

12. INCOMPLETE DATA. FEEDBACK CONTROL. QUADRATIC COST FUNCTIONAL.

12.1. Classical incomplete data.

Here, we consider here new type of models with unobservable for controlled process X_t a control $u(t)$ is created by observation Y_t , being a sum of the controlled process and some noise. For such a model, the control $u(t)$ might be taken for fixed t only as a function of the value of t and the past of observable values $Y_s, s \leq t$. That type of control is of a feedback form but with respect to observations:

$$u(t) = u(t, Y_0^t).$$

It is named the control with incomplete data.

To find a constructive solution for control with incomplete date, we assume that the pair X_t, Y_t is defined by the Itô equations with respect to independent Wiener processes W_t and W'_t :

$$\begin{aligned} dX_t &= a(t)X_t dt + c(t)u(t, Y_0^t)dt + b(t)dW_t \\ dY_t &= A(t)X_t dt + B(t)dW'_t \end{aligned} \quad (12.1)$$

subject to the random initial condition X_0, Y_0 being Gaussian random vector with known parameters. Continuous functions $a(t), A(t), b(t), B(t), c(t)$ are assumed to be known and $B^2(t)$ is strictly positive.

Remark 1. *Equations (12.1) form joint system. So that any admissible control $u(t, Y_0^t)$ has to be chosen such that to guarantee the existence and uniqueness of the solution for (12.1).*

As in the previous considerations, the control $u(t, Y_0^t)$ is chosen to minimize the quadratic cost functional

$$J(u) = \mathbf{E} \left\{ rX_T^2 + \int_0^T \left(p(t)X_t^2 + q(t)(u(t, Y_0^t))^2 \right) dt \right\}, \quad (12.2)$$

where $r \geq 0$, and $p(t), q(t)$ are continuous functions, $p(t) \geq 0$ while $q(t) \geq c > 0$.

A standard Bellman equation is not applicable here. That is why our first task is to adapt the model considered to “Dynamic Programming Method” (DPM). The main tool now is the generalized Kalman filter which creates $\widehat{X}_t = \mathbf{E}(X_t|Y_0^t)$ and $P(t) = \mathbf{E}(X_t - \widehat{X}_t)^2$. The generalized Kalman filter defines equations for \widehat{X}_t and $P(t)$:

$$\begin{aligned}
d\widehat{X}_t &= [a(t)\widehat{X}_t + c(t)u(t, Y_0^t)]dt + \frac{P(t)A(t)}{B^2(t)}(dY_t - A(t)\widehat{X}_t dt), \\
\dot{P}(t) &= 2a(t)P(t) + b^2(t) - \frac{A^2(t)P^2(t)}{B^2(t)}
\end{aligned} \tag{12.3}$$

subject to the initial conditions \widehat{X}_0 and $P(0)$, where \widehat{X}_0 is known linear function of Y_0 and $P(0)$ is known constant.

Introduce a random process

$$\widehat{W}_t = \int_0^t \frac{1}{B(s)}(dY_s - A(s)\widehat{X}_s ds). \tag{12.4}$$

From the Kalman filtering theory, it is well known that \widehat{W}_t is, so called, innovation Wiener process. Formally, trajectories of \widehat{W}_t created even by control (generally speaking, in nonlinear way, but nevertheless \widehat{W}_t is zero mean Gaussian process with the correlation function $t \wedge s$, i.e. \widehat{W}_t is the standard Wiener process.

Thus, \widehat{X}_t is defined by the Itô equation with respect to \widehat{W}_t :

$$d\widehat{X}_t = [a(t)\widehat{X}_t + c(t)u(t, Y_0^t)]dt + \frac{P(t)A(t)}{B(t)}d\widehat{W}_t. \tag{12.5}$$

Now, we transform the cost functional to a form in which X_t is replaced by \widehat{X}_t . Taking into the consideration properties of the conditional expectation, write

$$\begin{aligned}
\mathbf{E}X_T^2 &= \mathbf{E}\mathbf{E}(X_T^2|Y_0^T) = \mathbf{E}\left\{\widehat{X}_T^2 + (X_T - \widehat{X}_T)^2\right\} = \mathbf{E}\widehat{X}_T^2 + P(T) \\
\mathbf{E}X_t^2 &= \mathbf{E}\mathbf{E}(X_t^2|Y_0^t) = \mathbf{E}\left\{\widehat{X}_t^2 + (X_t - \widehat{X}_t)^2\right\} = \mathbf{E}\widehat{X}_t^2 + P(t).
\end{aligned}$$

Then the cost functional is transformed into

$$\begin{aligned}
J(u) &= \mathbf{E}\left\{r\widehat{X}_T^2 + \int_0^T \left(p(t)\widehat{X}_t^2 + q(t)(u(t, Y_0^t))^2\right)dt\right\} \\
&\quad + rP(T) + \int_0^T p(t)P(t)dt.
\end{aligned} \tag{12.6}$$

The second term in the right side of (12.6), $rP(T) + \int_0^T p(t)P(t)dt$, is independent of the control. Therefore, if some control minimizes

$$J_1(u) = \mathbf{E}\left\{r\widehat{X}_T^2 + \int_0^T \left(p(t)\widehat{X}_t^2 + q(t)(u(t, Y_0^t))^2\right)dt\right\}$$

then it also minimizes the original cost functional as well. Further, introduce a new cost functional

$$J_1(\widehat{X}_0, u) = \mathbf{E} \left\{ r\widehat{X}_T^2 + \int_0^T \left(p(t)\widehat{X}_t^2 + q(t)((u(t), Y_0^t))^2 \right) dt \middle| Y_0 \right\} \quad (12.7)$$

and note that $J_1(u) = \mathbf{E}J_1(\widehat{X}_0, u)$.

Remark also that \widehat{X}_0 is a linear transformation of Y_0 only and so, \widehat{X}_0 is independent of the control.

Proposition. *Assume $u^\circ(t, (Y^\circ)_0^t)$ is the optimal control for $J_1(\widehat{X}_0, u)$. Then the same control is optimal for the original model.*

Proof. By virtue of (12.6), we have

$$J(u) = \mathbf{E}J_1(\widehat{X}_0, u) + rP(T) + \int_0^T p(t)P(t)dt.$$

Taking an arbitrary admissible control $u(t, Y_0^t)$ and taking into account that Y_0 and $P(t)$ are independent of any control, we obtain, that

$$\begin{aligned} J(u) &= \mathbf{E} \left(J(Y_0, u) \right) + rP(T) + \int_0^T p(t)P(t)dt \\ &\geq \mathbf{E} \left(J(Y_0, u^\circ) \right) + rP(T) + \int_0^T p(t)P(t)dt \\ &= J(u^\circ). \end{aligned} \quad (12.8)$$

This lower bound allows to claim that u° is the optimal control. \square

To find the optimal control u° for the auxiliary model, we apply the result from Lecture 10. Formally, we apply the Bellman equation

$$\begin{aligned} -V'_t(t, x) &= \left[\frac{1}{2} \frac{(P(t)A(t))^2}{B^2(t)} V''_{xx}(t, x) + a(t)xV'_x(t, x) \right] \\ &\quad + \inf_u \left[c(t)uV'_x(t, x) + p(t)x^2 + q(t)u^2 \right] \\ V(T, x) &= rx^2 \end{aligned} \quad (12.9)$$

the solution of which is $V(t, x) = \Gamma(t)x^2 + Q(t)$, where $\Gamma(t)$ is defined by the Ricatti equation

$$-\dot{\Gamma}(t) = 2a(t)\Gamma(t) + p(t) - \frac{c^2(t)\Gamma^2}{q(t)} \quad (12.10)$$

subject to the boundary condition $\Gamma(T) = r$, where

$$Q(t) = \int_t^T \frac{(P(s)A(s))^2}{2q(s)} ds.$$

At all,

$$u^\circ(t, x) = -\frac{c(t)\Gamma(t)}{q(t)}x.$$

Hence, we arrive at the full description corresponding to the optimal control:

$$\begin{aligned} u^\circ(t, \widehat{X}_t^\circ) &= -\frac{c(t)\Gamma(t)}{q(t)}\widehat{X}_t^\circ \\ d\widehat{X}_t^\circ &= [a(t)\widehat{X}_t^\circ + c(t)u^\circ(t, (\widehat{X}_t^\circ)_0^t)]dt + \frac{P(t)A(t)}{B(t)}(dY_t^\circ - A(t)\widehat{X}_t^\circ dt) \\ \widehat{X}_0^\circ &= \mathbf{E}(X_0|Y_0) \\ dY_t^\circ &= A(t)X_t^\circ dt + B(t)dW_t' \\ Y_0^\circ &= Y_0 \\ dX_t^\circ &= [a(t)X_t^\circ + c(t)u^\circ(t, \widehat{X}_t^\circ)]dt + b(t)dW_t \\ X_0^\circ &= X_0, \end{aligned}$$

where $P(t)$ and $\Gamma(t)$ are solutions of two Ricatti's equations

$$\begin{aligned} \dot{P}(t) &= 2a(t)P(t) + b^2(t) - \frac{A^2(t)P^2(t)}{B^2(t)} \\ P(0) &= \mathbf{E}(X_0 - \mathbf{E}(X_0|Y_0))^2 \\ -\dot{\Gamma}(t) &= 2a(t)\Gamma(t) + p(t) - \frac{c^2(t)\Gamma^2(t)}{q(t)} \\ \Gamma(T) &= r \end{aligned}$$

which have to be solved in the direct (' \rightarrow ') and backward (' \leftarrow ') time respectively.

12.2. More controls. Controlled observations. Let us consider more complicated setting (with two controls $u(t, Y_0^t)$ and $v(t)$):

$$\begin{aligned} dX_t &= a(t)X_t dt + c(t)u(t, Y_0^t)dt + b(t)dW_t \\ dY_t &= v(t)X_t dt + B(t)dW_t'. \end{aligned} \tag{12.11}$$

The control $u(t, Y_0^t)$ is of a feedback form with respect to observations while the control $v(t)$ is deterministic. The control $v(t)$ allows to improve the signal-to-noise ratio

$$SN(t) := \frac{v^2(t, Y_0^t)}{B^2(t)}.$$

From a common sense of view, the signal-to-noise ratio can not be taken too large. That fact is expressed via an additional “payment” included in the cost functional, “ $f(t, v(t))$ ”. In other words, instead $J(u)$ instead the following cost functional is used:

$$J(u, v) = \mathbf{E} \left\{ rX_T^2 + \int_0^T (p(t)X_t^2 + q(t)(u(t, Y_0^t))^2 + f(t, v(t))) dt \right\} \quad (12.12)$$

where $p(t)$, $q(t)$ are continuous function, $r \geq 0$, $p(t) \geq 0$, and $q(t) \geq \text{const} > 0$, and where $f(t, z)$ is (bounded) positive continuous function.

Under this setting, we apply the generalized Kalman filter:

$$\begin{aligned} d\hat{X}_t &= [a(t)\hat{X}_t + c(t)u(t, Y_0^t)]dt + \frac{P(t)v(t, Y_0^t)}{B^2(t)}(dY_t - v(t)\hat{X}_t dt), \\ \dot{P}(t) &= 2a(t)P(t) + b^2(t) - \frac{v^2(t)P^2(t)}{B^2(t)} \end{aligned} \quad (12.13)$$

and use \hat{X}_t and $P(t)$ for a transformation of the original cost functional into

$$J(u, v) = \mathbf{E} \left\{ J_1(Y_0, u, v) + rP(t) + \int_0^T [p(t)P(t) + f(t, v(t))] dt \right\}, \quad (12.14)$$

where

$$J_1(Y_0, u, v) = \mathbf{E} \left(r\hat{X}_T^2 + \int_0^T (p(t)\hat{X}_t^2 + q(t)(u(t, Y_0^t))^2) dt \middle| Y_0 \right). \quad (12.15)$$

As previously, the innovation Wiener process

$$\bar{W}_t = \int_0^t \frac{1}{B(s)} [dY_s - v(s)\hat{X}_s ds],$$

formally depending on controls u, v . The \hat{X}_t process is defined with respect to \bar{W}_t as:

$$d\hat{X}_t = [a(t)\hat{X}_t + c(t)u(t, Y_0^t)]dt + \frac{v(t)}{B(t)} d\bar{W}_t,$$

that is v controls the intensity of the innovation “white noise”.

For every fixed function $v(t)$, the optimal control u^v is defined with the help of the Bellman equation

$$-\frac{\partial}{\partial t} V^v(t, x) = \min_u \left[p(t)x^2 + q(t)u^2 + (V^v)'_x(t, x)[a(t)x + c(t)u] + \frac{1}{2}(V^v)''_{xx}(t, x) \frac{v^2(t)}{B^2(t)} \right]$$

subject to the boundary condition $V^v(T, x) \equiv rx^2$, so that with the fixed v , the optimal control

$$u^v(t, x) = -\frac{c(t)\Gamma(t)}{q(t)}X_t^v,$$

where $\Gamma(T) = r$ and

$$-\frac{d}{dt}\Gamma(t) = 2a(t)\Gamma(t) + p(t) - \frac{c^2(t)\Gamma^2(t)}{q(t)},$$

and where

$$\begin{aligned} dX_t^v &= a(t)X_t^v dt + c(t)u^v(t, (Y^v)_0^t)dt + b(t)dW_t \\ dY_t^v &= v(t)X_t^v dt + B(t)dW_t'. \end{aligned}$$

Hence, the cost

$$J(u^v, v) = \mathbf{E}J_1(\widehat{X}_{v_0}^v, u^v) + rP^v(T) + \int_0^T [p(t)P^v(t) + f(t, v(t))]dt,$$

where $J_1(\widehat{X}_{v_0}^v, u^v) = V(0, \widehat{X}_{v_0}^v) = \Gamma(0)\widehat{X}_{v_0}^v{}^2 + \int_0^T \Gamma(t)\frac{v^2(t)}{B^2(t)}dt$ and $P^v(t)$ is the solution of the Ricatti equation involved in the filter

$$\dot{P}^v(t) = 2a(t)P^v(t) + b^2(t) - \frac{v^2(t)(P^v)^2(t)}{B^2(t)}. \quad (12.16)$$

Both initial conditions for the filter are independent of the control v , $\widehat{X}_{v_0}^v = \widehat{X}_0$, $P^v(0) = P(0)$. Consequently

$$J(u^v, v) = \Gamma(0)\mathbf{E}\widehat{X}_0^2 + \int_0^T \Gamma(t)\frac{v^2(t)}{B^2(t)}dt + rP^v(T) + \int_0^T [p(t)P^v(t) + f(t, v(t))]dt.$$

Hence, the control v has to be chosen such that to minimize the cost functional

$$I(v, P(0)) = rP^v(T) + \int_0^T \left\{ \Gamma(t)\frac{v^2(t)}{B^2(t)} + p(t)P^v(t) + f(t, v(t)) \right\} dt \quad (12.17)$$

provided that $P^v(t)$ is the solution of differential equation (12.16) in which the control v involves. This deterministic control problem can be solved separately, but an explicit solution for it is unknown.

Home work

1. Write the Bellman equation for the control problem with the controlled process $P^v(t)$ and the cost functional $I(v, P(0))$ given in (12.16) and (12.17) respectively.