12. Stochastic Itô integral

In Lect. 10, it is defined a stochastic integral $\int_0^t g(s)\dot{W}_s ds$ for any deterministic function $g(t)$ with $\int_0^t g^2(s)ds < \infty$, $t > 0$, with respect to Gaussian white noise $\dot{W}_t$ a generalized derivative of Wiener process. This integral was introduced by Wiener.

It is shown in Lect. 10 that the random process $\int_0^t g(s)\dot{W}_s ds$, $t \ge 0$ is Gaussian with zero expectation and variance $\int_0^t g^2(s)ds$. It can be effectively applicable for creating stochastic models described by linear differential equations driving by Gaussian white noises.

In this Lecture, we define another definition of the stochastic integral when $g(s)$ is also random process. This integral was introduced by Itô. To clarify the Itô method for creating the stochastic integral, we redefine the Wiener stochastic integral from point of view of the Itô method.

Let $g(t)$, $0 \le t \le T$ be a deterministic function from $L^2_{[0,T]}$, that is

$$\int_0^T g^2(t)dt < \infty. \tag{12.1}$$

It is well know that any function from $L^2_{[0,T]}$ might be approximated in L^2-norm by a sequence $g_k(t)$, $k \ge 1$, of piece-wise constant functions:

$$\lim_{k\to\infty} \int_0^t \big(g(t) - g_k(t)\big)^2 dt = 0. \tag{12.2}$$

For fixed k and $g_k(s) = g_k(t_k),\ t_k \le s < t_{k+1}$, set

$$J_t(g_k) = \int_0^t g_k(s)dW_s \quad := \sum_{k:t_k\le t} g(t_k)(W_{t_{k+1}\wedge t} - W_{t_k\wedge t}). \tag{12.3}$$

Notice that $J_t(g_k)$, $0 \le t \le T$ is zero mean Gaussian random process (continuous) with

$$\boldsymbol{E}J_t^2(g_k) = \sum_{k:t_k\le t} g^2(t_k)(t_{k+1}\wedge t - t_k\wedge t) = \int_0^t g_k^2(s)ds; \tag{12.4}$$

the latter is implied owing to the independence of increments of Wiener process for nonoverlapping intervals and

$$\boldsymbol{E}(W_{t_{k+1}\wedge t} - W_{t_k\wedge t})^2 = t_{k+1}\wedge t - t_k\wedge t.$$

We find now the correlation function for $J_t(g_k)$.

Since $g(t_k)(W_{t_{k+1}\wedge t} - W_{t_k\wedge t})), k = 0, 1, \dots$ forms a sequence of the orthogonal random variables, we get

$$\begin{aligned}\boldsymbol{E} J_t(g_k) J_{t'}(g_k) &= \sum_{k: t_k \le t\wedge t'} g^2(t_k) \boldsymbol{E}\big(W_{t_{k+1}\wedge t} - W_{t_k\wedge t}\big)^2 \\ &= \sum_{k: t_k \le t\wedge t'} g^2(t_k)\big(t_{k+1}\wedge t) - t_k \wedge t) = \int_0^{t\wedge t'} g^2(s)ds. \quad (12.5)\end{aligned}$$

Show now that that the sequence $J_t(g_k), k \ge 1$ is fundamental in the Cauchy sense, that is for any $k, \ell \to \infty$

$$\lim_{k,\ell} \boldsymbol{E}\big(J_t(g_k) - J_t(g_\ell)\big)^2 = 0.$$

In fact, whereas $J_t(g_k) - J_t(g_\ell) = J_t(g_k - g_\ell)$, by (12.4) we have

$$\boldsymbol{E}\big(J_t(g_k) - J_t(g_\ell)\big)^2 = \int_0^t (g_k(s) - g_\ell(s)\big)^2 ds,$$

so that the fundamental property holds true due to (12.2):

$$\begin{aligned}&\int_0^t (g_k(s) - g_\ell(s)\big)^2 ds \\ &\le \int_0^T (g_k(s) - g_\ell(s)\big)^2 ds \\ &\int_0^T (g_k(s) - g(s) + g(s) - g_\ell(s)\big)^2 ds \\ &\le 2\int_0^T (g_k(s) - g(s)\big)^2 ds \\ &+2\int_0^T (g_e ll(s) - g(s)\big)^2 ds \xrightarrow[k,\ell\to\infty]{} 0.\end{aligned}$$

Hence by the Cauchy criteria (see, Lect. 2, Theorem 2.2), the sequence $J_t(g_k), k \ge 1$ converges in L^2-norm to a limit which we denote by $J_t(g) := \int_0^t dW_s$: $\lim\limits_{k\to\infty} \boldsymbol{E}\big(J_t(g_k) - J_t(g)\big)^2 = 0, \ t \le T$.

The limit $J_t(g)$ is independent of a choice of an approximating sequence $g_k(s)$. If $g'_k(s)$ is a sequence, different from $g_k(s)$, approximating $g(s)$ in a sense (12.2) and $J'_t(g)$ is a corresponding limit different from $J_t(g)$, then $\boldsymbol{E}\big(J_t(g) - J'_t(g)\big)^2 = 0$. Indeed,

$$\begin{aligned}&\boldsymbol{E}\big(J_t(g) - J'_t(g)\big)^2 \\ &\quad = \boldsymbol{E}\big(J_t(g) - J_t(g_k) + J_t(g_k) - J_t(g'_k) + J_t(g'_k) - J'_t(g)\big)^2 \\ &\qquad = 3\boldsymbol{E}\big(J_t(g) - J_t(g_k)\big)^2 + 3\boldsymbol{E}\big(J_t(g) - J'_t(g)\big)^2 \\ &\qquad\qquad + 3\boldsymbol{E}\big(J_t(g_k) - J_t(g'_k)\big)^2.\end{aligned}$$

The first and last terms in the right sides of the above inequality converge to zero with $k \to \infty$. The middle term is equal to

$$\boldsymbol{E} J_t^2(g_k - g_k') = \int_0^t \big(g_k(s) - g_k'(s)\big)^2 ds \xrightarrow[k\to\infty]{} 0,$$

due to the inequality

$$(g_k(s) - g_k'(s))^2 \le 2(g(s) - g_k(s))^2 + 2(g(s) - g_k'(s))^2$$

and (12.2) being valid for both sequences $(g_k(s))$, $(g_k'(s))$.

Properties of $J_t(g)$.

(1) $J_t(f) + J_t(g) = J_t(f+g)$.

Proof. For f, g replaced by f_k, g_k, this property is valid and is preserved under thr converges in L^2-norm. □

(2) $\boldsymbol{E} J_t(g) \equiv 0$, $\boldsymbol{E} J_t^2(g) = \int_0^t g^2(s)ds$.

Proof. With g replaced by g_k, these properties are valid. They remain valid in a general case by Lemma 2.4 (Lect.2) which states that the convergence in L^2-norm preserves the convergence of first and second moments. □

(3) $\boldsymbol{E} J_t(g) J_t(f) = \int_0^t g(s)f(s)ds$.

Proof. Write

$$\begin{aligned} J_t(g)J_t(f) &= \frac{1}{4}\Big(\big(J_t(g) + J_t(f)\big)^2 - \big((J_t(g) - J_t(f)\big)^2\Big) \\ &= \frac{1}{4}\Big(J_t^2(g+f) - J_t^2(g-f)\Big). \end{aligned}$$

Hence,

$$\begin{aligned} \boldsymbol{E} J_t(g)J_t(f) &= \frac{1}{4}\Big(\int_0^t \Big(\big(g(s) + f(s)\big)^2 - (g(s) - \big(f(s)\big)^2\Big) \\ &= \int_0^t g(s)f(s)ds. \end{aligned}$$

□

(4) $\boldsymbol{E} J_t(g) J_{t'}(g) = \int_0^{t\wedge t'} g^2(s)ds$.

Proof. Assume for a definiteness that $t > t'$ and set

$$f(s) = I(t' \ge s)g(s)$$

and notice that $\boldsymbol{E}\big(\int_{t'}^t f(s)dW_s\big)^2 = \int_{t'}^t I(t' \ge s)g^2(s)ds = 0$. Consequently, $J_t(f) = J_{t'}(g)$. Then

$$\boldsymbol{E} J_t(g)J_{t'}(g) = \boldsymbol{E} J_t(g)J_t(f) = \int_0^t g(s)f(s) = \int_0^{t'} g^2(s)ds.$$

□

(5) Random process $(J_t(g))_{t\le T}$ is Gaussian process continuous in L^2-norm and P-a.s.

Proof. The Gaussianness is derived with a help of the following general fact.

If sequence of Gaussian random variables $(\xi_n)_{n\ge1}$ converges in probability to a limit ξ, then ξ is Gaussian random variable. The result remains valid for any converging Gaussian vectors and processes.

The proof for converging Gaussian sequence is given below.

Denote by m_n and σ_n^2 the mean and variance of ξ_n respectively and by $\varphi_n(\lambda)$ its characteristic function. Then

$$\varphi_n(\lambda) = \exp\Big(\imath\lambda m_n - \frac{\lambda^2}{2\sigma_n^2}\Big) \quad \text{and} \quad \lim_{n\to\infty}\varphi_n(\lambda) = \varphi(\lambda).$$

By an arbitrariness of λ, we therefore have $\lim\limits_{n\to\infty} m_n = m$, $\lim\limits_{n\to\infty}\sigma_n^2 = \sigma^2$, that is $\varphi(\lambda) = \exp\Big(\imath\lambda m - \frac{\lambda^2}{2\sigma^2}\Big)$.

The continuity in L^2-norm holds true, since, sat for $t > t'$, we have

$$\boldsymbol{E}\Big(\int_{t'}^t g(s)dW_s\Big)^2 = \int_{t'}^t g^2(s)ds \xrightarrow[t-t'\to0]{} 0.$$

The continuity P-a.s., is provided by the following well known fact: *Gaussian process with orthogonal increments (on nonoverlapping intervals) is continuous, if its expectation and variance are continuous function.*

In our case, $\boldsymbol{E}J_t(g) \equiv 0$, $\boldsymbol{E}J_t^2 = \int_0^t g^2(s)$ are continuous functions. Moreover, for $t_1 < t_2 < t_3 < t_t$, we have

$$\boldsymbol{E}\int_{t_1}^{t_2} g(s)dW_s \int_{t_2}^{t_4} g(s)dW_s = 0.$$

□

(6) If $g(s)$ is smooth function, then

$$g(t)W_t - \int_0^t \dot g(s)W_s ds = \int_0^t g(s)dW_s.$$

Proof. It suffices to show that

$$\boldsymbol{E}\Bigg(\int_0^t g(s)dW_s - g(t)W_t + \int_0^t \dot g(s)W(s)ds\Bigg)^2 = 0. \tag{12.6}$$

Since

$$\boldsymbol{E}\Big(\int_0^t g(s)dW_s\Big)^2 = \int_0^t g^2(s)ds$$
$$\boldsymbol{E}\big(g(t)W_t - \int_0^t \dot g(s)W_s ds\big)^2 = \int_0^t g^2(s)ds$$

(for the second part see, (10.6) in Lect. 10), it suffices to show

$$\boldsymbol{E}\int_0^t g(s)dW_s\Big\{g(t)W_t-\int_0^t \dot g(s)W_s ds\Big\}=\int_0^t g^2(s)ds. \qquad (12.7)$$

Notice that

$$\begin{aligned}&\boldsymbol{E}\int_0^t g(s)dW_s g(t)W_t\\ &\qquad = g(t)\boldsymbol{E}\int_0^t g(s)dW_s\int_0^t dW(s)=g(t)\int_0^t g(s)ds. \qquad (12.8)\end{aligned}$$

Further,

$$\begin{aligned}&\boldsymbol{E}\int_0^t g(s)dW_s\int_0^t \dot g(s)W_s ds\\ =\ &\boldsymbol{E}\int_0^t\Big\{\int_0^t g(s')dW_{s'}\Big\}\dot g(s)W_s ds.\end{aligned}$$

Notice also that

$$\begin{aligned}&\boldsymbol{E}\int_0^t\Big\{\int_0^t g(s')dW_{s'}\Big\}\dot g(s)W_s ds\\ &\qquad =\boldsymbol{E}\int_0^t\Big\{\int_0^s g(s')dW_{s'}\Big\}\dot g(s)W_s ds\\ &\qquad +\boldsymbol{E}\int_0^t\Big\{\int_s^t g(s')dW_{s'}\Big\}\dot g(s)W_s ds\\ &\qquad\qquad =\int_0^t \dot g(s)\int_0^s g(s')ds'ds.\end{aligned}$$

Therefore, integrating by parts, we get

$$g(t)\int_0^t g(s)ds+\int_0^t \dot g(s)\int_0^s g(s')ds'=\int_0^t g^2(s)ds$$

and (12.7). □

(7) If $f(t)$ is smooth function, then

$$f(t)J_t(g)=\int_0^t \dot f(s)J_s(g)ds+J_t(fg).$$

Proof. Write

$$\begin{aligned}f(t)J_t(g)&=\int_0^t f(t)g(s)dW_s\\ &=J_t(fg)+\int_s^t\big(f(t)-f(s)\big)g(s)dW_s\\ &=J_t(fg)+\int_0^t\int_s^t \dot f(s)ds'g(s')dW_{s'}\end{aligned}$$

and notice that

$$\int_0^t \int_0^t I(s > s')\dot{f}(s)dsg(s')dW_{s'} = \int_0^t \dot{f}(s)J_s(g)ds.$$

□

Example. Consider the linear differential Itô equation

$$dX(t) = a(t)X(t)dt + b(t)dW(t)$$

subject to a random initial condition $X(0)$ independent of Wiener process $W(t)$, where $a(t)$, $b(t)$ are bounded (not obligatory continuous) deterministic function. Applying (7) with $f(t) = e^{\int_0^t a(s)ds}$ and $g(t) = e^{-\int_0^t a(s')ds'}b(t)$, we find

$$X(t) = e^{\int_0^t a(s)ds}\Big(X(0) + \int_0^t e^{-\int_0^s a(s')ds'}b(s)dW(s)\Big).$$