, the shortcomings of both the continuum and the singular fracture model are most pronounced, as was recognized by Williams (1962). Williams modified the singular elastic stress distribution to be finite and constant over a length δ, with the load on the uncracked ligament being carried by a series of discrete Voigt elements. The shortcomings of homogeneous viscoelastic models are as follows: there is a weak dependence of energy dissipation on the crack velocity for slow crack speeds, the quasi-static limit for the resistance to crack propagation does not coincide with that for a stationary crack, and, if the relaxation time approaches zero, the local energy release vanishes as well (Nuismer, 1974; Knauss and Mueller, 1975). In the latter case, if one were to use an energy criterion for crack growth, there is no way that such growth can occur. These shortcomings are due to the fact that the strain rate is infinite at the propagating crack tip for any nonzero crack velocity.

To facilitate the discussion of discrete *versus* continuum studies, the term 'homogeneous' will be used in this paper to signify a continuum with no length scale; the term 'homogeneous viscoelastic model' will signify a combined continuum-singular fracture model.

Traditionally, the homogeneous viscoelastic models have been modified to incorporate a cohesive zone ahead of the physical crack. In these cohesive zone models, cohesive stresses compensate the singularity at the crack tip, and govern the crack opening profile. The support of these stresses is completely defined by such a dependence and the requirement that the strain and strain rate be bounded. Note that the cohesive zone model was initially introduced by Barenblatt for homogeneous elastic bodies. Cohesive zones do not influence the steady-state crack propagation criterion (Willis, 1967b), and are important in the case of viscoelastic fracture.

The necessary and sufficient formulation of a cohesive zone model has not been stated: each is, in fact, rather *ad hoc* and questions of uniqueness and realism are always in the background (Costanzo and Walton, 1998; Langer and Lobkovsky, 1998). In an attempt to both provide an alternative model and eventually explore the deficiencies and advantages of cohesive zone models, a lattice model for viscoelastic fracture is introduced in this paper. For instance, both approaches can provide a viscoelastic fracture model in which the near crack tip strains and the strain rates are finite, and the viscoelastic properties transition smoothly to the elastic behavior. Conversely, the viscoelastic lattice fracture model (VLFM) is not amenable to *ad hoc* or supplementary modifications. In cohesive zone models, the zone itself is a contiguous but separate entity, whilst in the proposed VLFM the location, orientation, and shape of the process zone are generally not prescribed *a priori*.

Quasi-static studies of viscoelastic fracture have been mainly devoted to polymers (Knauss, 1970, 1973, 1974, 1976, 1986, 1989, 1993; Knauss and Dietmann, 1970; Wnuk and Knauss, 1970; Mueller and Knauss, 1971; Schapery, 1975a, 1975b, 1975c; Kanninen and Popelar, 1985) and concrete (Bažant and Jirásek, 1993; Wu and Bažant, 1993; Bažant and Li, 1997; Bažant and Planas, 1998). Dynamic nonlattice studies of viscoelastic fracture include those by Willis (1967a), Kostrov and Nikitin (1970), Atkinson and List (1972), Atkinson and Popelar (1979), Popelar and Atkinson (1980), Sills and Benveniste (1981), Walton (1982, 1987, 1995), Lee and Knauss (1989), Herrmann and Walton (1991, 1994), Walton and Herrmann (1992), Ryvkin and Banks-Sills (1992, 1994) and Geubelle et al. (1998). Surveys of the latter studies were provided by Freund (1990) and Walton (1995).

An important type of cohesive-zone model originated with a paper by Hillerborg et al. (1976). Hillerborg required that the post-peak tensile softening behavior be incorporated by a fundamental but experimentally corroborated stress-separation curve. This type of model is now known as the 'fictitious crack model.' The fictitious crack model studies by Li and Liang (1986) and Mulmule and Dempsey (1997, 1998), which treat the bulk material behavior as either linearly elastic or linearly viscoelastic, portray the ability of this approach to analyze problems undergoing the growth of large-scale process zones. In the viscoelastic fictitious crack model (VFCM), the dependence of the cohesive stress on the crack opening displacement and the rate of the crack opening displacement is governed by a stress-separation law, which is, in effect, a constitutive equation for this particular cohesive crack model. Mulmule and Dempsey (1998) formulated and applied a VFCM model to the fracture of sea ice. There the weight function method was used to compute the required parameters such as the stress intensity factor and the crack opening displacements. The desired stress-separation curves were backed out by modeling the load *versus* crack opening displacements at several points.

The dynamic Mode III elastic fracture of a square lattice was considered by Slepyan (1981a, 1981b, 1982a) for sub-critical and super-critical crack speeds. The Modes I and II fracture of an elastic triangular lattice were studied by Kulakhmetova et al. (1984). In these works, the structure-dependent total energy dissipation was analytically found for the three modes as functions of the crack velocity. Similar relations were obtained by Slepyan (1976) and Marder and Gross (1995) for elastic lattice strips. The same problems for anisotropic lattices (lattices which correspond to anisotropic elastic media) were solved by Kulakhmetova (1985a, 1985b). Some general conclusions concerning the resistance to crack propagation in a complex medium are presented in Slepyan (1982b, 1984). Mikhailov and Slepyan (1986) investigated crack propagation in a composite material model. Slepyan and Kulakhmetova (1986) made use of this approach for a model of rock joints. Finally, the papers by Slepyan and Troyankina (1984, 1988) were devoted to fracture waves in piece-wise-linear and nonlinear chain structures. Such structures are used to simulate phase transition dynamics in structured media. Reviews of works devoted to the fracture of elastic lattices have been provided by Slepyan (1990, 1993, 1998). In addition, a number of works have been devoted to the stability of crack propagation in discrete elastic lattices (Fineberg et al., 1991, 1992; Marder, 1991; Marder and Xiangmin Liu, 1993; Marder and Gross, 1995).

In the present paper, steady-state crack propagation in viscoelastic lattices is considered. To be specific, consider an unbounded medium and a J-type circular contour surrounding the crack tip. The total energy flux through this contour can be expressed as the sum of two terms: the first being carried by long-wave/low frequency waves, as in the case of a homogeneous body, the second by high frequency waves associated with the discrete lattice structure. The first propagates from the far-field to the crack tip, the second away from the crack tip. The first inward traveling energy flux dissipates (in part) during propagation to the crack tip: this is dissipation by the viscoelastic behavior of the material itself. The second outward traveling energy flux dissipates as well (completely). This dissipation is also caused by the material's viscoelasticity. It is important to note that the second energy flux term does not arise in the fracture of a homogeneous nonlattice material model. If the radius of the contour is very large, only the first term is involved, and the corresponding energy release rate is termed from here on the total or far-field energy release rate. If the contour is shrunk onto the crack tip (in this paper, this contour would encircle one bond), both energy fluxes are present: the first is now less than the far-field, the second is actually maximum. The difference between the first and second is in fact the local energy release rate which goes to fracture itself. The definition of this local energy release rate can include only the elastic energy of the breaking bond, or its total energy. This will be discussed more specifically later in the paper.

The amount by which the far-field energy has decreased during propagation to the crack tip may be called the viscous resistance, while the second or outgoing energy flux may be called the wave resistance to crack propagation. In general, the wave and viscous resistances are interconnected. However, it is shown in an important asymptotic case of large viscosity ($C_\alpha \gg 1$), that they may be separated. For a nonzero crack velocity, the wave resistance is asymptotically defined by that in an elastic lattice with the glassy (short time) modulus, and the viscous resistance corresponds to a homogeneous viscoelastic material. In the case of a viscoelastic lattice, vanishingly small creep and relaxation times correspond to an elastic limit, whereas there is no such limit in the case of a homogeneous viscoelastic material.

In addition, the quasi-static limit for a viscoelastic lattice, in contrast to a homogeneous material, corresponds to the stationary crack. In the case of large viscosity, this leads to a pronounced influence on the resistance to crack propagation over the initial portion of the crack velocity regime: the resistance increases very fast with this velocity from the stationary value. The corresponding theory for a slow crack in a viscoelastic lattice is derived and relations for the resistance to the crack propagation *versus* the crack velocity are presented. For the unbounded lattice, such a dependence is expressed in an explicit analytical form.

The square-cell lattice considered in this paper represents Mode III fracture. The fracture Modes I and II based on the triangular-cell lattice will be considered separately.

2. General Formulation

A square-cell plane lattice is considered. The lattice is assumed to consist of point particles, each of mass M, connected by massless viscoelastic bonds, as portrayed in Figure 1. Let a, μ, σ and ε be the bond length, its stiffness, tensile force and strain, respectively.

Each bond is assumed to satisfy the standard viscoelastic material stress-strain relation:

$$\sigma + \beta \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \mu \left(\varepsilon + \alpha \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \right), \tag{1}$$

where t is time; α and β are creep and relaxation times, respectively, and $\beta/\alpha = \phi$. It is assumed that $\alpha \geq \beta \geq 0$. This means that a passive, stable material of the

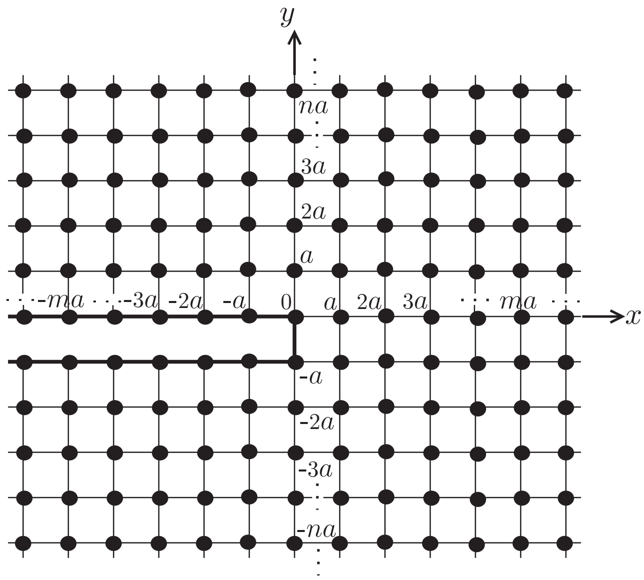

Figure 1. The square-cell unbounded lattice.

bonds is considered (see Appendix 1). Note that, under zero initial conditions $\sigma = \varepsilon = 0$ ($t = 0$), the case $\alpha = \beta$ corresponds to an elastic material. A more general viscoelastic stress-strain relation is considered in Appendix 2.

A crack, formed by the breakage of individual massless viscoelastic bonds, is assumed to propagate with a constant velocity, v, between two neighboring horizontal lines of particles: $y = 0$ and $y = -a$ (Figure 1). This means that the time interval between the breaks of two neighboring bonds, a/v, is assumed to be a constant. Accordingly, the displacement of each particle is represented in the form

$$\boldsymbol{u} = \boldsymbol{u}(\eta, y), \quad \eta = (x - vt)/a. \tag{2}$$

Note that x and y are discrete coordinates of the particles. Conversely, η can be viewed as a continuous variable because of continuous time, t. Dependencies of the same type are valid for the force and strain of each bond. The breakage of a bond placed on the crack line is assumed to occur at $\eta = 0$. Thus, the crack is placed at $\eta < 0$ and the intact bonds are placed in front of the crack, $\eta > 0$.

Symmetry of deformation of the lattice is assumed:

$$u(\eta, -a - y) = -u(\eta, y), \quad \varepsilon(\eta) = [u(\eta, 0) - u(\eta, -a)]/a = 2u/a, \tag{3}$$

where the crack opening displacement $u = u(\eta) = u(\eta, 0)$. Thus the viscoelastic relation (1) for $\eta > 0$ can be rewritten as

$$\sigma(\eta) - V_\beta \sigma'(\eta) = \mu(\varepsilon(\eta) - V_\alpha \varepsilon'(\eta)) = 2\mu[u(\eta) - V_\alpha u'(\eta)]/a, \tag{4}$$

where the parameters are introduced as

$$V_\alpha = \frac{\alpha v}{a}, \quad V_\beta = \frac{\beta v}{a}. \tag{5}$$

Note that the viscoelastic relation (4) is valid for a bond before it is broken, and that it does not incorporate a jump in σ at $\eta = 0$. Because of this consideration, the right-sided Fourier transform (identified from here on by the subscript '+') is used in the form

$$\sigma_+(k) = \int_{+0}^{\infty} \sigma(\eta)\, \mathrm{e}^{ik\eta}\, \mathrm{d}\eta, \quad \varepsilon_+(k) = \int_{+0}^{\infty} \varepsilon(\eta)\, \mathrm{e}^{ik\eta}\, \mathrm{d}\eta \quad (\operatorname{Im} k > 0), \tag{6}$$

where the symbol '+0' means the zero limit of a positive value. Similarly, the left-sided Fourier transform is defined as

$$\sigma_-(k) = \int_{-\infty}^{0} \sigma(\eta)\, \mathrm{e}^{ik\eta}\, \mathrm{d}\eta, \quad u_-(k) = \int_{-\infty}^{0} u(\eta)\, \mathrm{e}^{ik\eta}\, \mathrm{d}\eta \quad (\operatorname{Im} k < 0). \tag{7}$$

Note that for a broken bond the notion 'strain' at $\eta < 0$ has no meaning. In addition to the right-sided and left-sided Fourier transforms defined above, the double-sided Fourier transform, is required. In a generalized sense

$$u^F(k) = \int_{-\infty}^{\infty} u(\eta)\, \mathrm{e}^{ik\eta}\, \mathrm{d}\eta = \lim(u_+ + u_-) \quad (\operatorname{Im} k \to 0). \tag{8}$$

In (8) the Fourier transform is valid for ordinary and generalized functions of slow growth: functions that can grow with $\eta \to \pm\infty$, but not faster than a power of η. While being defined on the real k-axis, the Fourier transform can be analytically extended into the complex k-plane.

The right-sided Fourier transformation of relation (4) leads to

$$(1 + ikV_\beta)\sigma_+ + V_\beta\sigma(+0) = \mu[(1 + ikV_\alpha)\varepsilon_+ + V_\alpha\varepsilon(+0)]. \tag{9}$$

In view of the fact that $\sigma_+(k)$ and $\varepsilon_+(k)$ are regular functions in the upper half-plane of the complex variable k, it now follows that

$$\varepsilon_{+\beta} \equiv \varepsilon_+\left(\frac{i}{V_\beta}\right) = \frac{\phi}{\mu(1-\phi)}[\mu V_\alpha\varepsilon(+0) - V_\beta\sigma(+0)],$$
$$\sigma_{+\alpha} \equiv \sigma_+\left(\frac{i}{V_\alpha}\right) = \frac{1}{(1-\phi)}[\mu V_\alpha\varepsilon(+0) - V_\beta\sigma(+0)]. \tag{10}$$

Hence, from Equations (9) and (10),

$$\varepsilon_+ = \frac{2u_+}{a}, \quad E = \frac{1 + ikV_\alpha}{1 + ikV_\beta}, \quad \phi = \frac{\beta}{\alpha},$$

$$\varepsilon_+ = \frac{\sigma_+}{\mu} - (1-\phi)\,\frac{\sigma_{+\alpha} + ikV_\alpha\sigma_+}{\mu(1 + ikV_\alpha)} = \frac{\sigma_+}{\mu E} - \frac{(1-\phi)\sigma_{+\alpha}}{\mu(1 + ikV_\alpha)}, \tag{11}$$

where the point $k = i/V_\alpha$ is regular. Note that E and $1/E$ could have been marked by the subscript '$-$' because these functions have no singular point in the lower half-plane of the complex k-plane.

The Fourier transform of (4) leads to the relation

$$\sigma^F = \frac{2\mu}{a} E u^F. \tag{12}$$

If u^F is replaced by u_+, this equality gives us a different function, say σ_*^F:

$$\sigma_*^F = \frac{2\mu}{a} E u_+. \tag{13}$$

The term $\sigma_*(\eta)$ equals $\sigma(\eta)$ defined by Equation (12), but only for $\eta > 0$ because $E = E_-$. Thus σ_*^F is not equal to σ_+. However, in the following, a relation between u_+ and σ_+ is required, and this is the reason why Equation (11) is used but not Equation (13).

From Equation (11) it follows that

$$\varepsilon(+0) = \lim_{k \to i\infty} (-ik)\varepsilon_+(k) = \frac{\phi}{\mu}\sigma(+0) + (1-\phi)\,\frac{\sigma_{+\alpha}}{\mu V_\alpha}. \tag{14}$$

Equations (11) and (14) play a crucial role in the description of steady-state crack propagation through a layer of viscoelastic bonds. The limiting strain, $\varepsilon(+0)$, depends on only two parameters of the stress distribution: the limiting stress, $\sigma(+0)$, and the Fourier transform of stress at $k = i/V_\alpha$: $\sigma_{+\alpha}$. Note that when $V_\alpha, \to 0$

$$\sigma_{+\alpha} = \int_0^\infty \sigma(\eta)\,\mathrm{e}^{-\eta/V_\alpha}\,\mathrm{d}\eta \sim V_\alpha \sigma(+0). \tag{15}$$

In this case

$$\sigma_{+\alpha} + ikV_\alpha\sigma_+ \sim V_\alpha[ik\sigma_+ + \sigma(+0)] = -V_\alpha\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\eta}\right)_+ \to 0, \tag{16}$$

and as follows from Equation (11)

$$\varepsilon_+ \sim \sigma_+/\mu, \quad \varepsilon(+0) \sim \sigma(+0)/\mu. \tag{17}$$

The latter results correspond to an elastic material with an equilibrium (long time) modulus as expected. When $V_\alpha \to \infty$, assuming $\sigma(\eta) \to 0$ when $\eta \to \infty$, it is evident that

$$\sigma_{+\alpha}/V_\alpha \to 0, \quad \varepsilon(+0) \sim \sigma(+0)\phi/\mu, \tag{18}$$

that corresponds to the glassy modulus as it must.

Let c be the critical crack velocity in the corresponding homogeneous elastic material (c is the long shear wave velocity for Mode III fracture). The crack velocity v is said to be 'slow' if $v \ll c$. If $C_\alpha \equiv \alpha c/a$ is large, the strain decreases rapidly from σ/μ and approaches the lower value $\phi\sigma/\mu$ over the initial portion of the crack velocity regime.

The local energy release (the energy spent on fracture itself) may be defined in several ways. Consider, for instance, the total viscoelastic work, A_v, accumulated in a broken bond, its elastic energy, A_e, which corresponds to the equilibrium modulus and an effective elastic energy, A_0, based on the limiting stress, $\sigma(+0)$, and strain, $\varepsilon(+0)$. The viscoelastic per-bond energy is defined by

$$A_v = a\int_{-\infty}^{x/v} \sigma\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\,\mathrm{d}t = -a\int_0^{\infty} \sigma(\eta)\frac{\mathrm{d}\varepsilon}{\mathrm{d}\eta}\,\mathrm{d}\eta. \tag{19}$$

Using Parseval's relation for Fourier transforms of two real functions,

$$\int_{-\infty}^{\infty} f(x)g(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^F(k)\overline{g^F(k)}\,\mathrm{d}k, \tag{20}$$

one can rewrite this expression in the following forms.

$$\begin{aligned} A_v &= \frac{a}{2\pi}\int_{-\infty}^{\infty} \overline{\sigma_+(k)}[ik\varepsilon_+ + \varepsilon(+0)]\,\mathrm{d}k \\ &= \frac{a\sigma^2(+0)}{2\mu} + (1-\phi)\frac{a}{2\pi\mu}\int_{-\infty}^{\infty}\frac{ikV_\alpha}{1+ikV_\alpha}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\eta}\right)_+\overline{\sigma_+(k)}\,\mathrm{d}k \\ &= \frac{a\phi\sigma^2(+0)}{2\mu} - (1-\phi)\frac{a}{2\pi\mu}\int_{-\infty}^{\infty}\frac{1}{1+ikV_\alpha}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\eta}\right)_+\overline{\sigma_+(k)}\,\mathrm{d}k \end{aligned}$$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\eta}\right)_+ = -[ik\sigma_+ + \sigma(+0)]. \tag{21}$$

In the derivation of this formula, note that

$$\int_{-\infty}^{\infty}\frac{\overline{\sigma_+(k)}}{1+ikV_\alpha}\,\mathrm{d}k = 0$$

because the integrand is regular in the lower half of the k-plane and it is $o(1/|k|)$ for $|k| \to \infty$. In this connection, note also that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[ik\sigma_+(k) + \sigma(+0)]\overline{\sigma_+(k)}\,\mathrm{d}k$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\eta}\right)_{+}\overline{\sigma_{+}(k)}\,\mathrm{d}k = -\int_{0}^{\infty}\sigma\frac{\mathrm{d}\sigma}{\mathrm{d}\eta}\,\mathrm{d}\eta = \frac{\sigma^{2}(+0)}{2}. \qquad (22)$$

The expression (21) has the following asymptotes:

$$A_v \sim \frac{a\sigma^2(+0)}{2\mu} \quad (V_\alpha \to 0), \qquad A_v \sim \frac{a\phi\sigma^2(+0)}{2\mu} \quad (V_\alpha \to \infty). \qquad (23)$$

The associated elastic and effective elastic energies of a broken bond have much simpler expressions:

$$A_e = \frac{a\mu\varepsilon^2(+0)}{2} = \frac{2\mu u^2(+0)}{a}, \qquad A_0 = \frac{a\sigma(+0)\varepsilon(+0)}{2}. \qquad (24)$$

The local energy release rates are now given by

$$G_v = pA_v, \quad G_e = pA_e, \quad G_0 = pA_0, \qquad (25)$$

where p is a number of the breaking bonds per unit length of the crack. In the problem considered below, $p = 1/a$.

Note that for a realistic case when $\sigma \geq \mu\varepsilon$ and for $\eta > 0$, $\sigma(\eta) < \sigma(+0)$ (given that $\sigma(+0) = \sigma_{\max}$), it is clear that

$$\frac{\mu\varepsilon^2(+0)}{2} \leq -\int_0^\infty \sigma\frac{\mathrm{d}\varepsilon}{\mathrm{d}\eta}\,\mathrm{d}\eta < \sigma(+0)\varepsilon(+0). \qquad (26)$$

Thus,

$$G_e \leq G_v < 2G_0. \qquad (27)$$

The global, far-field energy release rate, G, corresponds to the low-rate modulus (or equivalently, the elastic homogeneous material). The local and global energy releases differ by energy dissipation:

$$G = G_v + D_0 = G_e + D, \qquad (28)$$

where D_0 is the total dissipation rate outside the breaking bonds and D is the same but including dissipation in the breaking bonds.

In this paper, no particular definition of the local energy release rate is favored as a crack extension criterion. The main goal is to derive comparative results for the local-to-global energy release ratios, stresses and elongation under the influence of the discrete structure and the viscoelasticity of the lattice. Simply note that an increase in a global-to-local energy release ratio is associated with an increase in the resistance to crack propagation.

Much of the discussion and portrayal of the results will involve the normalized local energy release rates, which are defined as

$$\mathcal{R}_v = \frac{G_v}{G}, \quad \mathcal{R}_e = \frac{G_e}{G}, \quad \mathcal{R}_0 = \frac{G_0}{G}. \qquad (29)$$

These parameters record the lattice influence and are from hereon referred to as 'lattice factors'. Note that a decrease in the resistance (G) actually implies an increase in $\mathcal{R}_0$, and vice versa.

3. Unbounded Square-Cell Lattice

The dynamic equation of the lattice shown in Figure 1 is

$$M\left(1+\beta\frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\mathrm{d}^2u_{m,n}}{\mathrm{d}t^2} = \frac{\mu}{a}\left(1+\alpha\frac{\mathrm{d}}{\mathrm{d}t}\right)(u_{m+1,n}+u_{m-1,n}+u_{m,n+1}+u_{m,n-1}-4u_{m,n}), \tag{30}$$

where m and n are horizontal and vertical numbers of a particle, respectively ($m \equiv x/a$, $n \equiv y/a$). This equation is valid for particles that are not connected by bonds across the crack path or on the cracked surfaces: for $n > 0$ and $n < -1$.

Via a long-wave (low-frequency) approximation, the lattice corresponds to a plane homogeneous body of density M/a^2 and shear modulus μ/a. Accordingly, the shear wave propagation velocity is given by $c = \sqrt{a\mu/M}$. The crack propagation problem is considered below for $0 \leq V = v/c < 1$.

Assuming $u_n = u_n(\eta)$, $\eta = m - vt/a$, where η is treated as a continuous variable for each m, one can rewrite equation (30) in the form

$$\frac{v^2}{c^2}\left(1-V_\beta\frac{\mathrm{d}}{\mathrm{d}\eta}\right)\frac{d^2u_n(\eta)}{\mathrm{d}\eta^2} = \left(1-V_\alpha\frac{\mathrm{d}}{\mathrm{d}\eta}\right)[u_n(\eta+1)+u_n(\eta-1)+u_{n+1}(\eta)+u_{n-1}(\eta)-4u_n(\eta)] \tag{31}$$

while in terms of the two-sided Fourier transform

$$(h^2+2E)u_n^F - E(u_{n+1}^F + u_{n-1}^F) = 0, \tag{32}$$

where

$$h^2 = 2E(1-\cos k) + (0+ikV)^2, \quad r^2 = h^2+4E,$$
$$E = \frac{1+ikV_\alpha}{1+ikV_\beta}, \quad V = \frac{v}{c}, \quad 0+ikv = \lim_{s\to+0}(s+ikv) \tag{33}$$

(see Appendix 4 in connection with the last limit).

Equation (32) is satisfied by the expression

$$u_n^F = u^F\lambda_{1,2}^n, \quad u^F = u_0^F \tag{34}$$

with

$$\lambda_1 \equiv \lambda = \frac{r-h}{r+h}, \quad \lambda_2 = \frac{1}{\lambda}. \tag{35}$$

For the problem of a crack in an unbounded lattice, given anti-symmetric deformations and that $u_n^F \to 0$ when $n \to \pm\infty$,

$$u_n^F = u^F \lambda^n \quad (n \geq 0), \quad u_n^F = -u^F \lambda^{-(n+1)} \quad (n \leq -1) \tag{36}$$

Note that $|\lambda| < 1$ if s in Equation (33) is positive. Indeed, if $s > 0$

$$\operatorname{sgn} \operatorname{Im} E = \operatorname{sgn} k \ (\alpha \geq \beta), \quad -\pi < \operatorname{Arg} h^2 < \pi,$$

$$\operatorname{Re} h > 0, \quad \operatorname{Re} r > 0, \quad \operatorname{sgn} \operatorname{Im} h = \operatorname{sgn} \operatorname{Im} r \tag{37}$$

and it follows from this that

$$|r - h| < |r + h|. \tag{38}$$

Consider now the line $n = 0$. Let σ_m be the stress that acts on the particle $(m, 0)$ from below. Then Equation (30) takes the form

$$\left(1 + \beta \frac{\mathrm{d}}{\mathrm{d}t}\right)\left(M \frac{\mathrm{d}^2 u_{m,0}}{\mathrm{d}t^2} + \sigma_m\right)$$
$$= \frac{\mu}{a}\left(1 + \alpha \frac{\mathrm{d}}{\mathrm{d}t}\right)(u_{m+1,0} + u_{m-1,0} + u_{m,1} - 3u_{m,0}). \tag{39}$$

From this it follows that

$$(\sigma_m)^F = \sigma_+ + \sigma_- = -(\mu/a)[(h^2 + E)u^F - E u_1^F] \tag{40}$$

or, using Equations (33–36),

$$\sigma_+ + \sigma_- = -\frac{\mu h(r+h)}{2a} u^F = -\frac{\mu h(r+h)}{2a}(u_+ + u_-). \tag{41}$$

Substituting u_+ from Equation (11) into Equation (41) gives

$$\frac{L}{2E}\sigma_+ + \frac{\mu}{a}u_- = -\frac{L-1}{2E}\sigma_- + (1-\phi)\frac{\sigma_{+\alpha}}{2(1 + ikV_\alpha)} \tag{42}$$

with

$$L = \frac{r}{h}. \tag{43}$$

It follows from Equation (37) that the index of this function is zero:

$$\operatorname{Ind} L(k) = \frac{1}{2\pi}[\operatorname{Arg} L(+\infty) - \operatorname{Arg} L(-\infty)] = 0. \tag{44}$$

In addition, $L(k) = 1 \ (k \to \pm\infty)$, and for real k

$$\ln L(k) = \ln |L(k)| + i \operatorname{Arg} L(k) \to 0 \quad (k \to \pm\infty) \tag{45}$$

with $\operatorname{Arg} L(0) = 0$,

$$\ln|L(-k)| = \ln|L(k)|, \quad \operatorname{Arg} L(-k) = -\operatorname{Arg} L(k). \tag{46}$$

The function L may now be represented by the product

$$L = L_+ L_-, \tag{47}$$

where

$$\begin{aligned} L_+(k) &= \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln L(\xi)}{\xi - k}\,\mathrm{d}\xi\right) \quad (\operatorname{Im} k > 0), \\ L_-(k) &= \exp\left(-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln L(\xi)}{\xi - k}\,\mathrm{d}\xi\right) \quad (\operatorname{Im} k < 0) \end{aligned} \tag{48}$$

with

$$\operatorname{Arg} L(\xi) = 0 \quad (\xi = 0, -\infty, +\infty). \tag{49}$$

In Equation (48), L_+ is a regular function of k in the upper half-plane, while L_- is a regular function of k in the lower half-plane of k. These functions have the following asymptotes:

$$\begin{aligned} L_+ &\sim \left(\frac{4}{1 - V^2}\right)^{1/4} \frac{\Omega}{\sqrt{0 - ik}} \quad (k \to 0), \\ L_- &\sim \left(\frac{4}{1 - V^2}\right)^{1/4} \frac{1}{\Omega\sqrt{0 + ik}}, \quad (k \to 0), \\ L_+ &= 1 \quad (k = i\infty), \quad L_- = 1 \quad (k = -i\infty). \end{aligned} \tag{50}$$

In the derivation of the expressions in (50), it has been noted that $|L|$ and $\operatorname{Arg} L$ are even and odd functions of ξ as stated in Equation (46). The constant Ω is defined by the equality

$$\Omega = \exp\left(\frac{1}{\pi}\int_0^{\infty}\frac{\operatorname{Arg} L(\xi)}{\xi}\,\mathrm{d}\xi\right). \tag{51}$$

Note that $\operatorname{Arg} r < \operatorname{Arg} h$ $(k > 0)$. Hence $\operatorname{Arg} L < 0$ and $\Omega < 1$.

Equation (42) may now be expressed in the form

$$\frac{L_+}{2}\sigma_+ + \frac{\mu E}{a L_-}u_- = -\frac{L_+\sigma_-}{2} + \frac{\sigma_-}{2L_-} + (1 - \phi)\,\frac{\sigma_{+\alpha} E}{2(1 + ikV_\alpha)L_-}. \tag{52}$$

Consider now the external loading σ^0 (with σ_- as its Fourier transform) to be of the form:

$$\sigma^0 = -q\ \exp(-ik_0\eta)H(-\eta), \quad \sigma_- = -\frac{q}{0 + i(k - k_0)}, \quad \operatorname{Im} k_0 \geq 0, \quad (53)$$

where H is the Heaviside unit step function. In this case the first term on the right-hand side of Equation (52) can be represented in the form:

$$-\frac{L_+\sigma_-}{2} = \frac{[L_+(k) - L_+(k_0 + i0)]q}{2[0 + i(k - k_0)]} + \frac{L_+(k_0 + i0)q}{2[0 + i(k - k_0)]}. \quad (54)$$

The point $k = k_0$ is regular in the first term on the right-hand side of Equation (54). This term is regular in the upper half-plane. The remaining terms on the right-hand sides of Equations (54) and (52) are regular in the lower half-plane. It now follows from Equations (52–54), given that $\sigma_+ \to 0$ when $k \to i\infty$,

$$\sigma_+ = \frac{[L_+(k) - L_+(k_0 + i0)]q}{[0 + i(k - k_0)]L_+(k)}, \quad (55)$$

$$u_- = \frac{[L_-(k)L_+(k_0 + i0) - 1]aq}{2\mu[0 + i(k - k_0)]E} + (1 - \phi)\,\frac{a\sigma_{+\alpha}}{2\mu(1 + ikV_\alpha)}, \quad (56)$$

where

$$\sigma_{+\alpha} = \frac{V_\alpha[L_+(k_0 + i0) - L_{+\alpha}]q}{L_{+\alpha}(1 + ik_0V_\alpha)},$$

$$L_{+\alpha} = \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln L(\xi)}{\xi - i/V_\alpha}\,\mathrm{d}\xi\right). \quad (57)$$

This solution is valid if $L_+(k_0) \neq \infty$. If $k_0 = 0$ this function is infinite (in this case of an unbounded lattice). This property provides a way to obtain the complementary solution.

The above particular solution is now used to derive the solution for homogeneous boundary conditions. Let

$$qL_+(k^0) = C, \quad (C = \text{const})\ \text{with}\ \lim q = 0 \quad (k^0 \to 0). \quad (58)$$

It now follows from Equation (55) that

$$\sigma_+ = \frac{1/L_+(k^0) - 1/L_+(k)}{0 + i(k - k^0)}C. \quad (59)$$

Because the point $k = k^0 \neq 0$ is regular, the denominator of this expression may be replaced by $-[0 - i(k - k^0)]$. Next, separate Equation (59) into two terms

$$\sigma_+ = \frac{C}{[0 - i(k - k^0)]L_+(k)} - \frac{C}{[0 - i(k - k^0)]L_+(k^0)}. \quad (60)$$

The limit ($k^0 \to 0$) of each term may now be considered separately. In the limit as $k^0 \to 0$, the second term vanishes, and the Fourier transform of the solution takes the form

$$\begin{aligned}
\sigma_{+\alpha} &= \frac{V_\alpha C}{L_{+\alpha}}, \\
\sigma_+(k) &= \frac{C}{(0 - ik)L_+(k)}, \\
u_-(k) &= \frac{aCL_-(k)}{2\mu E(0 + ik)} + (1 - \phi)\frac{a\sigma_{+\alpha}}{2\mu(1 + ikV_\alpha)}.
\end{aligned} \tag{61}$$

The limiting stress and strain at $\eta = +0$ are of a special interest. These values can be obtained by means of the formulas:

$$\begin{aligned}
\sigma(+0) &= \lim_{k \to i\infty} (-ik)\sigma_+(k), \\
\varepsilon(+0) &= 2 \lim_{k \to -i\infty} (ik)u_-(k)/a = 2u(-0)/a,
\end{aligned} \tag{62}$$

leading to

$$\sigma(+0) = C, \quad \varepsilon(+0) = CS/\mu, \quad S = \phi + (1 - \phi)/L_{+\alpha}. \tag{63}$$

Note that the expression for $\varepsilon(+0)$ is based on displacement continuity at the crack tip ($\varepsilon(+0) = 2u(+0)/a = 2u(-0)/a$), which is valid due to the presence of inertia (provided by the mass of the particles). The same expression for $\varepsilon(+0)$ also follows from Equation (14).

The unknown constant C can be expressed in terms of the far-field stress intensity factor. Indeed, its value corresponds to the long-wave approximation ($k \to 0$) which follows from Equations (50) and (61) and coincides with the classical solution for a homogeneous elastic body:

$$\begin{aligned}
\sigma_+ &\sim \left(\frac{1 - V^2}{4}\right)^{1/4} \frac{C}{\Omega}(0 - ik)^{-1/2}, \\
K_{\mathrm{III}} &= \left(1 - V^2\right)^{1/4} \frac{C}{\sqrt{a}\Omega},
\end{aligned} \tag{64}$$

where it is taken into account that the averaged stresses are equal to σ/a. The far-field energy release rate is (see, e.g., Freund, 1990)

$$G = \frac{aK_{\mathrm{III}}^2}{2\mu\sqrt{1 - V^2}}. \tag{65}$$

Thus

$$C = \sqrt{a}K_{\mathrm{III}}\Omega(1 - V^2)^{-1/4} = \Omega\sqrt{2G\mu}, \tag{66}$$

and

$$\sigma(+0) = C = \Omega\sqrt{2G\mu}, \tag{67}$$

$$\varepsilon(+0) = \frac{S\sigma(+0)}{\mu} = S\Omega\sqrt{\frac{2G}{\mu}}. \tag{68}$$

Now the energy release ratios can be written. For the viscoelastic lattice considered, the ratio of G_v/G can be expressed based on the expression in Equations (21) and (25). Taking into account Equations (61), (66) and (67) this ratio is given by

$$\mathcal{R}_v = \frac{G_v}{G} = \Omega^2\left[1 - (1-\phi)\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{V_\alpha}{1+ikV_\alpha}\frac{L_+(k)-1}{|L_+(k)|^2}\,\mathrm{d}k\right]. \tag{69}$$

The corresponding ratios based on the effective elastic energy and the purely elastic energy of the broken bond are

$$\mathcal{R}_e = \frac{G_e}{G} = \frac{\mu\varepsilon^2(+0)}{2G} = S^2\Omega^2, \quad \mathcal{R}_0 = \frac{G_0}{G} = \frac{\sigma(+0)\varepsilon(+0)}{2G} = S\Omega^2. \tag{70}$$

These expressions reduce to the results obtained by Slepyan (1981a, 1982b) for the elastic lattice (provided $\alpha \to \beta$):

$$\mathcal{R}_e = \mathcal{R}_0 = \Omega^2 \quad (E = 1). \tag{71}$$

Identical results are obtained in the limit as $\alpha \to 0$ $(0 \leq \beta \leq \alpha)$.

The role of the discrete structure of the lattice on the dimensionless viscoelastic parameters $C_\alpha = \alpha c/a$ and $\phi = \beta/\alpha$ are shown in Figures 2–7. These plots reveal the crack-speed-dependent dissipation by both high-frequency wave radiation and viscosity. The number of such waves, and the energy which they carry out of the propagating crack tip, depend mainly on the crack speed. In particular, only one wave mode is excited if the crack speed exceeds half the longitudinal shear wave speed (approximately), while the number of different wave modes is unbounded as the crack speed tends to zero. This results in a nonmonotone dependence on crack speed for low v/c values. At the same time, the influence of viscosity increases as $\phi = \beta/\alpha$ decreases. This, in turn, leads to an increase in the total resistance to crack propagation (which is inversely proportional to R_0) and damping of the dynamic effects due to the radiation.

For the case of relatively low dissipation, it can be seen that a minimum in the energy radiation exists (as a maximum of R_0). This minimum occurs at half the critical speed, as in the case of elastic lattices (see also Slepyan, 1998). Radiation increases without bound as the crack speed approaches the long shear wave speed, while remaining nonzero as the crack speed tends to zero. In the latter quasi-static

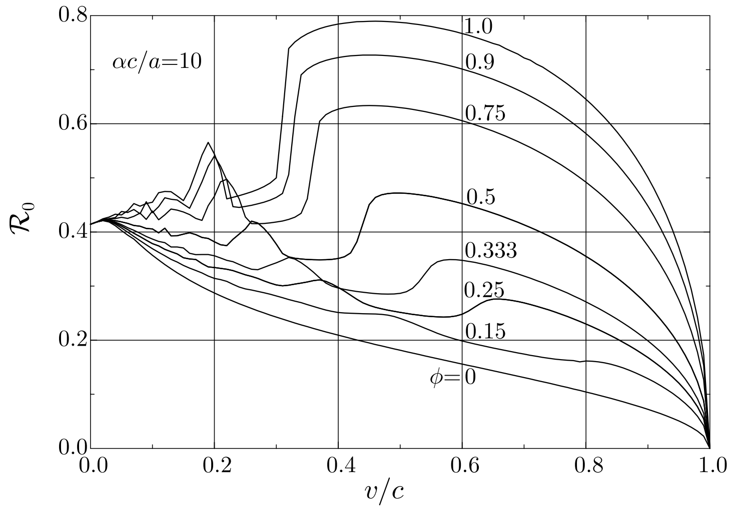

Figure 2. Effective energy release ratios *versus* velocity for a range of ϕ values, given $C_\alpha = \alpha c/a = 10$.

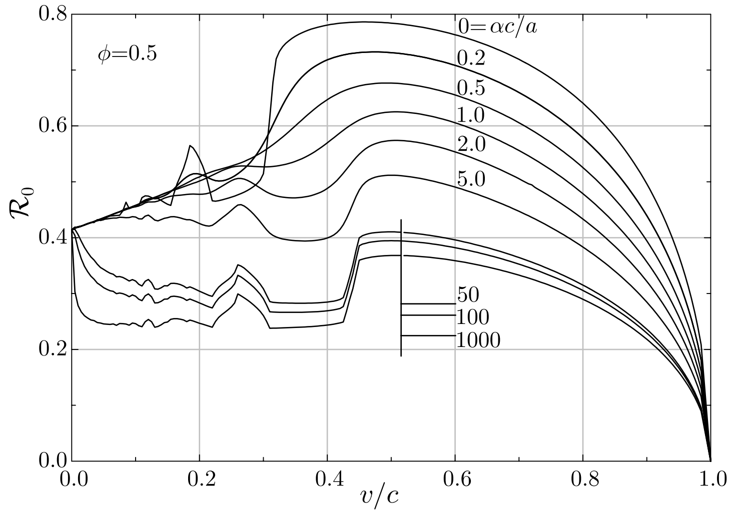

Figure 3. Effective energy release ratios *versus* velocity for a range of $C_\alpha = \alpha c/a$ values, given that $\phi = 0.5$.

case, the radiation energy, $D = D_0 = G - G_0 = \sqrt{2}G_0$, where G_0 is the fracture energy on the microscale.

For cases in which the resistance decreases (that is, $\mathcal{R}_0$ increases, as for the curves for $\phi = 0.75$, 0.9, and 1.0 in Figure 2) over the initial crack speed region, slow stable steady-state crack growth is not possible, given a limiting-strain criterion (see also Marder and Gross, 1995). However, for a large viscosity, C_α, and small ϕ, the energy release ratio first decreases with the crack speed. In other

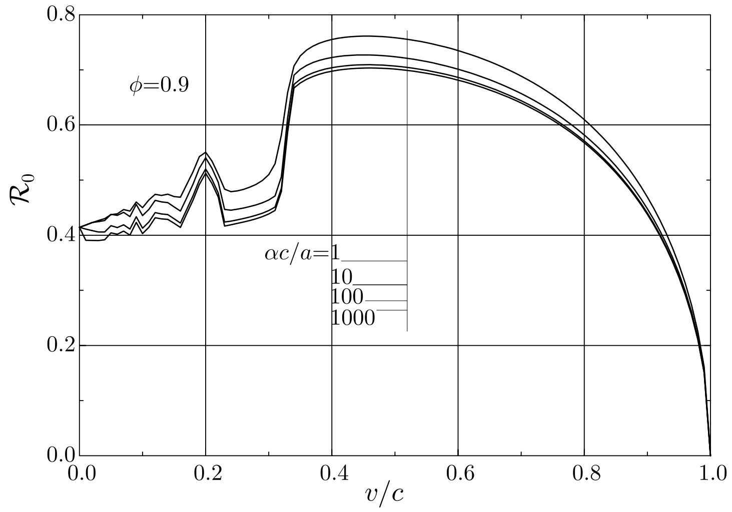

Figure 4. Effective energy release ratios *versus* velocity for a range of $C_\alpha = \alpha c/a$ values, given that $\phi = 0.9$.

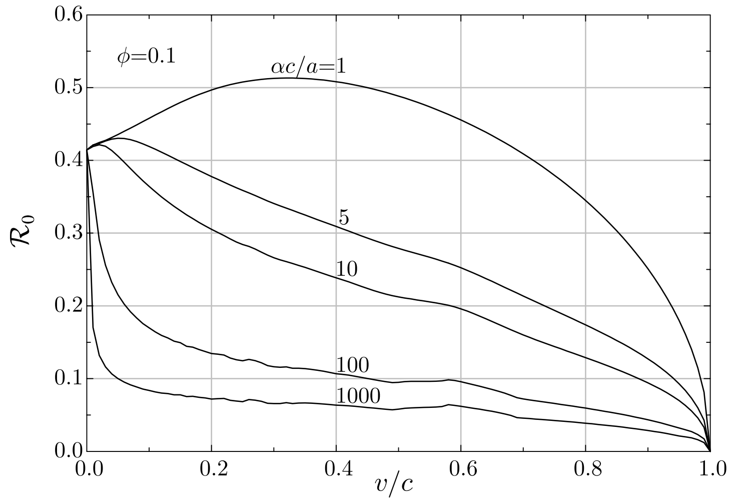

Figure 5. Effective energy release ratios *versus* velocity for a range of $C_\alpha = \alpha c/a$ values, given that $\phi = 0.1$.

words, the wave resistance increases and hence slow crack growth is possible. This is one of the most important phenomena exhibited by this viscoelastic lattice model. An increase in the relaxation time, β, leads to the elastic-type behavior of the energy release ratio, while an increase in the creep time, α, results in suppression of the dynamic effects.

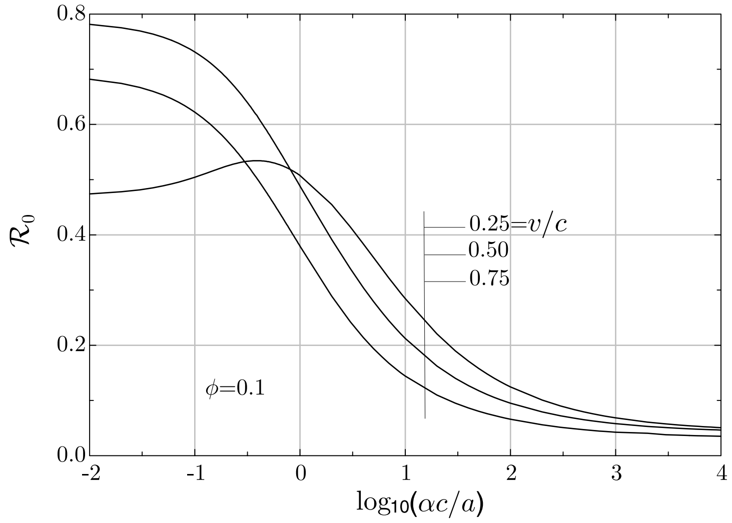

Figure 6. Effective energy release ratios *versus* $\log_{10}(\alpha c/a)$ for a range of v/c values, given that $\phi = 0.1$.

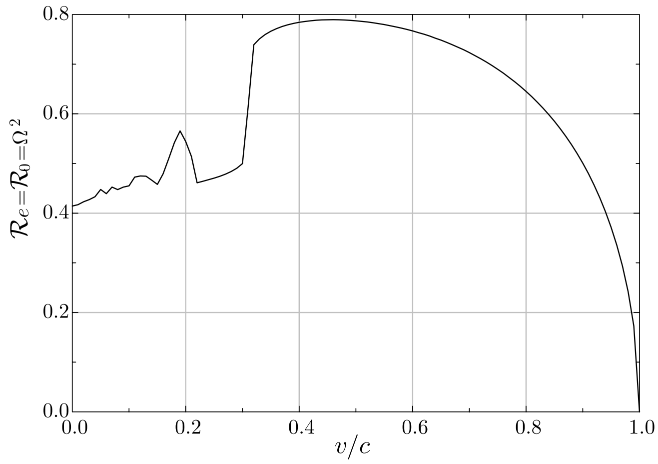

Figure 7. Energy release ratio for the elastic lattice ($\phi = 1$)

3.1. Transition to a Homogeneous Material

The Fourier-description of the problem for a homogeneous material follows from that for the lattice as an asymptote for $k \to 0$, by the substitution of $k^2/2$ for $1 - \cos k$ in h^2 and $4\hat{E}$ for r^2; $\hat{E}$ equals E under the condition that $a = 1$ (under these condition k becomes the dimensional parameter of the Fourier transformation over x). In addition, the second term in the expression for u_- in (61) is negligible in comparison with the first term because it is finite while the first term is $O(k^{-3/2})$ if $k \to 0$. Note that this term represents the crack opening displacements which arise due to the extension of the bonds between the particles on the lines $n = 0$ and

$n = -1$ in front of the crack. In the classical formulation for a homogeneous body, there is no such layer and there is no corresponding term. Thus

$$\begin{aligned}
\hat{E} &= \frac{1 + ik\alpha v}{1 + ik\beta v}, \\
h^2 &= \hat{E}k^2 + (0 + ikV)^2, \\
L &= 2\frac{\sqrt{\hat{E}}}{h}, \quad L_+ = \frac{1}{\sqrt{0 - ik}}, \\
\sigma_+(k) &= \frac{C_1}{\sqrt{0 - ik}}, \quad C_1 = \text{const}, \\
L_- &= 2\left(\frac{1 + i\alpha vk}{1 - V^2 + ikv(\alpha - \beta V^2)}\right)^{1/2} \frac{1}{\sqrt{0 + ik}}, \\
u_-(k) &= \frac{C_1/\hat{E}}{(0 + ik)^{3/2}} \left(\frac{1 + i\alpha vk}{1 - V^2 + ikv(\alpha - \beta V^2)}\right)^{1/2}. \qquad (72)
\end{aligned}$$

In the case of a homogeneous material, the expression for the energy release rate at the moving crack tip corresponds to an elastic body with the short term modulus because the stress/strain rates tend to infinity in the vicinity of the crack tip and is given by (Slepyan, 1990)

$$G_0 = \lim_{p\to\infty} p^2\sigma_+(ip)u_-(-ip) = C_1^2 \frac{\phi}{\sqrt{1 - V^2\phi}}. \qquad (73)$$

The far-field energy release rate G is given by the long-wave approximation ($k \to 0$) and corresponds to an elastic material loaded under the same conditions. From Equation (73), under the condition that $\alpha = \beta$,

$$G = \frac{C_1^2}{\sqrt{1 - V^2}}. \qquad (74)$$

Note that this expression coincides with that in Equation (65). Thus the energy release ratio G_0/G is given by

$$\mathcal{R}_0 = \frac{G_0}{G} = \phi\sqrt{\frac{1 - V^2}{1 - V^2\phi}}. \qquad (75)$$

Note that this ratio depends on $\phi = \alpha/\beta$ but not α and β separately. This conclusion, however, is valid only for a homogeneous viscoelastic material but not for a lattice.

3.2. Transition to an Elastic Lattice

Let $0 < V_\beta < V_\alpha \to 0$. In this case, it follows from Equation (57) that $L_{+\alpha} \to 1$. Now the relations (67–70) become as follows:

$$\sigma(+0) = \Omega\sqrt{2G}, \quad \varepsilon(+0) \to \Omega\sqrt{2G},$$
$$\frac{G_v}{G} \to \frac{G_e}{G} \to \frac{G_0}{G} \to \Omega^2. \tag{76}$$

At the same time, $E \to 1$ (so long as $|k|$ does not tend to infinity), and hence the function Ω defined by Equation (51) tends to that based on $E = 1$. Thus, in contrast to a homogeneous viscoelastic material, the results for a viscoelastic lattice tend to those for the elastic lattice when the creep and relaxation times, α and β, tend to zero. This conclusion is not unexpected. Indeed, in the case of a homogeneous viscoelastic material there is no time-unit besides α and β, and hence the associated energy release ratios can depend only on the ratio of these parameters. In contrast, a lattice model harbors a time-unit associated with the structure, and an elastic lattice retains a limit with respect to ratios of the creep and relaxation times to this time-unit.

Note that for an elastic lattice, the ratios considered in Equation (76) are independent of the elastic modulus μ/a. It manifests itself only in the dependence $\Omega(V)$.

3.3. Quasi-Static Limit

For a homogeneous material, Equation (75) provides that

$$\mathcal{R}_0 = \frac{G_0}{G} = \frac{\beta}{\alpha} \quad (v = +0). \tag{77}$$

Thus in a viscoelastic homogeneous material, there is a finite dissipation even for a vanishing crack velocity. This is a manifestation of the fact that the energy release at a moving crack tip corresponds to an elastic body with the glassy modulus, while the far field corresponds to the low-rate modulus.

In contrast, the quasi-static limit of the local energy release for a viscoelastic lattice obviously corresponds to an elastic lattice. Indeed, in the case of a large time-interval between the breakage of two neighboring bonds, the influence of viscosity on the lattice state has time to vanish. Thus, dissipation does not change the final strain energy of the bond before it breaks, and this energy is the same as in the elastic lattice. In this quasi-static case, the resistance caused by viscosity in the viscoelastic lattice is the same as the wave resistance in the elastic lattice. It can be determined by the formula (Slepyan, 1982a)

$$\mathcal{R}_0 = \exp\left(-\frac{1}{\pi}\int_0^\pi \ln L(k)\,\mathrm{d}k\right) \quad (v = +0,\ \operatorname{Arg} L = 0), \tag{78}$$

where the function L for the discrete Fourier transformation over x is the same as for the continuous transformation for $t = 0$ (see Appendix 3).

In the case of the square-cell lattice,

$$\mathcal{R}_0 = \exp\left(-\frac{1}{2\pi}\int_0^\pi \ln\frac{4+2(1-\cos k)}{2(1-\cos k)}\,\mathrm{d}k\right) = \sqrt{2}-1. \tag{79}$$

3.4. Viscoelastic Lattice with $\alpha \geq \beta = 0$

As first pointed out by Kostrov and Nikitin (1970), for crack propagation in a viscoelastic homogeneous material with $0 = \beta < \alpha$, the energy release at the propagating crack tip is zero. This can be easily seen in Equation (75). This effect is a consequence of the fact that in this case the short term modulus is infinite. A different conclusion is reached via a viscoelastic lattice model. Indeed, in the case of a lattice, the glassy modulus does not play such a dramatic role, and if $\beta = 0$, Equations (63), (67) and (68) give

$$\sigma(+0) = \Omega\sqrt{2G\mu}, \quad \varepsilon(+0) = \frac{\Omega}{L_{+\alpha}}\sqrt{\frac{2G}{\mu}} \tag{80}$$

and the energy release ratio (69) is still nonzero as are the ratios in (70).

3.5. Viscoelastic Lattice with $v_\alpha \to \infty$, $\alpha/\beta = \text{const}$

For a given $0 < V < 1$, when $V_\alpha \to \infty$, the function $L_{+\alpha}$ in Equation (57) behaves as

$$L_{+\alpha} \sim \left[\frac{4}{1-V^2}\right]^{1/4}\sqrt{V_\alpha}\Omega \to \infty \tag{81}$$

and hence (see Equation (70))

$$\mathcal{R}_e \sim \phi^2\Omega^2, \quad \mathcal{R}_0 \sim \phi\Omega^2. \tag{82}$$

To proceed, represent the integrand in (51) as the sum

$$\int_0^\infty \frac{\operatorname{Arg} L(\xi)}{\xi}\,\mathrm{d}\xi = I_{\mathrm{el}} + I_{\mathrm{hv}},$$

$$I_{\mathrm{el}} = \int_\delta^\infty \frac{\operatorname{Arg} L(\xi)}{\xi}\,\mathrm{d}\xi,$$

$$I_{\mathrm{hv}} = \int_0^\delta \frac{\operatorname{Arg} L(\xi)}{\xi}\,\mathrm{d}\xi, \quad \delta \to 0,\ C_\alpha\delta \to \infty, \tag{83}$$

where I_{hv} and I_{el} denote the 'homogeneous viscoelastic' and 'elastic lattice' portions, respectively. In I_{el}, one may replace E by α/β. With δ small in I_{hv}, one may replace $2(1-\cos k)$ by k^2 and r^2 by $4E$. That is,

$$
\begin{aligned}
I_{el} &\sim \int_0^\infty \operatorname{Arg} L(\xi)\,\frac{d\xi}{\xi} \qquad \left(E=\frac{\alpha}{\beta}\right),\\
I_{hv} &= \frac{1}{2}\int_0^\delta \operatorname{Arg}\frac{1+i\xi V_\alpha}{1-V^2+i\xi(V_\alpha-V^2V_\beta)}\,\frac{d\xi}{\xi}\\
&\sim \frac{1}{2}\int_0^\infty \operatorname{Arg}\frac{1+i\xi}{1-V^2+i(1-V^2\phi)\xi}\,\frac{d\xi}{\xi}\\
&= \frac{\pi}{4}\ln\frac{1-V^2}{1-V^2\phi}.
\end{aligned}
\tag{84}
$$

Now

$$\Omega_{el}=\exp(I_{el}/\pi),\quad \Omega_{hv}=\exp(I_{hv}/\pi)=\left(\frac{1-V^2}{1-V^2\phi}\right)^{1/4}. \tag{85}$$

The integral I_{el} corresponds to an elastic lattice with the instantaneous modulus:

$$\mathcal{R}_{el}=\Omega_{el}^2 \quad \left(E=\frac{\alpha}{\beta}\right), \tag{86}$$

while I_{hv}, corresponds to a homogeneous viscoelastic material (see Equations (82) and (75)),

$$\mathcal{R}_{hv}=\phi\sqrt{\frac{1-V^2}{1-V^2\phi}}=\phi\Omega_{hv}^2. \tag{87}$$

The energy release ratio for a homogeneous viscoelastic material is portrayed in Figure 8.

Under the conditions considered here, the function Ω in (51) may now be represented via the product

$$\Omega=\Omega_{el}\Omega_{hv} \tag{88}$$

and hence

$$\sigma(+0)=\Omega_{el}\Omega_{hv}\sqrt{2G\mu},\quad \varepsilon(+0)=\phi\Omega_{el}\Omega_{hv}\sqrt{2G/\mu}. \tag{89}$$

Since the second term in the last expression for A_v in (21) tends to zero for $V_\alpha\to 0$, it follows that

$$\mathcal{R}_e\sim\phi\mathcal{R}_{el}\mathcal{R}_{hv},\quad \mathcal{R}_v\sim\mathcal{R}_0\sim\mathcal{R}_{el}\mathcal{R}_{hv}. \tag{90}$$

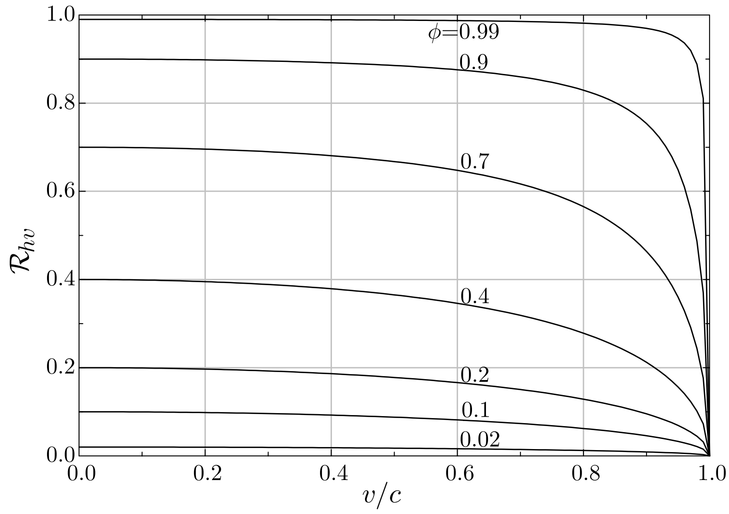

Figure 8. Energy release ratio *versus* velocity for a homogeneous viscoelastic material.

A natural separation of the lattice and viscosity effects has now been obtained. Note that this case ($V_\alpha \to \infty$ with $\phi = \text{const}$) corresponds to a large viscosity as well as a small size of the lattice cell under given viscosity. Indeed, an increase of the parameters V_α and V_β is equivalent to a decrease of the lattice cell size.

For finite steady-state crack speeds, the viscous resistance to crack propagation is high if the ratio α/β is large. At the same time, the quasi-static limit corresponds to the elastic lattice case which is independent of viscosity. Evidently, a pronounced influence of viscosity on the resistance to crack propagation in a lattice is possible: the resistance increases very fast with crack velocity in the low velocity regime.

The above separation may be useful even in numerical simulations of viscoelastic fracture based on a lattice model. When the crack speed variability corresponds to a large viscosity time-scale, the lattice effect can be separated and the coupling with a homogeneous viscoelastic material model becomes transparent.

4. Lattice Strip

Consider now a square-cell lattice strip. Let the particles on the lines $n = N$ and $n = -N - 1$ be fixed (see Figure 9). At the crack surfaces $\eta < 0$, an external loading, $\pm q = \text{const}$, is expected to act on the particles $n = 0$ and -1, respectively. The crack propagates between the lines $n = 0$ and $n = -1$ and the displacement field is anti-symmetric. In the case of this clamped strip

$$u_n^F = u_0^F \frac{\lambda^n - \lambda^{2N-n}}{1 - \lambda^{2N}} \quad (n \geq 0),$$

$$u_n^F = -u_0^F \frac{\lambda^{-1-n} - \lambda^{2N+1+n}}{1 - \lambda^{2N}} \quad (n \leq -1). \tag{91}$$

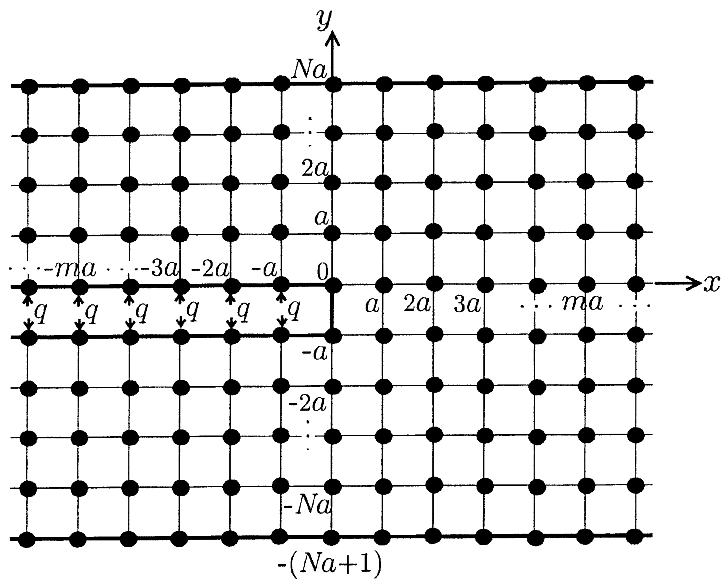

Figure 9. A square-cell clamped lattice strip.

The same dynamic equations (30) and (39) govern the deformations, where

$$\sigma_- = -q = \text{const} \quad (\eta < 0). \tag{92}$$

The following expression for the Fourier transform of σ_m follows from Equations (39) and (91),

$$\begin{aligned}\sigma_+ &= q_- - \frac{\mu}{a}[(h^2 + E)u^F - Eu_1^F] \\ &= q_- - \frac{\mu h}{2a\omega_1}(u_+ + u_-),\end{aligned} \tag{93}$$

in which

$$q_- = q^F = \frac{q}{0 + ik}, \quad \omega_1 = \frac{(r+h)^{2N} - (r-h)^{2N}}{(r+h)^{2N+1} + (r-h)^{2N+1}}. \tag{94}$$

Substituting for u_+ in Equation (93) by the expression in (11), an equation identical to (42) is obtained, but with a different expression for the function L:

$$L = \frac{r\omega_2}{h} = \frac{4E\omega_1}{h} + 1, \quad \omega_2 = \frac{(r+h)^{2N+1} - (r-h)^{2N+1}}{(r+h)^{2N+1} + (r-h)^{2N+1}}. \tag{95}$$

In contrast to the unbounded lattice, the coefficient, $L/(2E)$, and the right part of Equation (42) are now meromorphic functions: they do not contain branch points. Indeed

$$L = \frac{D_1}{D_2} \tag{96}$$

with

$$D_1 = \sum_{m=0}^{N} \binom{2N+1}{2m+1} (r^2)^{N-m} (h^2)^m,$$

$$D_2 = \sum_{m=0}^{N} \binom{2N+1}{2m} (r^2)^{N-m} (h^2)^m, \tag{97}$$

where $\binom{N}{m}$ is a binomial coefficient $(0 \leq m \leq N)$. Further

$$L \to 1 \ (k \to \pm\infty), \quad L \to 2N+1 \ (k \to 0), \quad \operatorname{Ind} L = 0. \tag{98}$$

The last equality follows from the inequality (37) which shows that $\operatorname{Re}\omega_2 > 0$. At the same time, $\omega_2 \to 1$ $(k \to \pm\infty)$. The trajectory for ω_2 in the complex k-plane is closed and the point $k = 0$ lies outside the area enclosed by this trajectory.

Equation (42) can now be rewritten in the form (52)

$$\frac{L_+}{2}\sigma_+ + \frac{\mu E}{aL_-}u_- = \frac{q}{2(0+ik)}\left(L_+ - \frac{1}{L_-}\right) + (1-\phi)\,\frac{\sigma_{+\alpha}}{2(1+ikV_\beta)L_-}, \tag{99}$$

where L_+ and L_- are defined by Equations (48) and (96). For the lattice strip, the latter decomposition gives the following asymptotic expressions:

$$L_+(k) \to 1 \quad (k \to +i\infty),$$
$$L_-(k) \to 1 \quad (k \to -i\infty),$$
$$L_+(k) \to \sqrt{2N+1}\,\Omega \quad (k \to 0),$$
$$L_-(k) \to \sqrt{2N+1}/\Omega \quad (k \to 0),$$
$$\Omega = \exp\left[\frac{1}{\pi}\int_0^\infty \frac{\operatorname{Arg} L(\xi)}{\xi}\,\mathrm{d}\xi\right]. \tag{100}$$

The first term on the right-hand side of Equation (99) must be expressed as the sum of two terms, one regular in the upper half-plane and one regular in the lower half-plane (remembering that $E = E_-$). This is done as follows:

$$\frac{q}{2(0+ik)}\left(L_+ - \frac{1}{L_-}\right) = C_+ + C_-,$$

$$C_+ = \frac{q[L_+(k) - L_+(0)]}{2(0+ik)}, \quad C_- = \frac{q}{2(0+ik)}\left(L_+(0) - \frac{1}{L_-(k)}\right). \tag{101}$$

The solution of Equation (99) is now given by

$$\begin{aligned}\sigma_+ &= \frac{q}{0+ik}\left(1-\frac{L_+(0)}{L_+(k)}\right),\\ u_- &= \frac{qa}{2\mu E(0+ik)}[L_+(0)L_-(k)-1]+(1-\phi)\,\frac{\sigma_{+\alpha}}{2(1+ikV_\alpha)}.\end{aligned} \tag{102}$$

Subsequently (compare with (14)),

$$\begin{aligned}\sigma(+0) &= \lim_{p\to\infty} p\sigma_+(ip) = q[L_+(0)-1],\\ \varepsilon(+0) &= \frac{2u(0)}{a} = \frac{2}{a}\lim_{p\to\infty} pu_-(-ip)\\ &= \frac{q}{\mu}\left\{\phi[L_+(0)-1]+(1-\phi)\left[\frac{L_+(0)}{L_{+\alpha}}-1\right]\right\},\\ u(-\infty) &= \lim_{p\to 0} pu_-(-ip) = qaN/\mu.\end{aligned} \tag{103}$$

The global energy release rate can be defined as the total work of the traction minus the elastic energy per unit length of the lattice strip far to the left of the crack tip. It is

$$G = 2[qu(-\infty) - qu(-\infty)/2] = q^2N/\mu. \tag{104}$$

For the clamped lattice strip, the energy release ratios are given by

$$\begin{aligned}\mathcal{R}_0 &= \frac{1}{2N}(\Psi-1)\left[\phi(\Psi-1)+(1-\phi)\left(\frac{\Psi}{L_{+\alpha}}-1\right)\right],\\ \mathcal{R}_e &= \frac{1}{2N}\left[\phi(\Psi-1)+(1-\phi)\left(\frac{\Psi}{L_{+\alpha}}-1\right)\right]^2,\\ \Psi &= \sqrt{2N+1}\Omega.\end{aligned} \tag{105}$$

These expressions tend to the corresponding expressions in (70) for the unbounded lattice when $N \to \infty$.

5. Quasi-Static Behavior

For both the unbounded lattice and the lattice strip, the case of slow steady-state crack propagation ($v \ll c$) is considered. In fact, the asymptotic behavior is examined for $V \to 0$ without restrictions respective to the parameter V_α: it can tend to zero, a nonzero value, or infinity. In this case, in the determination of the energy release ratios, the inertia of the lattice can be neglected because the corresponding term tends to zero for any finite k.

The function $L(k)$ has a static limit for $V \to 0$ which is independent of the creep and relaxation times α and β. In the limiting case, it is a periodic, nonnegative function, and the period $T = 2\pi$. The factorization of a Cauchy-type integral for a periodic function is required in this case. Eatwell and Willis (1982) and Slepyan (1982a) showed that any nonnegative, periodic, locally integrable function $L(k)$ may be factorized as follows (Arg $L = 0$), with $L = L_+ L_-$,

$$L_\pm(k) = \exp\left[\pm\frac{1}{2iT}\int_{-T/2}^{T/2} \operatorname{In} L(\xi) \cot\frac{\pi(\xi - k)}{T}\,\mathrm{d}\xi\right], \qquad (106)$$

where $\operatorname{Im} k > 0$ for L_+ and $\operatorname{Im} k < 0$ for L_-, respectively; T is the period. The limiting values are (noting that $L(-\xi) = L(\xi)$)

$$L_+(k) \to L_{+\infty} \quad (k \to i\infty),$$

$$L_-(k) \to L_{+\infty} \quad (k \to -i\infty),$$

$$L_{+\infty} = \exp\left[\frac{1}{T}\int_0^{T/2} \operatorname{In} L(\xi)\,\mathrm{d}\xi\right], \qquad (107)$$

and

$$L_{+\alpha} = \exp\left[\frac{1}{2iT}\int_{-T/2}^{T/2} \operatorname{In} L(\xi) \cot\frac{\pi}{T}\left(\xi - \frac{i}{V_\alpha}\right)\mathrm{d}\xi\right]. \qquad (108)$$

The expressions in (61) are unmodified.

For the unbounded square-cell lattice, the function L in Equation (106) can be re-expressed in the form

$$L = \left(\frac{4 + 2(1 - \cos k)}{2(1 - \cos k) + 0}\right)^{1/2} = \left(\frac{1 + \sin^2 k/2}{\sin^2 k/2 + 0}\right)^{1/2}. \qquad (109)$$

For this function, the following explicit factorization is valid:

$$L_+ = \left[\frac{\sin(k/2 + i\operatorname{Arsh} 1)}{\sin(k/2 + i0)}\right]^{1/2} = \left[\frac{\sqrt{2}\sin k/2 + i\cos k/2}{\sin(k/2 + i0)}\right]^{1/2},$$

$$L_- = \left[\frac{\sin(k/2 - i\operatorname{Arsh} 1)}{\sin(k/2 - i0)}\right]^{1/2} = \left[\frac{\sqrt{2}\sin k/2 - i\cos k/2}{\sin(k/2 - i0)}\right]^{1/2}. \qquad (110)$$

Note that this factorization differs from that derived above in Equation (48) in spite of the fact that the function L in Equation (109) is the limit of that defined by

Equation (43) for $V \to 0$. The asymptotes of $L_\pm$ are

$$L_+ \sim \sqrt{\frac{2}{0 - ik}}, \quad L_- \sim \sqrt{\frac{2}{0 + ik}} \quad (k \to 0),$$

$$L_+ \sim \sqrt{\sqrt{2} + 1} \quad (k \to i\infty),$$

$$L_- \sim \sqrt{\sqrt{2} - 1} \quad (k \to -i\infty), \tag{111}$$

and

$$L_{+\alpha} = \left(\sqrt{2} + \coth \frac{1}{2V_\alpha}\right)^{1/2}. \tag{112}$$

On the basis of these results and the general solution in (61), the far-field stress is given by

$$\sigma_+ \sim \frac{C}{\sqrt{2(0 - ik)}}. \tag{113}$$

From this it follows that

$$K_{\mathrm{III}} = C/\sqrt{a}, \quad C = \sqrt{2G\mu}. \tag{114}$$

The Fourier transforms of the stress and crack opening displacement can now be explicitly written in terms of the far-field energy release rate:

$$\sigma_{+\alpha} = \frac{V_\alpha \sqrt{2G\mu}}{L_{+\alpha}},$$

$$\sigma_+ = \frac{\sqrt{2G\mu}}{0 - ik}\left[\frac{\sin(k/2 + i0)}{\sqrt{2}\sin k/2 + i\cos k/2}\right]^{1/2},$$

$$u_- = \frac{a}{2E}\frac{\sqrt{2G/\mu}}{0 + ik}\left[\frac{\sqrt{2}\sin k/2 - i\cos k/2}{\sin(k/2 - i0)}\right]^{1/2} + \frac{a\,(1 - \phi)\,\sigma_{+\alpha}}{2\mu(1 + ikV_\alpha)}. \tag{115}$$

Note that only the crack opening displacement depends on the crack speed (due to the presence of the second term); the stresses do not. By neglecting inertia, the stress distribution is independent of viscosity.

Displacement continuity is not maintained in the vicinity of a crack tip a massless viscoelastic lattice:

$$u(+0) \neq u(-0). \tag{116}$$

In this case, the limiting strain of the breaking bond is defined by the formula in (14) only.

The limiting stress, strain and displacement discontinuity are

$$\begin{aligned}\sigma(+0) &= \vartheta\sqrt{2G\mu},\\ \varepsilon(+0) &= \vartheta Z\sqrt{2G/\mu},\\ u(-0)-u(+0) &= \lim_{k\to -i\infty}(iku_-) - a\varepsilon(+0)/2 = a\vartheta\phi\sqrt{G/\mu}\end{aligned} \quad (117)$$

in which

$$\begin{aligned}\vartheta &= \sqrt{\sqrt{2}-1},\\ Z &= \phi + (1-\phi)\left[(\sqrt{2}-1)\left(\sqrt{2}+\coth\frac{1}{2V_\alpha}\right)\right]^{-1/2}.\end{aligned} \quad (118)$$

For the slow steady-state fracture of an unbounded viscoelastic lattice, the energy release ratios are

$$\mathcal{R}_e = \frac{G_e}{G} = (\sqrt{2}-1)Z^2, \quad \mathcal{R}_0 = \frac{G_0}{G} = (\sqrt{2}-1)Z. \quad (119)$$

Each ratio approaches the value $\sqrt{2}-1$ at zero crack velocity (compare with Equation (79)).

These results, derived independently of the dynamic treatment, present exact quasi-static asymptotes for low crack velocities ($V \ll 1$) in an unbounded lattice, valid for any value of the parameter V_α. While the dynamic asymptote for $V \to 0$, and the quasi-static solution itself are different, the difference manifests itself just after the breakage of a bond. Then, due to a large time-interval between the breakage of neighboring bonds, the dynamic state quickly approaches the quasi-static state. When the parameter $V_\alpha \ll 1$ the energy release ratios

$$\mathcal{R}_e \sim \mathcal{R}_0 \sim (\sqrt{2}-1). \quad (120)$$

However, if it happens that $V \ll 1$, $C_\alpha \gg 1$, $\phi \ll 1$, such that for a small increase in the crack velocity the parameter V_α becomes large, these same ratios are much reduced

$$\mathcal{R}_e \sim (\sqrt{2}-1)\phi^2, \quad \mathcal{R}_0 \sim (\sqrt{2}-1)\phi. \quad (121)$$

This reduction, which correlates with an increase in the resistance to crack propagation, occurs over a small portion of the steady-state crack speed regime. Thus, in this model, the speed of a slowly propagating crack will depend very strongly on the applied load or the far-field energy release rate. In these considerations, it has been assumed that a stable crack obeys a criterion like $G_0 \le G_c$, yet in the case of a slowly propagating crack, it is the same as a limiting strain criterion: $\varepsilon(+0) \le \varepsilon_c$.

Consider now the case of slow steady-state crack propagation ($v \ll c$) in a clamped square-cell lattice strip. The quasi-static solution may be deduced based on the formulas (102) and (14) for σ_+ and ε_+, respectively, and by noting (96–97), with

$$\begin{aligned} L &= \frac{D_{1s}}{D_{2s}}, \\ \Lambda &= \frac{\sin^2 k/2 + 0}{1 + \sin^2 k/2}, \\ D_{1s} &= \sum_{m=0}^{N} \binom{2N+1}{2m+1} \Lambda^m, \\ D_{2s} &= \sum_{m=0}^{N} \binom{2N+1}{2m} \Lambda^m. \end{aligned} \tag{122}$$

In this case,

$$\begin{aligned} L_+(0) &= \sqrt{2N+1}, \\ L_{+\alpha} &\to L_{+\infty} \quad (V_\alpha \to 0), \\ L_{+\alpha} &\to L_+(0) \quad (V_\alpha \to \infty) \end{aligned} \tag{123}$$

and

$$\sigma(+0) = q(\sqrt{2N+1}/L_{+\infty} - 1), \quad \varepsilon(+0) = qZ_1/\mu. \tag{124}$$

in which

$$Z_1 = \phi\left(\frac{\sqrt{2N+1}}{L_{+\infty}} - 1\right) + (1-\phi)\left(\frac{\sqrt{2N+1}}{L_{+\alpha}} - 1\right). \tag{125}$$

For the slow steady-state fracture of an clamped viscoelastic lattice, the energy release ratios are

$$\mathcal{R}_e = Z_1^2/2N, \quad \mathcal{R}_0 = (\sqrt{2N+1}/L_{+\infty} - 1)Z_1/2N. \tag{126}$$

These ratios have the following asymptotes:

$$\begin{aligned} \mathcal{R}_e &\sim \mathcal{R}_0 \sim (\sqrt{2N+1}/L_{+\infty} - 1)^2/2N \quad (V_\alpha \to 0), \\ \mathcal{R}_e &\sim \phi^2(\sqrt{2N+1}/L_{+\infty} - 1)^2/2N \quad (V_\alpha \to \infty), \\ \mathcal{R}_0 &\sim \phi(\sqrt{2N+1}/L_{+\infty} - 1)^2/2N, \quad (V_\alpha \to \infty). \end{aligned} \tag{127}$$

6. Discussion

The role of the discrete structure of the lattice and the dimensionless viscoelastic parameters $V_\alpha = \alpha v/a$ and $\phi = \beta/\alpha$ are shown in Figures 2–7. From the macro-level point of view, both the radiation by high-frequency waves due to the structure response and dissipation itself can be called 'dissipation'. The latter quantity is the difference between the total energy release, G, as the energy flux from infinity and the energy lost in the breaking bonds, G_0. For these plots, the latter energy is defined in terms of the limiting tensile force and strain: $G_0 = \sigma(+0)\varepsilon(+0)/2$ and the ratio, $R_0 = G_0/G$, is shown.

The curve $\phi = 1$ shown in Figure 2 corresponds to an elastic lattice (shown separately in Figure 7). The dependence for $\phi = 1$ is characterized by the following distinctive features. The dissipation is finite for a vanishing crack speed: $R_0(0) = \sqrt{2}-1$. This is due to radiation by high-frequency waves which are excited by each break of the bond. In the presence of viscosity ($\phi < 1$), these waves dissipate but the long-wave energy flux from infinity does not (since $v = +0$), and this initial point, $R_0(0)$, is the same for any dissipation. Next, R_0 possesses a maximum for each value of ϕ (if ϕ is not too low), and hence the dissipation (by this radiation) has a minimum at approximately half the shear wave speed ($v/c \approx 1/2$). Further, R_0 has a nonmonotone dependence on crack speed for low v/c values. This is a manifestation of a strong dependence on the number of different waves, and the energy which they carry away from the propagating crack tip, on the crack speed. Finally, the energy release ratio tends to zero and hence the resistance to crack propagation increases without bound as the crack speed approaches the shear wave speed. As can be seen in Figure 2, the influence of viscosity increases should $\phi = \beta/\alpha$ decrease further; this results in a monotonic increase of the resistance over the whole crack speed range ($0 < v/c < 1$).

An influence of $C_\alpha = \alpha c/a$ on the considered dependencies for $\phi = 0.5, 0.9$ and 0.1 are presented in Figures 3, 4 and 5, respectively. The viscosity has almost no influence for $\phi = 0.9$ (Figure 4), cannot prevent the nonmonotonic behavior when $\phi = 0.5$ (Figure 3), and has a strong influence in the case of ϕ small (Figure 5). The influence of C_α for some values of the crack speed in the latter case is shown in Figure 6.

The energy release ratio for a homogeneous viscoelastic material is portrayed in Figure 8. In contrast to a lattice, the result here depends on ϕ only and there is no pronounced increase of the resistance to crack propagation over an initial region of the crack speed. Hence slow crack growth cannot occur within the framework of a homogeneous viscoelastic material model.

The role of the discrete structure of the lattice strip, and the nondimensional parameters V_α and ϕ, and the strip width is shown in Figures 10–15. In each figure, the curves differ by the parameter N which characterizes the strip width, $(2N+1)a$. These plots correspond to $\phi = 0.5$ and C_α = 0.1, 1, 2, 10 and 100 in Figures 10, 11, 12, 13 and 14, respectively. The results for the elastic lattice strip are presented

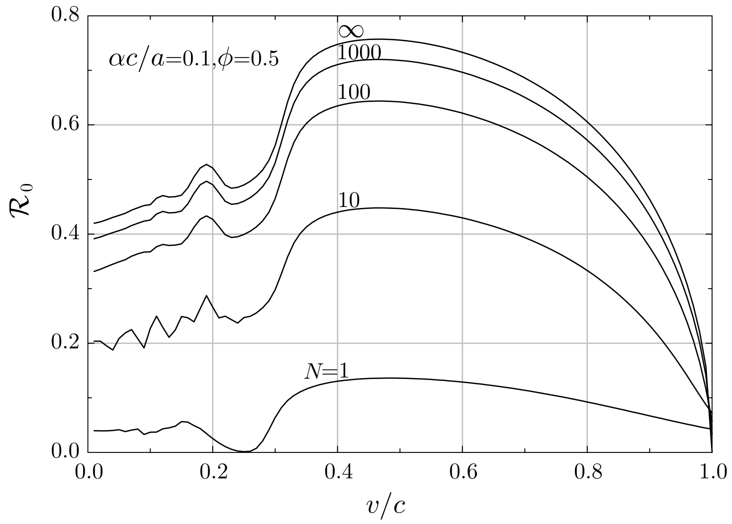

Figure 10. Effective energy release ratio *versus* velocity for the lattice strip with $C_\alpha = 0.1, \phi = 0.5$.

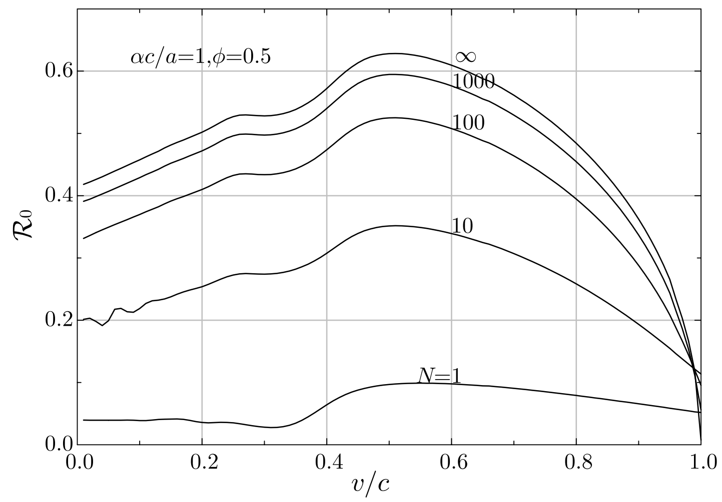

Figure 11. Effective energy release ratio *versus* velocity for the lattice strip with $C_\alpha = 1, \phi = 0.5$.

in Figure 15. For $\phi = 0.5$, an increase in C_α does not eliminate the nonmonotonic behavior of the energy release ratio *versus* crack speed. Also, the lattice strip results differ from those for the unbounded lattice even for rather large values of N. At the same time, qualitatively, the plots for $N = \infty$ and $N = 10$ are similar. These conclusions are important as regards the interpretation of numerical modeling using lattice strips of finite width. The energy release ratio decreases with a decrease of the strip width (considering that the bond length, a, remains the same). The

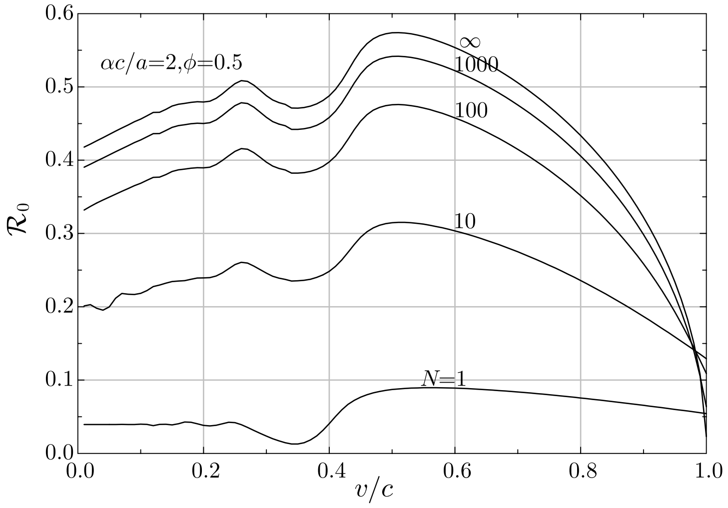

Figure 12. Effective energy release ratio *versus* velocity for the lattice strip with $C_\alpha = 2, \phi = 0.5$.

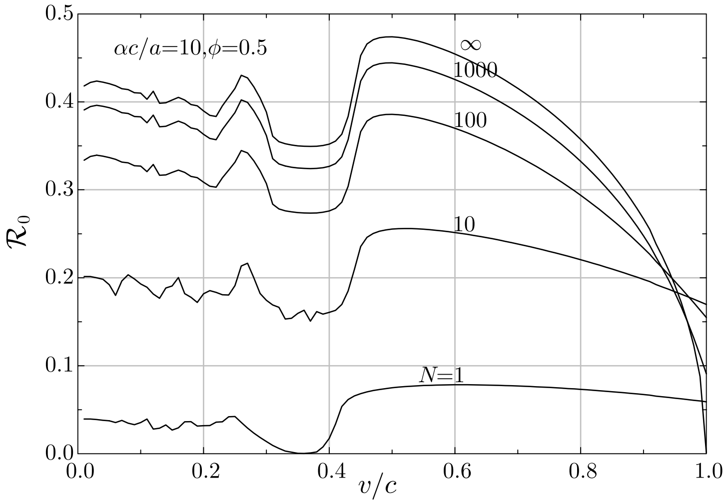

Figure 13. Effective energy release ratio *versus* velocity for the lattice strip with $C_\alpha = 10, \phi = 0.5$.

dependencies $R_0(v/c)$ approach that for the unbounded lattice with an increase of the width, but not rapidly (see Figures 10–15).

The normalized resistance to quasi-static crack growth in a lattice strip is shown in Figure 16 for $0 \leq V_\alpha \leq 1000$ for a range of ϕ-values, and in Figure 17 for $0 \leq V_\alpha \leq 10$. Note that a boundary-layer type of stable dependence for G/G_0 *versus* V_α arises over an initial portion of the first region, with a decrease in the value of $\phi = \beta/\alpha$.

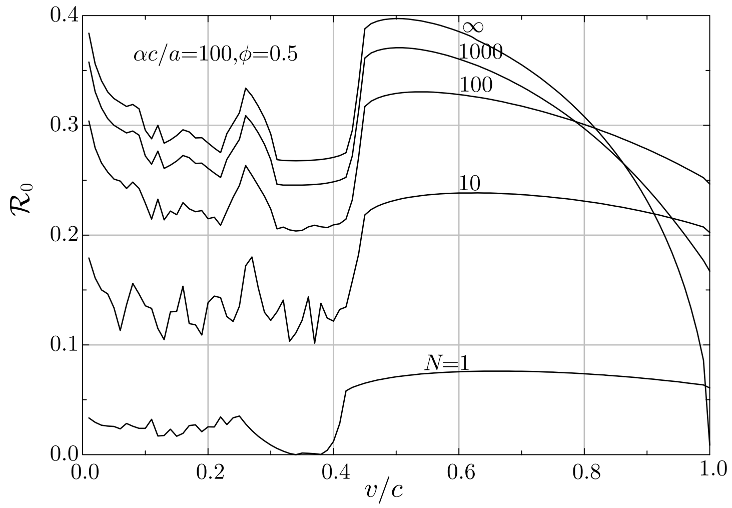

Figure 14. Effective energy release ratio *versus* velocity for the lattice strip with $C_\alpha = 100$, $\phi = 0.5$.

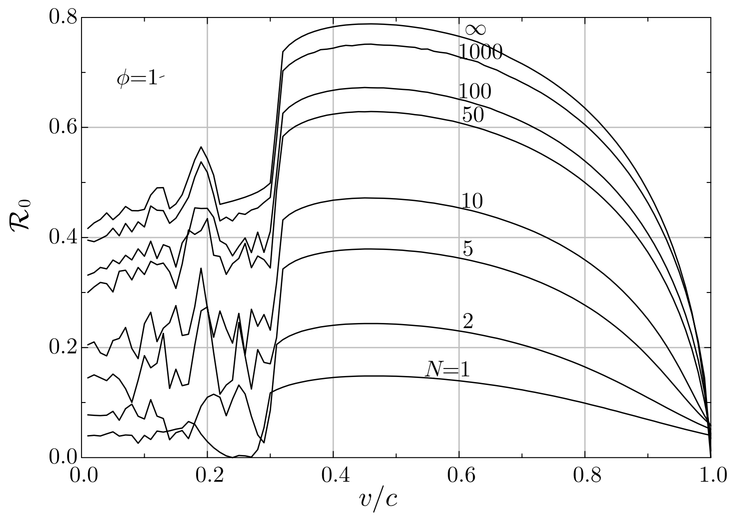

Figure 15. Energy release ratios for the elastic lattice strip ($\phi = 1$).

7. Conclusions

In this paper, the main aspects of steady-state crack propagation in a square-cell viscoelastic lattice have been studied. In an elastic lattice, there is structure but no viscosity. In viscoelastic homogeneous and cohesive-zone models, there is viscosity but no material structure. In this viscoelastic lattice model, there are both material structure and viscosity. Coupling the latter two factors causes a diverse array of crack propagation phenomena. As shown, these factors can be separated in the case of pronounced viscosity, $C_\alpha = c\alpha/a \to \infty$, $\phi = \beta/\alpha =$ const.

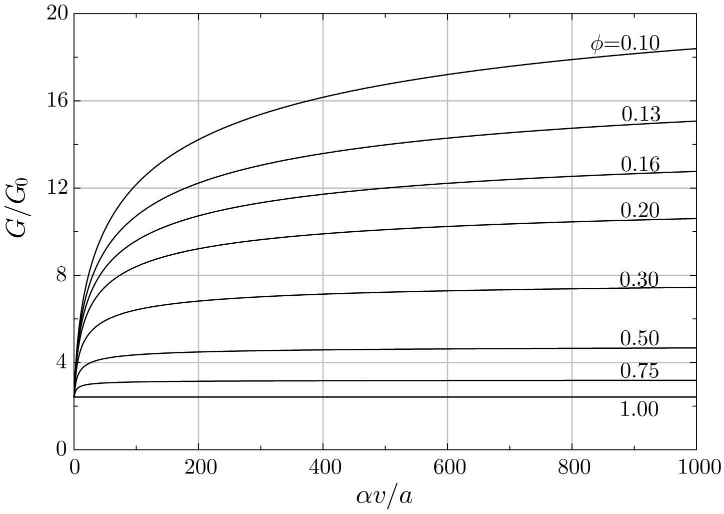

Figure 16. Normalized resistance to quasi-static crack growth in a lattice strip *versus* $0 \leq V_\alpha \leq 1000$ for a range of ϕ-values.

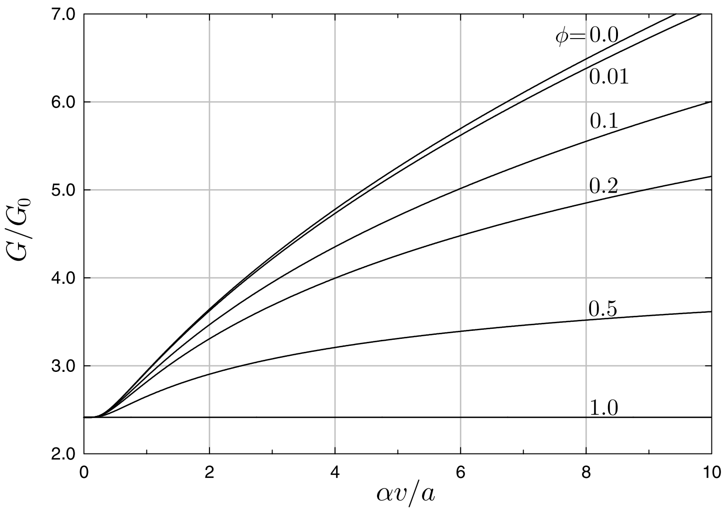

Figure 17. Normalized resistance to quasi-static crack growth in a lattice strip *versus* $0 \leq V_\alpha \leq 10$ for a range of ϕ-values.

No dynamic effects such as radiation given steady-state crack propagation exist in the cohesive zone models. Also, the lattice differs (from cohesive zone models) by a regular structure, the same in the crack path and in the bulk of the material. This creates no artificial restrictions on the crack-speed-dependent distribution of dissipation caused by viscosity.

The dynamic viscoelastic lattice model reduces to the requisite quasi-static limit and homogeneous viscoelastic material behavior. The first limit coincides with the static state. However, during slow crack growth, radiation still occurs because of discrete bond-rupture events. In the homogeneous material case, an important boundary-layer-type dependence, which is revealed by the lattice model in the case of pronounced viscosity, is lost as follows: in a viscoelastic lattice, the rapid rise in resistance for early crack acceleration allows slow cracks to grow stably. Formally, this phenomenon manifests itself in a very rapid rise in the value of $V_\alpha = v\alpha/a$ due solely to a small increase of the normalized crack speed $V = v/c$ from $v = 0$ (if the parameter C_α is large). In homogeneous materials, $V_\alpha = \infty$ for any $v > 0$ because $a = 0$, and this physically important behavior cannot be modeled.

In contrast to an elastic lattice, the high-frequency waves cannot propagate as oscillations of the lattice structure in the viscoelastic lattice from infinity because of dissipation due to viscosity. The only structure-associated waves which exist are excited by the propagating crack. For moderate viscosity, the amount of energy carried away from the crack tip by these waves decreases with an increase of the crack speed (for slow crack speeds). In this case, slow crack growth is not possible. With increasing viscosity, that is, for an increase of C_α given that ϕ is low, the resistance by viscosity increases and the role of the radiation resistance becomes less important. The latter observation, however, does not concern the zero-speed limit where the influence of high frequency wave radiation on the resistance to crack propagation remains important.

The solutions derived in this paper give the relations between the global (far-field) and local energy release rates, and relations between the far-field energy release rate and the breaking bond strain. These crack-speed and viscosity-dependent relations can be used for the crack propagation determination under given conditions and fracture criterion. However, in the formulation adopted in this paper, the crack speed is prescribed and the energy release ratios are considered under a given speed. In general, to come to a conclusion whether a steady-state solution exists, and if it does, at what crack speed one has to invoke a material-dependent fracture criterion and to trace whether it is satisfied by the solution (over each successive time-interval from one periodic bond-fracture to the next). The analysis shows, in particular, that for large viscosity the limiting strain considered is maximal at the end of each such time-interval. Were one to use the limiting strain criterion as a fracture criterion, the solutions derived in this paper could be used to determine the associated crack speed. This observation applies equally to slow stable crack growth.

Note that the lattice model can also be looked upon as a finite-element approximation of a continuous material. Numerical simulations of this type of lattice model, including various nonlinear extensions, requires no additional finite-element approximation. On the other hand, the analytical solutions derived for the infinite lattice and lattice strip, and phenomena revealed by this model, can serve as benchmark solutions for finite-element analyses.

Finally, the solutions derived are expressed in terms of nondimensional parameters $C_\alpha = \alpha c/a$, $V_\alpha = \alpha v/a$ and ϕ. This suggests a structure-associated size effect. Indeed, for given relaxation and creep times and the crack speed, the size of the parameters C_α and V_α is determined by the size a. Thus, under the same conditions, the viscosity parameters governing fracture increase with a decrease in the structure size.

Acknowledgements

This research was supported by the Ministry of Science, Israel (for the first two authors, Grant No. 9673-1-96), and by U.S. Office of Naval Research (for the third author, Grant No. N00014-96-1-1210).

Appendix

1. Causality, Stability and Passivity

Consider a viscoelastic stress-strain relation

$$\sigma(t) = E_t(t) * \varepsilon(t), \tag{128}$$

where the symbol $*$ means the convolution

$$E_t(t) * \varepsilon(t) = \int_{-\infty}^{t} E_t(t-\tau)\varepsilon(\tau)\,\mathrm{d}\tau \tag{129}$$

and $E_t(t)$ is a generalized function. In the convolution integral it is taken into account that a future strain does not influence the current stress and hence $E_t(t) = 0$ for $t < 0$. This is the *causality principle*.

If the relation (128) is valid for any t this equality can be also expressed in terms of the Fourier transforms:

$$\sigma^F(k) = E(k)\varepsilon^F(k), \quad E(k) = E_t^F(k), \tag{130}$$

where the Fourier transformation is defined by the relation as

$$\sigma^F(k) = \int_{-\infty}^{\infty} \sigma(t)\,\mathrm{e}^{ikt}\,\mathrm{d}t. \tag{131}$$

Along with the causality principle, the viscoelastic modulus should obey the requirements of *stability* and *passivity* if the material is really stable and passive. The first requirement means that stress does not increase exponentially under a fixed strain and vice versa. Hence in the Fourier representation, singular and zero points of E are placed in the upper half-plane of k. Note that in the case of the Fourier transformation over η, the Fourier transform, E has no such points in the lower half-plane, that is $E = E_-$.

The feature of passivity means that work cannot be negative, that is

$$A = \int_0^t \sigma(t)\dot{\varepsilon}\,\mathrm{d}t \geq 0 \quad \left(\dot{\varepsilon} = \frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\right). \tag{132}$$

Consider a closed path of strain: $\varepsilon = 0$ for $t < 0$ and $t > T < \infty$. In this case, the Parseval equality can be used in the form

$$\begin{aligned} A &= \int_0^T \sigma(t)\dot{\varepsilon}\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(k)\varepsilon^F(k)\overline{(\dot{\varepsilon})^F(k)}\,\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E(k)ik|\varepsilon^F(k)|^2\,\mathrm{d}k = -\frac{1}{\pi}\int_{-\infty}^{0} \operatorname{Im} E(k)|\varepsilon^F(k)|^2 k\,\mathrm{d}k \\ &= -\frac{1}{\pi}\int_0^{\infty} \operatorname{Im} E(k)|\varepsilon^F(k)|^2 k\,\mathrm{d}k. \end{aligned} \tag{133}$$

It follows from this that for a passive material

$$\operatorname{Im} kE(k) \leq 0. \tag{134}$$

Note that this inequality is changed for the opposite one in the case of the Fourier transformation over η. For the above-considered constitutive equation (1), Equation (134) leads to the inequality $\alpha \geq \beta$.

2. Generalized Standard Model

Consider a more general one-dimensional force-strain relation for a viscoelastic bond:

$$\prod_n \left(1 + \beta_n \frac{\mathrm{d}}{\mathrm{d}t}\right)\sigma = \prod_n \left(1 + \alpha_n \frac{\mathrm{d}}{\mathrm{d}t}\right)\varepsilon, \quad 1 \leq n \leq n^*. \tag{135}$$

Here, note the restriction that no two values among α_n, β_n are the same. For a steady-state problem in which the considered functions depend on η only, this relation takes the form

$$\prod_n \left(1 - V_{\beta n} \frac{\mathrm{d}}{\mathrm{d}\eta}\right)\sigma = \prod_n \left(1 - V_{\alpha n} \frac{\mathrm{d}}{\mathrm{d}\eta}\right)\varepsilon. \tag{136}$$

Using the right-sided Fourier transformation it can be found that

$$\sigma_+ \prod_n (1 + ikV_{\beta n}) - 2u_+ \prod_n (1 + ikV_{\alpha n}) = \sum_n a_n k^{n-1}, \tag{137}$$

where the polynomial of the power $n^* - 1$ in the right part arises as a result of this transformation; it depends on values of $\sigma(+0)$ and $u(0)$ and their derivatives up to the order $n^* - 1$. The constants a_n are defined by the requirement that σ_+ and u_+ do not contain poles and zeros in the upper half-plane of k, i.e., they really can be marked by the subscript $+$. These conditions are satisfied by the expressions

$$\begin{aligned} \sigma_+(k) &= 2\frac{\prod_n(1+ikV_{\alpha n})}{\prod_n(1+ikV_{\beta n})}u_+(k) \\ &\quad - 2\sum_{m=1}^{n^*}\frac{\prod_n(1-\alpha_n/\beta_m)u_+[i/V_{\beta m}]}{(1+ikV_{\beta m})\prod_{n\neq m}(1-\beta_n/\beta_m)}, \\ u_+(k) &= \frac{\prod_n(1+ikV_{\beta n})}{2\prod_n(1+ikV_{\alpha n})}\sigma_+(k) \\ &\quad - \sum_{m=1}^{n^*}\frac{\prod_n(1-\beta_n/\alpha_m)\sigma_+[i/V_{\alpha m}]}{2(1+ikV_{\alpha m})\prod_{n\neq m}(1-\alpha_n/\alpha_m)}. \end{aligned} \tag{138}$$

In the simplest case $n^* = 1$, $\prod_{n\neq m} = 1$, and the relation in (11) is obtained.

3. Continuous and Discrete Fourier Transforms

Consider a function $f(x - vt)$, $x = an$, $n = 0, \pm 1, \ldots$. The Fourier transformation over $(-vt/a)$ leads to

$$f^{(-vt/a)}(k) = \int_{-\infty}^{\infty} f(x - vt)\, \mathrm{e}^{ik\eta - ikn}\, \mathrm{d}(-vt/a) = \mathrm{e}^{-ikn} f^F(k), \tag{139}$$

where

$$f^F(k) = \int_{-\infty}^{\infty} f(\eta)\, \mathrm{e}^{ik\eta}\, \mathrm{d}\eta, \quad \eta = (x - vt)/a. \tag{140}$$

The discrete Fourier transform of (139) is

$$g(k, q) = \sum_{n=-\infty}^{\infty} f^{(-vt/a)}\, \mathrm{e}^{iqn} = 2\pi f^F(k)\delta(k - q), \tag{141}$$

where δ is the Dirac delta-function.

The discrete Fourier transform of f follows from using the inverse transformation of (141) over k:

$$f^n(q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(k, q)\, \mathrm{e}^{ikvt/a}\, \mathrm{d}k = f^F(q)\, \mathrm{e}^{iqvt/a}. \tag{142}$$

In the problems considered in this paper, the discrete Fourier transform is needed for $t = 0$, when the limiting stress and strain are achieved in the bond with the coordinate $x = 0$. For this time the discrete and continuous transformations give us the same result:

$$f^n(q) = f^F(q). \tag{143}$$

4. Causality Principle for Steady-State Solutions

A steady-state solution for a domain unbounded in the x-direction can be non-unique due to the existence of one or a set of inherent solutions as free waves which can be considered as produced by sources at infinity. The existence of such waves reflects itself by singular and zero points on the real k-axis of the Fourier transforms of the steady-state solutions. If such sources are not allowed by the problem formulation, they must be excluded from the solution. This can be achieved in various ways, in particular by the use of a rule based on the causality principle. Under this principle, the steady-state solution is considered as a limit (time $t \to \infty$) of the solution to the corresponding transient problem with zero initial conditions.

Consider the inverse Laplace (with respect to time t) and Fourier (with respect to the coordinate x) transformations, and let it be $u^{LF}(s, k)$, of a function $u(t, x)$

$$u(t, x) = \frac{1}{2\pi}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{-i\infty+0}^{i\infty+0} u^{LF}(s, k)\,\mathrm{e}^{st-ikx}\,\mathrm{d}s\,\mathrm{d}k. \tag{144}$$

The solution is required to become steady-state in the coordinate system moving along the x-coordinate with velocity v, that is in the coordinate $\eta = x - vt$. Substitute $x = \eta + vt$ in the representation (144), giving

$$\begin{aligned} u(t, x) &= w(t, \eta) \\ &= \frac{1}{2\pi}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{-i\infty+0}^{i\infty+0} u^{LF}(s, k)\,\mathrm{e}^{(s-ikv)t-ik\eta}\,\mathrm{d}s\,\mathrm{d}k. \end{aligned} \tag{145}$$

Now denote $s = ikv + p$, where k is assumed to be real, Re $p = +0$:

$$w(t, \eta) = \frac{1}{2\pi}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{-i\infty+0}^{i\infty+0} u^{LF}(p + ikv, k)\,\mathrm{e}^{pt-ik\eta}\,\mathrm{d}p\,\mathrm{d}k. \tag{146}$$

The last equality is the inverse Laplace transform (with respect to the explicitly written t) and Fourier transform (with respect to η) transformations of $w^{LF}(p, k) = u^{LF}(p + ikv, k)$ of the original $w(t, \eta)$. From this it follows that the double transformation in the moving coordinate system is

$$w^{LF\eta}(p, k) = u^{LF}(p + ikv, k). \tag{147}$$

It is assumed that a limit of the function $w(t,\eta)$ exists when $t \to \infty$, $\eta = \text{const}$. The well-known limiting theorem states:

If a limit of a function $f(t)$ with $t \to \infty$ exists, it is

$$\lim_{t\to\infty} f(t) = \lim_{p\to 0} pf^L(p), \tag{148}$$

where $f^L(p)$ is the Laplace transform of $f(t)$. For our problem this means that

$$\lim_{t\to\infty} w^{F\eta}(t,k) = \lim_{p\to +0} pu^{LF}(p+ikv,k). \tag{149}$$

However, if a nonzero limit exists, its Laplace transform contains the multiplier $1/p$ as for each constant original. Thus the causality-principle-based rule states:

For a steady-state solution, in the Fourier transform corresponding to the moving coordinate system, each product ikv must be supplemented by the term $+0$ which means the limit from the right.

This makes the solution uniquely defined. Mathematically, it makes the real k-axis free of the singular and zero points.

References

Atkinson, C. and List, R.D., 'A moving crack problem in a viscoelastic solid', *Internat. J. Engrg. Sci.* **10**, 1972, 309–322.

Atkinson, C. and Popelar, C.H., 'Antiplane dynamic crack propagation in a viscoelastic strip', *J. Mech. Phys. Solids* **27**, 1979, 431–439.

Bažant, Z.P. and Jirásek, M., 'R-curve modeling of rate and size effects in quasibrittle fracture', *Internat. J. Fracture* **62**, 1993, 355–373.

Bažant, Z.P. and Li, Y.-N., 'Cohesive crack model with rate-dependent crack opening and viscoelasticity: I. Mathematical model and scaling', *Internat. J. Fracture* **86**, 1997, 247–265.

Bažant, Z.P. and Planas, J., *Fracture and Size Effect in Concrete and Other Quasibrittle Materials*, CRC Press, Boca Raton, FL, 1998.

Costanzo, F. and Walton, J.R., 'A study of dynamic crack growth in elastic materials using a cohesive zone model', *Internat. J. Engrg. Sci.* **35**, 1997, 1085–1114.

Costanzo, F. and Walton, J.R., 'Numerical simulations of a dynamically propagating crack with a nonlinear cohesive zone', *Internat. J. Fracture* **91**, 1998, 373–389.

Eatwell, G.P. and Willis, J.R., 'The Excitation of a fluid-loaded plate stiffened by a semi-infinite array of beams', *IMA J. Appl. Mech.* **29**(3), 1982, 247–270.

Fineberg, J., Gross, S.P., Marder, M. and Swinney, H.L., 'Instability in dynamic fracture', *Phys. Rev. Lett.* **67**(4), 1991, 457–460.

Fineberg, J., Gross, S.P., Marder, M. and Swinney, H.L., 'Instability in the propagation of fast cracks', *Phys. Rev. B* **45**(10), 1992, 5146–5154.

Freund, L.B., *Dynamic Fracture Mechanics*, Cambridge University Press, Cambridge, 1990.

Geubelle, P.H., Daniluk, M.J. and Hilton, H.H., 'Dynamic mode III fracture in viscoelastic media', *Int. J. Solids Structures* **35**, 1998, 761–782.

Herrmann, J.M. and Walton, J.R., 'A comparison of the dynamic transient antiplane shear crack energy release rate for standard linear solid and power law type viscoelastic materials', *ASME J. Energy Res. Tech.* **113**, 1991, 222–229.

Herrmann, J.M. and Walton, J.R., 'On the energy release rate for dynamic transient mode 1 crack propagation in a general viscoelastic body', *Quart. Appl. Math.* **52**, 1994, 201–228.

Hillerborg, A., Modéer, M. and Petersson, P.E., 'Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements', *Cem. & Concrete Res.* **6**, 1976, 773–782.

Kanninen, M.F. and Popelar, C.H., *Advanced Fracture Mechanics*, Oxford University Press, Oxford, 1985.

Knauss, W.G., 'Delayed failure – The Griffith problem for linearly viscoelastic materials', *Internat. J. Fracture* **6**, 1970, 7–20.

Knauss, W.G., 'The mechanics of polymer fracture', *Appl. Mech. Rev.* **26**, 1973, 1–17.

Knauss, W.G., 'On the steady propagation of a crack in a viscoelastic sheet: experiments and analysis', in *Deformation and Fracture of High Polymers*, H.H. Kausch, J.A. Hassel and R.I. Jaffe (eds.), Plenum Press, New York, 1974, 501–541.

Knauss, W.G., 'Fracture of solids possessing deformation rate sensitive material properties', in *Deformation and Fracture of High Polymers*, H.H. Kausch, J.A. Hassel and R.I. Jaffe (eds.), Plenum Press, New York, 1976, 501–541.

Knauss, W.G., 'The mechanics of polymer fracture', in *Applied Mechanics Update*, C.R. Steele and G.S. Springer (eds.), ASME, New York, 1986, 211–216.

Knauss, W.G., 'Time dependent fracture of polymers', in *Advances in Fracture Research*, Vol. 4, K. Salama, K. Ravi-Chandar, D.M.R. Taplin and P. Rama Rao (eds.), Pergamon Press, New York, 1989, 2683–2711.

Knauss, W.G., 'Time dependent fracture and cohesive zones', *J. Engrg. Mat. & Tech.* **115**, 1993, 262–267.

Knauss, W.G. and Dietmann, H., 'Crack propagation under variable load histories in linearly viscoelastic solids', *Internat. J. Engrg. Sci.* **8**, 1970, 643–656.

Knauss, W.G. and Mueller, H.K., 'On the governing equations for quasi-static crack growth in linearly viscoelastic materials', *ASME J. Appl. Mech.* **42**, 1975, 520–521.

Kostrov, B.V. and Nikitin, L.V., 'Some general problems of mechanics of brittle fracture', *Arch. Mech. Stos.* **22**, 1970, 749–775.

Kulakhmetova, Sh.A., 'Influence of anisotropy of a lattice on an energy outflow from a propagating crack', *Vestnik LGU* **22**, 1985a, 51–57 [in Russian].

Kulakhmetova, Sh.A., 'Dynamics of a crack in an anisotropic lattice', *Sov. Phys. Dokl.* **30**(3), 1985b, 254–255.

Kulakhmetova, Sh.A., Saraikin, V.A. and Slepyan, L.I., 'Plane problem of a crack in a lattice', *Mech. Solids* **19**(3), 1984, 101–108.

Langer, J.S. and Lobkovsky, A.E., 'Critical examination of cohesive-zone models in the theory of dynamic fracture,' *J. Mech. Phys. Solids* **46**, 1998, 1521–1556.

Lee, O.S. and Knauss, W.G., 'Dynamic crack propagation along a weakly bonded plane in a polymer', *Exp. Mech.* **29**, 1989, 342–345.

Li, V.C. and Liang, E., 'Fracture processes in concrete and fiber reinforced cementitious composites', *ASCE J. Engrg. Mech. Div.* **112**, 1986, 566–586.

Marder, M., 'New dynamical equation for cracks, *Phys. Rev. Lett.* **66**, 1991, 2484–2487.

Marder, M. and Gross, S., 'Origin of crack tip instabilities', *J. Mech. Phys. Solids* **43**, 1995, 1–48.

Marder, M. and Xiangmin Liu, 'Instability in lattice fracture', *Phys. Rev. Lett.* **71**(15), 1993, 2417–2420.

Mikhailov, A.M. and Slepyan, L.I., 'Steady-state motion of a crack in a unidirectional composite', *Mech. Solids* **21**(2), 1986, 183–191.

Mueller, H.K. and Knauss, W.G., 'Crack propagation in a linearly viscoelastic strip', *J. Appl. Mech.* **38**, 1971, 483–488.

Mulmule, S.V. and Dempsey, J.P., 'Stress-separation curve for saline ice using the fictitious crack model', *ASCE J. Engrg. Mech.* **123**, 1997, 870–877.

Mulmule, S.V. and Dempsey, J.P., ‘A viscoelastic fictitious crack model for the fracture of sea ice’, *Mech. Time-Dependent Mater.* **1**, 1998, 331–356.

Nuismer, R.J., ‘On the governing equations for quasi-static crack growth in linearly viscoelastic materials’, *ASME J. Appl. Mech.* **41**, 1974, 631–634.

Popelar, C.H. and Atkinson, C., ‘Dynamic crack propagation in a viscoelastic strip’, *J. Mech. Phys. Solids* **28**, 1980, 79–93.

Ryvkin, M. and Banks-Sills, L., ‘Steady-state mode iii propagation of an interface crack in an inhomogeneous viscoelastic strip’, *Internat. J. Solids and Structures* **30**, 1992, 483–498.

Ryvkin, M. and Banks-Sills, L., ‘Mode III delamination of a viscoelastic strip from a dissimilar viscoelastic half-plane’, *Internat. J. Solid and Structures* **31**, 1994, 551–566.

Schapery, R.A., ‘A theory of crack initiation and growth in viscoelastic media I. Theoretical development’, *Internat. J. Fracture* **11**, 1975a, 141–159.

Schapery, R. A., ‘A theory of crack initiation and growth in viscoelastic media II. Approximate method of analysis’, *Internat. J. Fracture* **11**, 1975b, 369–388.

Schapery, R.A., ‘A theory of crack initiation and growth in viscoelastic media III. Analysis of continuous growth’, *Internat. J. Fracture* **11**, 1975c, 549–562.

Sills, L.B. and Benveniste, Y., ‘Steady state propagation of a mode III interface crack in an inhomogeneous viscoelastic media’, *Internat. J. Engrg. Sci.* **19**, 1981, 1255–1268.

Slepyan, L.I., ‘Dynamics of a crack in a lattice’, *Sov. Phys. Dokl.* **26**, 1981a, 538–540.

Slepyan, L.I., ‘Crack propagation in high-frequency lattice vibrations’, *Sov. Phys. Dokl.* **26**(9), 1981b, 900–902.

Slepyan, L.I., ‘Antiplane problem of a crack in a lattice’, *Mech. Solids* **17**(5), 1982a, 101–114.

Slepyan, L.I., ‘The relation between the solutions of mixed dynamic problems for a continuous elastic medium and a lattice’, *Sov. Phys. Dokl.* **27**(9), 1982b, 771–772.

Slepyan, L.I., ‘Dynamics of brittle fracture in media with a structure’, *Mech. Solids* **19**(6), 1984, 114–122.

Slepyan, L.I., ‘Dynamics of brittle fracture in media with a structure – Inhomogeneous problems’, in *Mathematical Methods of Mechanics of Solids*, R.V. Goldstein and O.A. Oleinik (eds.), Nauka, Moscow, 1986, 143–149 [in Russian].

Slepyan, L.I., *Mechanics of Cracks*, 2nd edition, Sudostroenie, Leningrad, 1990 [in Russian].

Slepyan, L.I., ‘Role of microstructure in fracture and phase transition dynamics’, in *MECAMAT’93: International Seminar on Micromechanics of Materials*, Editions EYROLLES, Moret-sur-Loing, France, 1993, 246–257.

Slepyan, L.I., ‘Crack dynamics’, in *Fracture: A Topical Encyclopedia of Current Knowledge Dedicated to Alan Arnold Griffith*, G. Cherepanov (ed.), Krieger Publishing Company, Melbourne, U.S.A., 1998, 620–661.

Slepyan, L.I. and Kulakhmetova, Sh.A., ‘Crack propagation in a bed consisting of rigid blocks with elastic interlayers’, *Phys. Solid Earth* **22**(12), 1986, 977–982.

Slepyan, L.I. and Troyankina, L.V., ‘Fracture wave in a chain structure’, *J. Appl. Mech. Techn. Phys.* **25**(6), 1984, 921–927.

Slepyan, L.I. and Troyankina, L.V., ‘Shock waves in a nonlinear chain’, in *Strength and Viscoplasticity*, R.V. Gol’dstein (ed.), Nauka, Moscow, 1988, 301–305 [in Russian].

Walton, J.R., ‘On steady-state propagation of anti-plane shear crack in an infinite general linearly viscoelastic body’, *Quart. Appl. Math.* **40**, 1982, 37–52.

Walton, J.R., ‘The dynamic energy release rate for a steadily propagating antiplane shear crack in a linear viscoelastic body’, *J. Appl. Mech.* **54**, 1987, 635–641.

Walton, J.R., ‘Dynamic viscoelastic fracture’, in *Crack and Contact Problems for Viscoelastic Bodies*, G.A.C. Graham and J.R. Walton (eds.), Springer-Verlag, Wien, 1995, 259–311.

Walton, J.R. and Herrmann, J.M., ‘A new method for solving dynamically accelerating crack problems: Part 1. The case of a semi-infinite mode III crack in elastic material revisited’, *Quart. Appl. Math.* **50**, 1992, 373–387.

Williams, M.L., ‘The fracture of viscoelastic material’, in *Fracture of Solids*, D.C. Drucker and J.J. Gilman (eds.), Interscience Publishers, 1962, 157–188.
Willis, J.R., ‘Crack propagation in viscoelastic media’, *J. Mech. Phys. Solids* **15**, 1967a, 229–240.
Willis, J.R., ‘A comparison of the fracture criteria of Griffith and Barenblatt’, *J. Mech. Phys. Solids* **15**, 1967b, 151–162.
Wnuk, M.P. and Knauss, W.G., ‘Delayed fracture in viscoelastic-plastic solids’, *Internat. J. Solids and Structures* **6**, 1970, 995–1009.
Wu, Z.S. and Bažant, Z.P., ‘Finite element modeling of rate effect in concrete fracture with influence of creep’, in *Creep and Shrinkage of Concrete*, Z.P. Bažant and I. Carol (eds.), E & FN Spon, London, 1993, 424–432.