6. Kalman filter implementation for linear algebraic equations. Karhunen-Loeve decomposition

6.1. **Solvable linear algebraic systems. Probabilistic interpretation.** Let A be a quadratic matrix (not obligatory nonsingular). Consider a system of solvable linear algebraic equation

$$Ax = y, \tag{6.1}$$

that is under singular matrix A system (6.1) possesses a solution. It is well known that all solutions of (6.1) under the singular matrix A are described as follows:

$$x = A^{\oplus}y + \big(I - A^{\oplus}A\big)h, \tag{6.2}$$

where $A^{\oplus}$ is the Moore-Penrose pseudoinverse matrix of A and I is the unite matrix. The vectors $A^{\oplus}y$ and $\big(I - A^{\oplus}A\big)h$ are orthogonal in a sense their inner product is equal to zero. In fact, since $A^{\oplus}A$ is symmetric matrix and $A^{\oplus}AA^{\oplus} = A^{\oplus}$ (see Section 5.2 in Lect. 5) we have

$$\begin{aligned}
\big\langle A^{\oplus}y, \big(I - A^{\oplus}A\big)h\big\rangle &= h^T(I - A^{\oplus}A)^T A^{\oplus}y \\
&= h^T(I - A^{\oplus}A)A^{\oplus}y \\
&= h^T(A^{\oplus} - A^{\oplus}AA^{\oplus})y = 0.
\end{aligned}$$

Hence, owing to an arbitrariness of h, the solution $A^{\oplus}y$ has the minimal Euclidean norm. Without of a' prior information, it is impossible to find a solution corresponding to a specific h (unless A is nonsingular).

We give now a probabilistic description of the solution $x^{\circ} = A^{\oplus}y$. Let X be a random vector with $\boldsymbol{E}X = 0$ and $\mathsf{Cov}(X, X) = I$ (here I is the unite matrix) and $Y = AX$. Assume the random vector Y is observable (measurable), the matrix A is known and it is required to find the best estimate of X. Clearly that as such estimate

$$\widehat{X} = \widehat{\boldsymbol{E}}(X|Y)$$

can be taken. Notice that $\mathsf{Cov}(X, Y) = \mathsf{Cov}(X, X)A^T = A^T$ and $\mathsf{Cov}(Y, Y) = A\mathsf{Cov}(X, X)A^T = AA^T$. Therefore, by Theorem 5.1 (Lect. 5), $\widehat{X} = A^T(AA^T)^{\oplus}$. On the other hand, since $A^T(AA^T)^{\oplus} = A^{\oplus}$ (see, Section 5.2, (6)), we have

$$\widehat{X} = A^{\oplus}Y. \tag{6.3}$$

Consider now the solution $x^{\circ} = A^{\oplus}y$. Set

$$z = X^T x^{\circ}. \tag{6.4}$$

Then

$$\boldsymbol{E}\widehat{X}z = \boldsymbol{E}A^{\oplus}Yz = \boldsymbol{E}A^{\oplus}AXX^Tx^{\circ} = A^{\oplus}A\mathsf{Cov}(X, X)x^{\circ}.$$

Further, since $\mathsf{Cov}(X, X) = I$ and $x^\circ = A^\oplus y$ and $A^\oplus A A^\oplus = A^\oplus$, we have

$$\boldsymbol{E}\widehat{X}z = x^\circ. \tag{6.5}$$

6.2. **Kalman filter for $\widehat{\boldsymbol{X}}$.** For a definiteness, assume that $A = A_{n\times n}$ and denote by a_k, $k = 1, \dots, n$ rows of A. By Y_k, $k = 1, \dots, n$ denote entries of Y. For notational convenience, introduce also a sequence X_k, $k = 0, 1, \dots, n$ with $X_0 = X$ and $X_k = X_{k-1}$. Obviously, $X_k \equiv X$. Nevertheless, these notions allow us to present the relation $Y = AX$ in a form compatible with "noiseless version" of (5.8) (Lect. 5)

$$\begin{aligned} X_k &= X_{k-1} \\ Y_k &= a_k X_{k-1}. \end{aligned} \tag{6.6}$$

Since the Kalman filter is also valid for the noiseless model, required only the use of pseudoinverse matrices, for $\widehat{X}_k = \widehat{\boldsymbol{E}}(X_k|\mathcal{M}_k)$, where $\mathcal{M}_k$ is the linear space generated by "$1, Y_1, \dots, Y_k$", and

$$P_k = \boldsymbol{E}(X_k - \widehat{X}_k)(X_k - \widehat{X}_k)^T$$

the following Kalman filter occurs

$$\begin{aligned} \widehat{X}_k &= \widehat{X}_{k-1} + P_{k-1}a_n^T\big(a_k P_{k-1} a_k^T\big)^\oplus\big(Y_k - a_k\widehat{X}_{k-1}\big) \\ P_k &= P_{k-1} - P_{k-1}a_k^T\big(a_k P_{k-1} a_k^T\big)^\oplus a_k P_{k-1} \end{aligned} \tag{6.7}$$

subject to the initial conditions $\widehat{X}_0 = 0$ and $P_0 = I$.

Notice that, whereas $X_k \equiv X$, we have $\widehat{X}_k = \widehat{\boldsymbol{E}}(X|\mathcal{M}_k)$. Hence $\widehat{X}_n = \widehat{X}$. Denote by A_k the sub-matrix of A containing first k rows of A and by Y^k the sub-vector of Y containing first k entries of Y. Owing to

$$Y^k = A_k X,$$

we have $\widehat{X}_k = A_k^\oplus Y^k$. So, the Kalman filter generates a sequence of projections

$$A_1^\oplus Y^1,\ A_2^\oplus Y^2,\ \dots,\ A_n^\oplus Y^n (= A^\oplus Y).$$

At the same time, since

$$\begin{aligned} X_k - \widehat{X}_k &= X - A_k y^k \\ &= X - A_k^\oplus A_k X \\ &= (I - A_k^\oplus A_k)X, \end{aligned}$$

we find

$$P_k = (I - A_k^\oplus A_k); \tag{6.8}$$

recall that $\mathsf{Cov}(X, X) = I$, $(I - A_k^\oplus A_k)$ is symmetric matrix, and $(A_k^\oplus A_k)^2 = A_k^\oplus A_k$ (see, Section 5.2, Lect. 5).

6.3. **Kalman filter for x°.** Denote by y_k, $k = 1, \ldots, y_k$ entries of $y = Ax^\circ$ and by y^k the sub-vector of y containing first k entries of y. Similar to (6.5), with z defined in (6.4), we find

$$\begin{aligned}\boldsymbol{E}\widehat{X}_k z = \boldsymbol{E}\widehat{X}_k X^T x^\circ = \boldsymbol{E}A_k^{\oplus} y^k X^T x^\circ \\ = \boldsymbol{E}A_k^{\oplus} A_k X X^T x^\circ = A_k^{\oplus} A_k \mathsf{Cov}(X,X) x^\circ \\ = A_k^{\oplus} A_k x^\circ = A_k^{\oplus} y^k := \widehat{x}_k. \quad (6.9)\end{aligned}$$

Notice that

$$x^\circ = \widehat{x}_n.$$

and

$$\begin{aligned}\boldsymbol{E}Y^k z = \boldsymbol{E}A_k \mathsf{Cov}(X,X) x^\circ = A_k x^\circ = y^k, \\ \boldsymbol{E}Y_k z = \boldsymbol{E}a_k \mathsf{Cov}(X,X) x^\circ = a_t x = y_k.\end{aligned} \quad (6.10)$$

Now, taking into the consideration (6.9) and (6.10) and multiplying the left- and right-hand sides of the first equation in (6.7) by z and then taking the expectation of the expressions obtained, we transform (6.7) into the Kalman filter generating $\widehat{x}_k$, $k = 1, \ldots, n$:

$$\begin{aligned}\widehat{x}_k &= \widehat{x}_{k-1} + P_{k-1} a_n^T \big(a_k P_{k-1} a_k^T\big)^{\oplus} \big(y_k - a_k \widehat{X}_{k-1}\big) \\ P_k &= P_{k-1} - P_{k-1} a_k^T \big(a_k P_{k-1} a_k^T\big)^{\oplus} a_k P_{k-1},\end{aligned} \quad (6.11)$$

where $\widehat{x}_0 = 0$, $P_0 = I$.

A natural question arises: why it makes sense to use (6.11) for a computation of $A^{\oplus} y$? Our arguments are the following. The algorithm given in (6.11) applies the simplest pseudoinverse operation for $\big(a_k P_{k-1} a_k^T\big)^{\oplus}$ for the scalar $a_k P_{k-1} a_k^T$, so that

$$\big(a_k P_{k-1} a_k^T\big)^{\oplus} = \begin{cases} \frac{1}{a_k P_{k-1} a_k^T}, & \text{if } a_k P_{k-1} a_k^T > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (6.12)$$

It is clear that (6.12), as any pseudoinverse operation, is ill-posed, since $a_k P_{k-1} a_k^T$ decreases with increasing k and being positive might be too close to zero that the fractal $\frac{1}{a_k P_{k-1} a_k^T}$ can not be computed correctly. Nevertheless, a structure of (6.12) allows to control the correctness of such type computation. A verification of the computation correctness heavily uses the fact that $a_k P_{k-1} a_k^T$ is the trace of the matrix $P_{k-1} a_k^T a_k P_{k-1}$ (in fact, by (6.8), $\mathsf{trace}(P_{k-1} a_k^T a_k P_{k-1}) = a_k P_{k-1}^2 a_k^T = a_k P_{k-1} a_k^T$). Consequently, we have

$$\mathsf{trace} P_{k-1} a_k^T \big(a_k P_{k-1} a_k^T\big)^{\oplus} a_k P_{k-1} = \begin{cases} 1, & \text{if } a_k P_{k-1} a_k^T > 0 \\ 0, & \text{otherwise} \end{cases}$$

and any "essential deviation" from 0 or 1 fixes the incorrectness.

The next question is what indicates $a_k P_{k-1} a_k^T = 0$?

Lemma 6.1. $a_kP_{k-1}a_k^T = \min\limits_{c_1,\dots,c_{k-1}} \|a_k - \sum\limits_{j=1}^{k-1} c_ja_j\|^2$. *Particularly, if* $a_kP_{k-1}a_k^T = 0$, *then the row* a_k *is a linear combination of* $a_1, \dots, a_k$.

Proof. Set $\triangle = a_k - \sum_{j=1}^{k-1} c_ja_j$, where numbers a_j's are chosen such that to minimize the Euclidean norm $\|\triangle\|^2$. Let c be the vectorrow with entries $c_1, \dots, c_k$ and notice that $\triangle = a_k - cA_{k-1}$ and so

$$\|\triangle\|^2 = \|a_k\|^2 - 2a_kA_{k-1}^Tc^T + cA_{k-1}A_{k-1}^Tc^T.$$

Since $\|\triangle\|^2$ is the minimal in c, the vector c solves the linear equation $\nabla_c\|\triangle\| = 0$, that is $c(A_{k-1}A_{k-1}^T) = a_kA_{k-1}^T$ and so

$$c = a_kA_{k-1}^T(A_{k-1}A_{k-1}^T)^{\oplus} = a_kA_{k-1}^{\oplus}.$$

Hence, $\triangle = a_k(I - A_{k-1}^{\oplus}A_{k-1})$ and therefore

$$\begin{aligned}\|\triangle\|^2 &= a_k(I - 2A_{k-1}^{\oplus}A_{k-1} + (A_{k-1}^{\oplus}A_{k-1})^2)a_k^{\oplus} \\ &= a_k(I - A_{k-1}^{\oplus}A_{k-1})a_k^T = a_kP_{k-1}a_k^T\end{aligned}$$

(for more details see Section 5.2 Lect. 5). □

Karhunen-Loeve decomposition

Let $\xi_k, k = 1, \dots, n$ be zero mean sequence of random variables with the correlation function $R(k, \ell)$. From point of view of a theoretical consideration and simulation as well, it is useful if ξ_k' are generated with a help of linear transformation of a white noise similar to different models of stationary random sequences in the wide sense given in Lect. 1.

6.4. **Orthonormal eigenvectors vectors of correlation matrix.** A matrix $\mathbf{R} = \mathbf{R}_{n\times n}$ with entries $R(k, \ell)$ is the correlation matrix of $\xi = \begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_n\end{pmatrix}$, that is $\mathbf{R} = \boldsymbol{E}\xi\xi^T$ and so $\mathbf{R}$ is nonnegative definite matrix (in fact, $x^T\mathbf{R}x = \boldsymbol{E}x^{\xi}\xi^Tx = \boldsymbol{E}\|x^T\xi\|^2 \ge 0$). It is well know from "Theory of matrices" that eigenvectors of a nonnegative definite matrix can be transformed in a system of orthonormal vectors. In other words, eigenvectors of $\mathbf{R}$

$$\begin{aligned}\varphi_1 &= (\varphi(1,1), ..., \varphi(1,n) \\ \varphi_2 &= (\varphi(2,1), ..., \varphi(2,n) \\ ..... &= .... \\ \varphi_N &= (\varphi(N,1), ..., \varphi(n,n)\end{aligned}$$

can be chosen such that for any k, l

$$\langle \varphi_k, \varphi_l \rangle = \sum_{j=1}^{n} \varphi(k,j)\varphi(\ell,j) = \begin{cases} 1, & k = \ell \\ 0, & \text{otherwise.} \end{cases} \tag{6.13}$$

Denote S the quadratic matrix with rows $\varphi_1, \varphi_2, \ldots, \varphi_n$ and notice that (6.13) provides $S^T S = I$, where I is the unite matrix. Hence, $S^T = S^{-1}$ and S is called "orthogonal matrix". Denote by λ_j the right eigenvector of $\mathbf{R}$ corresponding to φ_j: $\mathbf{R}\varphi_j = \lambda_j \varphi_j$. Owing to properties of S, we have

$$S^T \mathbf{R} S = S^T \mathbf{R} \begin{pmatrix} \lambda_1 \varphi_1 \\ \lambda_2 \varphi_2 \\ \vdots \\ \lambda_n \varphi_n \end{pmatrix} = \mathsf{diag}(\lambda_1, \ldots, \lambda_n), \tag{6.14}$$

where $\mathsf{diag}(\lambda_1, \ldots, \lambda_N)$ is the diagonal matrix with eigenvalues of $\mathbf{R}$ on the diagonal. Since $\mathbf{R}$ is the nonnegative matrix, its eigenvalues are nonnegative as well.

6.5. **Karhunen-Loeve theorem.**

Theorem 6.2. *There exists zero mean random vector with orthogonal entries $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ with $\boldsymbol{E}\varepsilon_j^2 \equiv 1$ such that for every $k = 1, 2, \ldots, n$*

$$\xi_k = \sum_{j=1}^{n} \sqrt{\lambda_j} \varphi(k,j) \varepsilon_j. \tag{6.15}$$

Proof. Assume first that $\mathbf{R}$ is nonsingular matrix, that is all λ_j's are positive. Then $\mathsf{diag}^{-1}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \mathsf{diag}(\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1})$.

Set

$$\varepsilon = \mathsf{diag}(\lambda_1^{-1/2}, \lambda_2^{-1/2}, \ldots, \lambda_n^{-1/2}) S^T \xi. \tag{6.16}$$

Notice that $\boldsymbol{E}\xi = 0$ and

$$\begin{aligned} \boldsymbol{E}\xi\xi^T = \mathsf{diag}&(\lambda_1^{-1/2}, \lambda_2^{-1/2}, \ldots, \lambda_n^{-1/2}) \\ &\times S^T \mathbf{R} S \mathsf{diag}(\lambda_1^{-1/2}, \lambda_2^{-1/2}, \ldots, \lambda_n^{-1/2}) \\ &= \mathsf{diag}(\lambda_1^{-1/2}, \lambda_2^{-1/2}, \ldots, \lambda_n^{-1/2}) \mathsf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \\ &\qquad \times \mathsf{diag}(\lambda_1^{-1/2}, \lambda_2^{-1/2}, \ldots, \lambda_n^{-1/2}) = I, \end{aligned}$$

that is entries of random vector ε are zero mean and orthogonal.

Further, since $S^T = S^{-1}$ and

$$\mathsf{diag}^{-1}\big(\lambda_1^{-1/2}, \lambda_2^{-1/2}, \ldots, \lambda_n^{-1/2}\big) = \mathsf{diag}\big(\lambda_1^{1/2}, \lambda_2^{1/2}, \ldots, \lambda_n^{1/2}\big),$$

(6.15) is implied by (6.16) under nonsingular $\mathbf{R}$, that is

$$\xi = \mathsf{diag}\big(\lambda_1^{1/2}, \lambda_2^{1/2}, \ldots, \lambda_n^{1/2}\big) S\varepsilon. \tag{6.17}$$

If $\mathbf{R}$ is singular, then, instead of (6.16), let us introduce

$$\varepsilon' = \mathsf{diag}^{\oplus}(\lambda_1^{1/2}, \lambda_2^{1/2}, \ldots, \lambda_n^{1/2}) S^T \xi, \tag{6.18}$$

here $\mathsf{diag}^{\oplus}$ is defined similarly to diag^{-1} with zeros located on same place where λ's are zeros. Now, introduce zero mean random vector (column of ξ size) η with orthogonal entries of the unite variance. Also assume that η and ξ are orthogonal and introduce

$$\varepsilon'' = \Big(I - \mathsf{diag}^{\oplus}\big(\lambda_1^{1/2}, \lambda_2^{1/2}, \dots, \lambda_n^{1/2}\big)\mathsf{diag}^{\oplus}\big(\lambda_1^{1/2}, \lambda_2^{1/2}, \dots, \lambda_n^{1/2}\big)\Big). \tag{6.19}$$

Henceforth, omitting arguments in $\mathsf{diag}(\lambda_1^{1/2}, \lambda_2^{1/2}, \dots, \lambda_n^{1/2})$, write

$$\varepsilon = \varepsilon' + \varepsilon'' = \mathsf{diag}^{\oplus} S^T\xi + \big(I - \mathsf{diag}^{\oplus}\mathsf{diag}^{\oplus}\big)\eta. \tag{6.20}$$

Taking into consideration of the pseudoinverse matrix properties (see, Section 5.2 in Lect. 5) it is readily to check that

$$\boldsymbol{E}\varepsilon = 0 \quad \text{and} \quad \boldsymbol{E}\varepsilon\varepsilon^T = I.$$

Under singular $\mathbf{R}$, the formula, given in (6.17), remains with ε from (6.20). Indeed, denote by $\widetilde{\xi}$ the right hand of (6.17) with that ε and show that $\xi = \widehat{\xi}$ with probability one. For this purpose, it suffices to prove that $\boldsymbol{E}\|\xi - \widetilde{\xi}\|^2 = 0$. Write

$$\begin{aligned}\boldsymbol{E}\|\xi - \widetilde{\xi}\|^2 &= \boldsymbol{E}(\xi - \widetilde{\xi})(\xi - \widetilde{\xi})^T \\ &= \big(I - S\mathsf{diag}(\cdot)\mathsf{diag}^{\oplus}(\cdot)S^T\big)\mathbf{R}\big(I - S\mathsf{diag}(\cdot)\mathsf{diag}^{\oplus}(\cdot)S^T\big)^T \\ &= \big(I - S\mathsf{diag}(\cdot)\mathsf{diag}^{\oplus}(\cdot)S^T\big)S\mathsf{diag}^2(\cdot)S^T\big(I - S\mathsf{diag}(\cdot)\mathsf{diag}^{\oplus}(\cdot)S^T\big) = 0.\end{aligned}$$

□

Remark 6.3. The Karhunen-Loeve decomposition holds true for a random vector ξ with entries from a countable set $\xi_1, \xi_2, \dots, \xi_n, \dots$, provided that

$$\sum_{k=1}^{\infty} R(k,k) < \infty.$$