10. Wiener process. Gaussian white noise

Brownian motion $(B_t)_{t\ge 0}$, described by the botanist Brown, is known also as the Wiener process $(W_t)_{t\ge 0}$, called in a honor of the mathematician Wiener who gave its mathematical "design".

Wiener process is zero mean Gaussian random process, so, as any zero mean Gaussian process, it is completely defined by the correlation function. In formulas: $\boldsymbol{E}W_t \equiv t$, with $x \wedge y = \min(x,y)$

$$\boldsymbol{E}W_tW_s = t \wedge s. \tag{10.1}$$

We derive now a few simple properties of Wiener process.

1. $\boldsymbol{E}W_t^2 \equiv t$ and $W_0 = 0$.

Proof. The first statement follows from (10.1) with $t = s$. Particularly, $\boldsymbol{E}W_0^2 = 0$ and the second statement holds true. □

2. $\boldsymbol{E}(W_t - W_s)^2 = |t - s|$.

Proof. Write

$$\boldsymbol{E}(W_t - W_s)^2 = t + s - 2(t \wedge s) = \begin{cases} t-s, & t > s \\ 0, & t = s \\ s-t, & t < s. \end{cases}$$

□

3. Increments of Wiener process are Gaussian random variables and are independent for nonoverlapping intervals. Their distributions coincide, if intervals are the same length (homogeneous increments).

Proof. Since (W_t) is Gaussian process, any subvector $W_{t_1}, \dots, W_{t_n}$ is Gaussian random vector. Consequently, also $(W_{t''} - W_{t'}), (W_{s''} - W_{s'}), \dots$ is Gaussian random vector. Hence, $W_{t''} - W_{t'}$ is zero mean Gaussian random variable with the variance $t'' - t'$, so that all increments defined on an interval of the length $t'' - t'$ have the same distribution. If $[s', s'']$ and $[t', t'']$ are nonoverlapping intervals. Then, for instance $t' > s''$, we find

$$\boldsymbol{E}(W_{t''} - W_{t'})(W_{s''} - Ws') = s'' - s' - s'' + s' = 0,$$

that is these increments are orthogonal. Consequently, they are independent (see, Lect. 9). □

4. Paths of Wiener process are continuous functions.

Proof. We apply Kolmogorov's condition guaranteeing the continuous of paths of a random process with the continuous time: *if for some* $\gamma > 0,\ \beta > 0$ *and any any* t, s *there is a positive constant* C *such that*

$$\boldsymbol{E}|X_t - X_s|^\gamma \le C|t - s|^{1+\beta}, \tag{10.2}$$

then paths of X_t *are continuous functions.*

We verify the Kolmogorov condition for $\gamma = 4$ and $\beta = 1$. To this purpose, let us recall the following property of zero mean Gaussian random variables ξ with $\boldsymbol{E}\xi^2 = \sigma^2$:

$$\boldsymbol{E}\xi^4 = 3\sigma^4. \tag{10.3}$$

Write $\boldsymbol{E}|W_t - W_s|^4 = 3|t-s|^2$, i.e. (10.2) holds true with $C = 3$. □

5. Paths of Wiener process are not differentiable function.

Proof. For $h > 0$, set $\Delta(h) = \frac{W_{s+h} - W_s}{h}$. The random variable $\Delta(h)$ is zero mean Gaussian with $\boldsymbol{E}(\Delta(h))^2 = \frac{1}{h}$. If $\Delta(h),\ h \to 0$ converges in some sense to a limit, then a sequence of characteristic functions $\boldsymbol{E}e^{\lambda\Delta(h)},\ h \to 0$ converge to a limit which is continuous function in the argument λ. In the case considered, we have (see, Lect. 9)

$$\boldsymbol{E}e^{\lambda\Delta(h)} = e^{-\lambda^2/(2h^2)} \xrightarrow[h\to 0]{} \begin{cases} 1, & \lambda = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the assumed convergence is lost and the derivative does not exist. □

Gaussian White Noise

In spite of trajectories of Wiener process are not differentiable, a generalized derivative exists.

Let us recall a definition of the generalized derivative for a deterministic function $w(t)$: for any smooth and compactly supported function $g = g(t)$ the generalized derivative $\dot{w}(t)$ of $w(t)$ (not obligatory differentiable function) is defined symbolically

$$\int_0^\infty g(t)\dot{w}(t)dt = -\int_0^\infty \dot{g}(t)w(t)dt.$$

For a smooth $w(t)$, when $\dot{w}(t)$ exists, the above formula is nothing but "integration by parts" formula.

The generalized derivative $\dot{W}(t)$ of Wiener process, defined similarly to integration by parts with a smooth $g(t)$

$$g(t)W_t = \int_0^t g(s)\dot{W}_s ds + \int_0^t \dot{g}(s)W_s ds \tag{10.4}$$

is called "Gaussian white noise".

We will analyse now main properties of Gaussian white noise.

1. $\boldsymbol{E}\dot{W}_t \equiv 0$.

Proof. Taking the expectation from both sides of (10.3), we find

$$\boldsymbol{E}\int_0^t g(s)\dot{W}_s ds = 0$$

and the result. □

2. A (generalized) correlation function of $\dot{W}_t$ is δ-function.

Proof. Write

$$\begin{aligned}
&\int_0^t \int_0^t g(s)g(s')\boldsymbol{E}\big(\dot{W}_s\dot{W}_{s'}\big)dsds' \\
&\quad := \boldsymbol{E}\left(\int_0^t g(s)\dot{W}_s ds \int_0^t g(s)\dot{W}_s ds\right) \\
&\quad = \boldsymbol{E}\left(g(t)W_t - \int_0^t \dot{g}(s)W_s ds\right)^2 \\
&\quad = tg^2(t) - 2g(t)\int_0^t \dot{g}(s)sds \\
&\qquad + \int_0^t \int_0^t \dot{g}(s)\dot{g}(s')(s \wedge s')dsds' := \Psi(t). \qquad (10.5)
\end{aligned}$$

Further,

$$\begin{aligned}
\frac{d\Psi(t)}{dt} &= g^2(t) + 2tg(t)\dot{g}(t) - 2tg(t)\dot{g}(t) \\
&\quad -2\dot{g}(t)\int_0^t \dot{g}(s)sds + 2\dot{g}(t)\int_0^t \dot{g}(s)sds \\
&= g^2(t).
\end{aligned}$$

Hence,

$$\Psi(t) = \int_0^t g^2(s)ds \qquad (10.6)$$

On the other hand,

$$\int_0^t \int_0^t g(s)g(s')\delta(s-s')ds'ds = \int_0^t g^2(s)ds.$$

□

3. $\int_0^t g(s)\dot{W}(s)ds$ is well defined, if only $\int_0^t g^2(s)ds < \infty$. Moreover,

$$\begin{aligned}
&\boldsymbol{E}\int_0^t g(s)\dot{W}(s)ds = 0 \\
&\boldsymbol{E}\left(\int_0^t g(s)\dot{W}(s)ds\right)^2 = \int_0^t g^2(s)ds.
\end{aligned} \qquad (10.7)$$

Proof. Here, a smoothness of $g(s)$ is not obligatory supposed. We assume only that $\int_0^t g^2(s)ds < \infty$. The latter assumption allows to choose a sequence $g_n(s)$, $n \geq 1$ of smooth functions such that

$$\lim_{n\to\infty}\int_0^t \big(g(s) - g_n(s)\big)^2 ds = 0. \qquad (10.8)$$

For every n, $\int_0^t g_n(s)\dot{W}(s)ds$ is well defined. Since $\int_0^t g_n(s)\dot{W}(s)ds$ determines a linear transform in g_n's we have

$$\int_0^t g_n(s)\dot{W}(s)ds - \int_0^t g_m(s)\dot{W}(s)ds = \int_0^t \big(g_n(s) - g_m(s)\big)\dot{W}(s)ds.$$

By (10.5) and (10.6), we have

$$\begin{aligned}
&\boldsymbol{E}\bigg(\int_0^t g_n(s)\dot{W}(s)ds - \int_0^t g_m(s)\dot{W}(s)ds\bigg)^2 \\
&\quad= \boldsymbol{E}\bigg(\int_0^t \big(g_n(s) - g_m(s)\big)\dot{W}(s)ds\bigg)^2 \\
&\quad= \int_0^t \big(g_n(s) - g_m(s)\big)^2 ds \\
&\quad\le 2\int_0^t \big(g_n(s) - g(s)\big)^2 ds \\
&\qquad+2\int_0^t \big(g_m(s) - g(s)\big)^2 ds \xrightarrow[n,m\to\infty]{} 0.
\end{aligned}$$

Hence, under $\int_0^t g^2(s)ds < \infty$ only, by the Cauchy criteria

$$\int_0^t g(s)\dot{W}(s)ds = \underset{n\to\infty}{\text{l.i.m.}} \int_0^t g(s)\dot{W}(s)ds. \tag{10.9}$$

Others statements are valid if $g(s)$ is smooth function and inherited under the convergence in L^2-norm. □

Linear differential equation driven by white noise.

We consider now a stochastic differential equation[1]

$$\dot{X}_t = a(t)X_t + b(t)\dot{W}(t) \tag{10.10}$$

subject to the initial condition X_0 being a random variable (say, with $\boldsymbol{E}X_0^2 < \infty$) independent of the white noise $\dot{W}(t)$, where $a(t)$ and $b(t)$ are bounded functions. Formally, a solution of (10.10) possesses a form as, if $b(t)\dot{W}(t)$ is a regular function

$$X_t = e^{\int_0^t a(s)ds}\Big(X_0 + \int_0^t e^{-\int_0^s a(s')ds'} b(s)\dot{W}(s)ds\Big). \tag{10.11}$$

[1]an integral form $X_t = X_0 + \int_0^t a(s)X_s ds + \int_0^t b(s)\dot{W}_s ds$ of this equation is completely legal as well as its solution

$$X_t = \Big(X_0 + \int_0^t b(s)\dot{W}_s ds\Big) + \int_0^t e^{\int_s^t a(s')ds'}\Big(X_0 + \int_0^s b(s')\dot{W}_{s'}ds'\Big)ds.$$

Since $\boldsymbol{E}\int_0^t e^{-\int_0^s a(s')ds'}b(s)\dot{W}(s)ds = 0$, taking the expectation from both sides of (10.11) we find

$$\boldsymbol{E}X_t = e^{\int_0^t a(s)ds}\boldsymbol{E}X_0. \tag{10.12}$$

Consequently

$$X_t - \boldsymbol{E}X_t = e^{\int_0^t a(s)ds}\Big(\big(X_0 - \boldsymbol{E}X_0\big) + \int_0^t e^{-\int_0^s a(s')ds'}b(s)\dot{W}(s)ds\Big)$$

and then

$$\begin{aligned}\mathsf{Var}(X_0) &= e^{2\int_0^t a(s)ds}\left(\mathsf{Var}(X_0) + \boldsymbol{E}\Big(\int_0^t e^{-\int_0^s a(s')ds'}b(s)\dot{W}(s)ds\Big)^2\right)\\ &= e^{2\int_0^t a(s)ds}\left(\mathsf{Var}(X_0) + \int_0^t e^{-2\int_0^s a(s')ds'}b^2(s)ds\Big)^2\right). \quad (10.13)\end{aligned}$$

Obviously, (10.12) and (10.13) define solutions of differential equations respectively

$$\begin{aligned}\dot{m}(t) &= a(t)m(t)\\ \dot{V}(t) &= 2a(t)V(t) + b^2(t)\end{aligned}$$

subject to $m(0) = \boldsymbol{E}X_0$, $V(0) = \mathsf{Var}(X_0)$.

In additional, for $t > t'$ the correlation function K(t,t') of the random process X_t is defined as:

$$K(t,t') = V(t')e^{\int_{t'}^t a(s)ds}.$$

If X_0 is a Gaussian random variable, then $(X_t))t \geq 0$ is Gaussian random process.