11. Poisson process. Poissonian White Boise. Telegraphic signal

11.1. **Poisson process.** First, recall a definition of Poisson random variable π.

A random random variable π, valued in the set $\{0,1,\cdots,\}$, is called Poisson random variable with parameter λ (>0), if its distribution function

$$P(\pi=k)=\frac{\lambda^k e^{-\lambda}}{k!},\ k=0,1,\dots,.$$

Notice that

$$\begin{aligned}\boldsymbol{E}\pi &= \sum_{k=1}^{\infty} k\frac{\lambda^k e^{-\lambda}}{k!}\\ &= e^{-\lambda}\lambda\sum_{k=1}^{\infty} k\frac{\lambda^{k-1}}{k!}\\ &= e^{-\lambda}\lambda\sum_{k=0}^{\infty}(k+1)\frac{\lambda^k}{(k+1)!}=\lambda.\end{aligned}$$

Poisson random process $\Pi=(\Pi_t)_{t\ge 0}$ is characterized as follows:

1. $\Pi_0=0$; paths of Poisson process are piece-wise constant right continuous functions (having limits to the left) and jumps of the unite size.

2. Increments $\Pi_{t''}-\Pi_{t'}$ are Poisson random variables with parameter $\lambda(t''-t')$, i.e. $\boldsymbol{E}(\Pi_{t''}-\Pi_{t'})=\lambda(t''-t')$; Increments of Poisson process from non-overlapping intervals are independent random variables.

It is possible to simulate Poisson process with a help of i.i.d. sequence exponentially distributed random variables $(\xi_j)_{j\ge 1}$ with

$$P(\xi_1\le t)=\begin{cases}1-e^{-\lambda t}, & t\ge 0\\ 0, & t<0,\end{cases}$$

$\tau_0=0$, $\tau_k=\sum\limits_{j=1}^{k}\xi_j$, $k\ge 1$ and set (here $I_{\{t\ge\tau_k\}}$ is the indicator function)

$$\Pi_t=\sum_{k\ge 1} I_{\{t\ge\tau_k\}}. \tag{11.1}$$

Such defined process Π_t has required paths and, being the inverse function of the sum of i.i.d. random variables τ_k's, and independent increments distributed as Poisson random variables.

As an exercise, we show that for Π_t, defined in (11.1),

$$P(\Pi_t=k)=\frac{(\lambda t)^k e^{-\lambda t}}{k!}. \tag{11.2}$$

By (11.1)), $\{\Pi_t = 0\} = \{t < \tau_1\}$ and so

$$P(\Pi_t = 0) = P(t < \tau_1) = 1 - P(\tau_1 \le t) = e^{-\lambda t}.$$

If $k \ge 1$, we have $\{\Pi_t = k\} = \{\tau_k \le t < \tau_{k+1}\}$ and, noticing that $\tau_{k+1} = \tau_k + \xi_{k+1}$ while τ_k and ξ_{k+1} are independent random variables, we find

$$\begin{aligned} P(\Pi_t = k) &= P(\tau_k \le t < \tau_{k+1}) \\ &= \boldsymbol{E} I(\tau_k \le t < \tau_{k+1}) = \boldsymbol{E} I(\tau_k \le t) I(\tau_{k+1} > t) \\ &= \boldsymbol{E} I(\tau_k \le t) \boldsymbol{E}\Big(I(\xi_{k+1} > t - \tau_k) \big| \tau_k \Big) \\ &= \boldsymbol{E} I(\tau_k \le t) P\big(\xi_{k+1} > t - \tau_k | \tau_k\big) \\ &= \boldsymbol{E} I(\tau_k \le t) e^{-\lambda(t-\tau_k)} = \int_0^t e^{-\lambda(t-s)} dP(\tau_k \le s). \end{aligned}$$

Hence, the resulting formula is the following

$$\boldsymbol{P}(\Pi_t = k) = \int_0^t e^{-\lambda(t-s)} d\boldsymbol{P}(\tau_k \le s) \tag{11.3}$$

In the next step, we find the distribution $P(\tau_k \le s)$. For $k = 1$,

$$\frac{d\boldsymbol{P}(\tau_1 \le t)}{dt} = \lambda e^{-\lambda t}. \tag{11.4}$$

Let $k \ge 2$. Write

$$\begin{aligned} P(\tau_k \le t) &= \boldsymbol{E} P(\tau_k \le t | \tau_{k-1}) = \boldsymbol{E} P(\xi_k \le t - \tau_{k-1} | \tau_{k-1}) \\ &= \int_0^t (1 - e^{-\lambda(t-s)}) dP(\tau_{k-1} \le s) \end{aligned}$$

and then, obviously,

$$\frac{dP(\tau_k \le t)}{dt} = \lambda \int_0^t e^{-\lambda(t-s)} \frac{dP(\tau_{k-1} \le s)}{ds} ds.$$

Now, starting from (11.4), we obtain

$$\frac{dP(\tau_k \le t)}{dt} = \lambda \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} \tag{11.5}$$

and, with a help of (11.3), the required distribution.

Expectation. The direct computation gives

$$\begin{aligned} \boldsymbol{E}\Pi_t &= e^{-\lambda t} \sum_{k=1}^{\infty} k \frac{(\lambda t^k)}{k!} \equiv \lambda t \\ \boldsymbol{E}\Big(\Pi_t - \Pi_s\Big) &= \lambda(t - s). \end{aligned} \tag{11.6}$$

Variance. Write $\mathsf{Var}(\Pi_t) = \boldsymbol{E}\Pi_t^2 - (\boldsymbol{E}\Pi_t)^2 = \boldsymbol{E}\Pi_t^2 - (\lambda t)^2$. Further,

$$\begin{aligned}\boldsymbol{E}\Pi_t^2 &= \sum_{k=1}^{\infty} k^2 \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\ &= \sum_{k=1}^{\infty} k \frac{(\lambda t)^k}{(k-1)!} e^{-\lambda t} \\ &= \sum_{j=0}^{\infty} (1+j) \frac{(\lambda t)^{(j+1)}}{j!} e^{-\lambda t} \\ &= \lambda t + (\lambda t)^2.\end{aligned}$$

Hence

$$\mathsf{Var}(\Pi_t) = \lambda t. \tag{11.7}$$

Correlation function. Write

$$\mathsf{Cov}(\Pi_t, \Pi_s) = \boldsymbol{E}(\Pi_t - \lambda t)(\Pi_s - \lambda s) = \boldsymbol{E}(\Pi_t \Pi_s) - \lambda^2 ts.$$

Further, it was mentioned above (without proof) that increments of Poisson process for nonoverlapping intervals are independent random variables, so that to prove that

$$\boldsymbol{E}(\Pi_t - \Pi_s)\Pi_s = \boldsymbol{E}(\Pi_t - \Pi_s)\boldsymbol{E}\Pi_s = \lambda^2(t-s), \ t \geq s. \tag{11.8}$$

Finally, for $t \geq s$

$$\begin{aligned}\mathsf{Cov}(\Pi_t, \Pi_s) &= \boldsymbol{E}(\Pi_t - \Pi_s + \Pi_s)\Pi_s - \lambda^2 ts \\ &= \boldsymbol{E}(\Pi_t - \Pi_s)\boldsymbol{E}\Pi_s + \boldsymbol{E}\Pi_s^2 - \lambda^2 ts \\ &= \lambda^2(t-s)s + \lambda^2 s^2 + \lambda s - \lambda^2 ts = \lambda s\end{aligned}$$

and, thus,

$$\mathsf{Cov}(\Pi_t, \Pi_s) = \lambda(t \wedge s). \tag{11.9}$$

Remark. With $\lambda = 1$, the correlation function of the Poisson process coincides with the correlation function of the Wiener process, that is from point of view of "Correlation Theory" both Wiener process and centered by the expectation Poisson process are "indistinguishable".

Poisson process is continuous in L^2 norm. For $t > s$, we have

$$\begin{aligned}\boldsymbol{E}(\Pi_t - \Pi_s)^2 &= \boldsymbol{E}\Pi_t^2 + \boldsymbol{E}\Pi_s^2 - 2\boldsymbol{E}\Pi_t\Pi_s \\ &= \boldsymbol{E}\Pi_t^2 + \boldsymbol{E}\Pi_s^2 - 2\boldsymbol{E}(\Pi_t - \Pi_s)\Pi_s - 2\boldsymbol{E}\Pi_s^2 \\ &= \boldsymbol{E}\Pi_t^2 - \boldsymbol{E}\Pi_s^2 - 2\boldsymbol{E}(\Pi_t - \Pi_s)\Pi_s \\ &= (\lambda t)^2 + \lambda t - (\lambda s)^2 - \lambda s - 2\lambda^2(t-s)s \\ &= \lambda^2(t-s)^2 - \lambda(t-s) \to 0, (t-s) \to 0.\end{aligned}$$

Hence, Poisson process is continuous in L^2-norm and in probability as well, whereas by the Chebyshev inequality

$$P\big(|\Pi_t - \Pi_s| > \varepsilon\big) \leq \frac{1}{\varepsilon^2}\boldsymbol{E}(\Pi_t - \Pi_s)^2 \to 0, \ |t-s| \to 0, \ \forall\, \varepsilon > 0.$$

Evidently, paths of Poisson process are discontinuous functions.

11.2. **Poissonian white noise.** As was mentioned above, under $\lambda = 1$, the correlation functions of Poisson and Wiener processes are the same. Henceforth, $\lambda = 1$ and $\overline{\Pi}_t = \Pi_t - t$. Paths of $\overline{\Pi}_t$ are not differentiable functions. But the generalized derivative is well defined: for smooth function $g(t)$

$$\int_0^t g(s)\dot{\overline{\Pi}}_s ds := g(t)\overline{\Pi}_t - \int_0^t \dot{g}(s)\overline{\Pi}_s ds.$$

As in the case of Wiener process, the integral $\int_0^t g(s)\dot{\overline{\Pi}}_s ds$ is also well defined, if $\int_0^t g^2(s)ds < \infty$ and

$$\boldsymbol{E}\int_0^t g(s)\dot{\overline{\Pi}}_s ds \equiv 0 \tag{11.10}$$

$$\boldsymbol{E}\Big(\int_0^t g(s)\dot{\overline{\Pi}}_s ds\Big)^2 \equiv \int_0^t g^2(s)ds. \tag{11.11}$$

In contrast to a Wiener case the stochastic integral $\int_0^t g(s)\dot{\overline{\Pi}}_s ds$ coincides with

$$\int_0^t g(s)d\Pi_s - \int_0^t g(s)ds = \sum_{k:\tau_k \le t} g(\tau_k) - \int_0^t g(s)ds,$$

where

$$\int_0^t g(s)d\Pi_s = \sum_{k:\tau_k \le t} g(\tau_k)$$

is the Stieltjes integral.

Owing to

$$\int_0^t g(s)\dot{\overline{\Pi}}_s ds = \int_0^t g(s)d\Pi_s - \int_0^t g(s)ds$$

and (11.10), we obtain

$$\boldsymbol{E}\int_0^t g(s)d\Pi_s = \int_0^t g(s)ds. \tag{11.12}$$

Remark. For $\lambda \neq 1$,

$$\boldsymbol{E}\int_0^t g(s)d\Pi_s = \lambda\int_0^t g(s)ds. \tag{11.13}$$

Linear stochastic differential equation w.r.t. Poisson white noise

The fact that the stochastic integral w.r.t. Poissonian white noise coincides with the Stieltjes integral w.r.t. $\overline{\Pi}_t \equiv \Pi_t - t$, allows to consider a linear differential equation

$$dX_t = a(t)X_t dt + b(t)d\overline{\Pi}_t \tag{11.14}$$

subject to a random initial condition X_0 independent of Poisson process. The stochastic equation given in (11.14) possesses the unique solution

$$X_t = e^{\int_0^t a(u)du}\Big(X_0 + \int_0^t e^{-\int_0^s a(u)du} b(s)d\overline{\Pi}_s\Big).$$

The expectation, variance and correlation function of the random process X_t coincide with the same objects corresponding to that case of Gaussian white noise $\dot{W}_t$ (see, the end of Lect. 10).

11.3. **Telegraphic process.** Let X_0 be a random variable values in $\{0, 1\}$ with distribution function $P(X_0 = 0) = p$, $P(X_0 = 1) = 1 - p$. Let (Π'_t), (Π''_t) be Poisson processes with parameters λ and μ. Assume that $X_0, (\Pi'_t), (\Pi''_t)$ are independent random objects.

Let us defined a random process as a solution of a stochastic equation

$$X_t = X_0 + \int_0^t (1 - X_{s-})d\Pi'_s - \int_0^t X_{s-}d\Pi''_s, \tag{11.15}$$

where X_{s-} is the limit to the lest, that is $X_{s-} = \lim_{s\uparrow t} X_s$.

Further, an important role plays the following statement.

Proposition 11.1. *Independent Poisson processes have disjoint jumps.*

Proof. Denote by Π'_{t-}, Π''_{t-} the left limits and by $\triangle\Pi'_t = \Pi'_t - \Pi'_{t-}$, $\triangle\Pi'_t = \Pi'_t - \Pi'_{t-}$ jumps of Poisson processes. An obvious formula

$$\Pi'_t\Pi''_t - \Pi'_{t-}\Pi''_{t-} = \Pi'_{t-}\triangle\Pi''_t + \Pi''_{t-}\triangle\Pi'_t + \triangle\Pi'_{t-}\triangle\Pi''_t$$

and the telescopic sum provide the following presentation for the product of Poisson processes:

$$\Pi'_t\Pi''_t = \int_0^t \Pi'_{s-}d\Pi''_s + \int_0^t \Pi''_{s-}d\Pi'_s + \sum_{s\le t} \triangle\Pi'_s\triangle\Pi''_s. \tag{11.16}$$

Notice that $\triangle\Pi'_s\triangle P''_s \ge 0$ and so the disjointness of jumps of (Π'_t), (Π''_t) holds true if with probability one $\sum_{s\le t} \triangle\Pi'_s\triangle\Pi''_s = 0$. So, it suffices to show that

$$\boldsymbol{E}\sum_{s>0} \triangle\Pi'_s\triangle\Pi''_s = 0. \tag{11.17}$$

We use (11.16), to prove the validity of (11.17). Since Poisson processes are assumed to be independent, we have

$$\boldsymbol{E}\Pi'_t\Pi''_t = \boldsymbol{E}\Pi'_t\boldsymbol{E}\Pi''_t = \lambda\mu t^2.$$

So, (11.17) is valid, if

$$\boldsymbol{E}\Bigg(\int_0^t \Pi'_{s-}d\Pi''_s + \int_0^t \Pi''_{s-}d\Pi'_s\Bigg) = \lambda\mu t^2. \tag{11.18}$$

Denote by τ'_k, τ''_k time values of jumps (Π'_t), (Π''_t) respectively. Then , we have

$$\int_0^t \Pi'_{s-} d\Pi''_s = \sum_{k=1}^{\infty} I(\tau''_k \le t)\Pi'_{\tau''_k -}.$$

Since Poisson processes are independent, (Π'_t) and τ''_k's are independent as well. So that $\boldsymbol{E}(\Pi'_{\tau''_k -}|\tau''_k) = (\boldsymbol{E}\Pi'_t)\big|_{t=\tau''_k} = \lambda\tau''_k$. Therefore

$$\begin{aligned}
\boldsymbol{E}\sum_{k=1}^{\infty} I(\tau''_k \le t)\Pi'_{\tau''_k -} &= \boldsymbol{E}\sum_{k=1}^{\infty} I(\tau''_k \le t)\boldsymbol{E}\big(\Pi'_{\tau''_k -}|\tau''_k\big) \\
&= \lambda\boldsymbol{E}\sum_{k=1}^{\infty} I(\tau''_k \le t)\tau''_k \\
&= \lambda\boldsymbol{E}\int_0^t s d\Pi''_s = \lambda\mu\int_0^t s ds = \lambda\mu\frac{t^2}{2}.
\end{aligned}$$

Consequently, $\boldsymbol{E}\int_0^t \Pi'_{s-} d\Pi''_s = \lambda\mu\frac{t^2}{2}$ and $\boldsymbol{E}\int_0^t \Pi''_{s-} d\Pi'_s = \lambda\mu\frac{t^2}{2}$ (the second equality is proved similarly). Hence, (11.18) holds true. $\square$

The disjointness of jumps of Poisson process provides that for any time value t, $X_t = 1$ or 0 (that property inspires the name "telegraphic signal").