2. LINEAR QUADRATIC DETERMINISTIC PROBLEM

Notations:

- For a vector Z, $\|Z\| = \sqrt{\langle Z, Z\rangle}$ is the Euclidean norm (here $\langle Z, Z\rangle = \sum_i Z_i^2$ is the inner product);
- For a vector Z and nonnegative definite matrix Q, $\|Z\|_Q = \sqrt{\langle Z, QZ\rangle}$ is $\mathbb{L}^2$-norm with the Kernel Q:

$$\langle Z, QZ\rangle = \sum_{i,j} Z_i Q_{ij} Z_j.$$

- * is transposition symbol;
- "grad" is symbol of gradient;

A controlled process X_t is defined by a linear vector differential equation with control action $U(t)$:

$$\dot{X}_t = a(t)X_t + c(t)U(t) \tag{2.1}$$

subject to the fixed initial condition $X_0 = x$.

Here X_t and $U(t)$ are vectors of sizes k and r, and $a(t)_{|k\times k}$, $c(t)_{|k\times r}$ are matrix valued functions of the argument t.

The control $U(t)$ has to be chosen such that to track a smooth vector valued function $\phi(t)$, of the size k, in a sense of the minimization of cost functional

$$J(x, \phi; U) = \|X_T - \phi(T)\|_h^2 + \int_0^T \Big(\|X_t - \phi(t)\|^2_{H(t)} + \|U(t)\|^2_{R(t)}\Big)dt, \tag{2.2}$$

where $h_{|k\times k}$, $H(t)_{|k\times k}$ and $R(t)_{|r\times r}$ are nonnegative definite matrices ($R(t)$ is uniformly in t nonsingular).

2.1. **Preliminaries.** Set $Y_t = X_t - \phi(t)$. Since $\phi(t)$ is assumed to be differentiable, by (2.1) we find the differential equation for Y_t

$$\dot{Y}_t = a(t)Y(t) + \big[a(t)\phi(t) - \dot{\phi}(t)\big] + c(t)U(t) \tag{2.3}$$

subject to $Y_0 = X_0 - \phi(0) := y$.

Notice also that the cost functional $J(x, \phi : U)$ is transformed to

$$J(y, \phi; U) = \|Y_T\|_h^2 + \int_0^T \Big(\|Y_t\|^2_{H(t)} + \|U(t)\|^2_{R(t)}\Big)dt. \tag{2.4}$$

As in Lecture 1, we introduce the Bellman function

$$V(t, y) = \min_{U(s):s\le t\le T} \|Y_T\|_h^2 + \int_t^T \Big(\|Y_s\|^2_{H(s)} + \|U(s)\|^2_{R(s)}\Big)ds$$

and apply Bellman's principle of optimality:

$$V(t,y) = \min_{U(s):t\le s\le t+\delta}\Big(\int_t^{t+\delta}\Big(\|Y_s\|^2_{H(s)} + \|U(s)\|^2_{R(s)}\Big)ds + V(t+\delta, Y_{t+\delta})\Big), \tag{2.5}$$

where $Y_t = y$ and for $s > t$

$$\dot{Y}_s = a(s)Y_s + \big[a(s)\phi(s) - \dot{\phi}(s)\big] + c(s)U(s).$$

With a help of (2.5) we derive heuristically the Bellman equation

$$\begin{aligned} -\frac{\partial V(t,y)}{\partial t} = \min_U \Bigg(& \|y\|^2_{H(t)} + \|U\|^2_{R(t)} \\ & + \operatorname{grad}_y V(t,y)\Big[a(t)y + \big[a(t)\phi(t) - \dot{\phi}(t)\big] + c(t)U\Big]\Bigg) \end{aligned} \tag{2.6}$$

subject to boundary condition $V(T,y) = \|y\|^2_h$.

2.2. **The optimal control.** We find now

$$U^\circ(t,y) = \operatorname*{argmin}_U\Bigg(\|y\|^2_{H(t)} + \|U\|^2_{R(t)} + \operatorname{grad}_y V(t,y)\Big[a(t)y + \big[a(t)\phi(y) - \dot{\phi}(t)\big] + c(t)U\Big]\Bigg)$$

Obviously, this procedure is reduced to a minimization in U of the quadratic form

$$Q(U) := \|U\|^2_{R(t)} + \operatorname{grad}_y V(t,y)c(t)U.$$

Since $Q(U)$ is the quadratic form U° solves the equation

$$\operatorname{grad}_U Q(U) = 0.$$

Notice that $\operatorname{grad}_U \|U\|^2_{R(t)} = 2U^*R(t)$, so that

$$\operatorname{grad}_U Q(U) = 2U^*R(t) + \operatorname{grad}_y V(t,y)c(t).$$

Hence,

$$U^\circ(t,y) = -\frac{1}{2}R^{-1}(t)c^*(t)\Big(\operatorname{grad}_y V(t,y)\Big)^*. \tag{2.7}$$

Then, the Bellman equation is transformed to

$$\begin{aligned} -\frac{\partial V(t,y)}{\partial t} = \Bigg(& \|y\|^2_{H(t)} + \frac{1}{4}\Big\|\operatorname{grad}_y V(t,y)c(t)R^{-1}\Big\|^2_{R(t)} \\ & + \operatorname{grad}_y V(t,y)\Big\{a(t)y + \big[a(t)\phi(y) - \dot{\phi}(t)\big]\Big\} \\ & - \frac{1}{2}\operatorname{grad}_y V(t,y)c(t)R^{-1}(y)c^*(t)\Big(\operatorname{grad}_y V(t,y)\Big)^*\Bigg). \end{aligned}$$

Owing to

$$\mathrm{grad}_y V(t,y)c(t)R^{-1}(t)c^*(t)\Big(\mathrm{grad}_y V(t,y)\Big)^* = \frac{1}{4}\Big\|\mathrm{grad}_y V(t,y)c(t)R^{-1}(t)\Big\|^2_{R(t)}$$

we find that

$$\begin{aligned}-\frac{\partial V(t,y)}{\partial t} = \Bigg(&\|y\|^2_{H(t)} - \frac{1}{4}\Big\|\mathrm{grad}_y V(t,y)c(t)R^{-1}\Big\|^2_{R(t)} \\ &+ \mathrm{grad}_y V(t,y)\Big\{a(t)y + \big[a(t)\phi(t) - \dot{\phi}(t)\big]\Big\}.\end{aligned} \tag{2.8}$$

As in Lecture 1, we shall find a solution of (2.8) as quadratic form in y with nonnegative definite matrix $\Gamma(t)$:

$$V(t,y) = y^*\Gamma(t)y + y^*B(t) + Q(t);$$

in particular, then $\mathrm{grad}_y V(t,y) = 2y^*\Gamma(t) + B^*(t)$. Substituting that $V(t,y)$ and $\mathrm{grad}_y V(t,y)$ in (2.8) we arrive at the identity

$$\begin{aligned}&- y^*\dot{\Gamma}(t)y + y^*\dot{B}(t) + \dot{Q}(t)\\ &\equiv \|y\|^2_{H(t)}\\ &- \frac{1}{4}\Big(2y^*\Gamma(t) + B^*(t)\Big)c(t)R^{-1}(t)c^*(t)\Big(2y^*\Gamma(t) + B^*(t)\Big)^*\\ &+ \Big(2y^*\Gamma(t) + B^*(t)\Big)\Big(a(t)y + \big[a(t)\phi(t) - \dot{\phi}(t)\big]\Big)\end{aligned}$$

which provide, with $2y^*\Gamma(t)a(t)y \equiv y^*\Gamma(t)a(t)y + y^*a^*(t)\Gamma(t)y$, the differential equations

$$\begin{aligned}&- \dot{\Gamma}(t) = H(t) - \Gamma(t)c(t)R^{-1}c^*(t)\Gamma(t) + \Gamma(t)a(t) + a^*(t)\Gamma(t)\\ &- \dot{B}(t) = \Big(a^*(t) - \Gamma(t)c(t)R^{-1}c^*(t)\Big)B(t) + 2\Gamma(t)\big[a(t)\phi(t) - \dot{\phi}(t)\big]\\ &- \dot{Q}(t) = -\frac{1}{4}B^*(t)c(t)R^{-1}(t)c^*(t)B(t)\end{aligned} \tag{2.9}$$

subject to the boundary conditions $\Gamma(T) = h$, $Q(T) = 0$ and $Q(T) = 0$.

Hence, by (2.7),

$$U^\circ(t,y) = -R^{-1}(t)c^*(t)\Big(\Gamma(t)y + B(t)\Big). \tag{2.10}$$

Thus, the optimal control for the original problem is defined as follows

$$U^\circ(t,X_t^\circ) = -R^{-1}(t)c^*(t)\Big(\Gamma(t)\big\{X_t^\circ - \phi(t)\big\} + B(t)\Big)$$

where

$$\dot{X}_t^\circ = a(t)X_t^\circ + c(t)U^\circ(t,X_t^\circ).$$

Remark. If $\dot{\phi}(t) \equiv a(t)\phi(t)$, then $B(t) \equiv 0$ and $Q(t) \equiv 0$.

2.3. **Infinite horizon.** From application point of view, it makes sense to analyze a case of very large time T, that is $T = \infty$.

To simplify an analysis, we assume that all matrices are time-independent, i.e.

$$\dot{X}_t = aX_t + cU(t), \tag{2.11}$$

$\phi(t) \equiv 0$, and

$$J(x, U) = \int_0^\infty \Big(\|X_t\|_H^2 + \|U(t)\|_R^2 \Big) dt. \tag{2.12}$$

The main problem for this setting is a requirement that there exists a control action $U(t)$ for which $J(x, U) < \infty$. We give conditions, guaranteeing the latter, expressed in terms of matrices a, c and H. Recall that matrix R is nonsingular.

With $T_k \nearrow \infty$, we consider a family of cost functionals $(J_{T_k}(x, U))_{k\ge1}$. For every k, we have

$$J_{T_k}(x, U) = \int_0^{T_k} \Big(\|X_t\|_H^2 + \|U(t)\|_R^2 \Big) dt.$$

For fixed k, denote the optimal control by $U^k(t)$. From the obtained above result we know that (by Remark $B(t) \equiv 0, Q(t) \equiv 0$) $U^k(t) = -R^{-1}c^*\Gamma^k(t)$, where $\Gamma^k(t)$ solves the Riccati equation

$$-\dot{\Gamma}^k(t) = a^*\Gamma^k(t) + \Gamma^k a + H - \Gamma^k(t)cR^{-1}c^*\Gamma^k$$

subject to the boundary condition $\Gamma^k(T) = 0$. Moreover,

$$\min_{U(s):0\le s\le T_k} J_{T_k}(x, U) = \|x\|^2_{\Gamma^k(0)}.$$

We show that $\big(\|x\|^2_{\Gamma^k(0)}\big)_{k\ge1}$ is an increasing sequence.

Denote by X_t^k the controlled process associated with optimal control $U^k(t)$ on $[0, T_k]$. Then

$$\|x\|^2_{\Gamma^k(0)} = \int_0^{T_k} \Big(\|X_t^k\|_H^2 + \|U^k(t)\|_R^2 \Big) dt$$

and so

$$\begin{aligned}
\|x\|^2_{\Gamma^{k+1}(0)} &= \int_0^{T_{k+1}} \Big(\|X_t^{k+1}\|_H^2 + \|U^{k+1}(t)\|_R^2 \Big) dt \\
&\ge \int_0^{T_k} \Big(\|X_t^{k+1}\|_H^2 + \|U^{k+1}(t)\|_R^2 \Big) dt \quad \text{(since } T^{k+1} > T^k) \\
&\ge \int_0^{T_k} \Big(\|X_t^k\|_H^2 + \|U^k(t)\|_R^2 \Big) dt \quad \text{(since } U^k \text{ is optimal)} \\
&= \|x\|^2_{\Gamma^k(0)}.
\end{aligned}$$

Consequently, $\lim_{k\to\infty} \|x\|^2_{\Gamma^k(0)}$ exists but we can not be sure that this limit is finite. To prove

$$\lim_{k\to\infty} \|x\|^2_{\Gamma^k(0)} < \infty,$$

notice that it suffices to choose some control action $\widetilde{U}(t)$ such that $J(x, \widetilde{U}) < \infty$. Then, whereas $\|x\|^2_{\Gamma^k(0)} \le J_{T_k}(x, \widetilde{U}) \le J(x, \widetilde{U})$, for any k, we get the desired property.

Assuming the matrix

$$A = \Big(\int_0^1 e^{-as} cc^* e^{-a^*s} ds\Big)^{-1}$$

is nonsingular. Then, taking

$$\widetilde{U}(t) = \begin{cases} -c^* e^{a^*t}\Big(\int_0^1 e^{-as} cc^* e^{-a^*s} ds\Big)^{-1} x & 0 \le t \le 1 \\ 0, & t > 1, \end{cases} \tag{2.13}$$

where $x = \widetilde{X}_0$ and $\dot{\widetilde{X}}_t = a\widetilde{X}_t + c\widetilde{U}(t)$, we find

$$\begin{aligned} \widetilde{X}_1 &= e^a\Big(x + \int_0^1 e^{-at} c\widetilde{U}(t) dt\Big) \\ &= e^a\Big[I - \int_0^1 e^{-at} cc^* e^{-a^*t} dt \Big(\int_0^1 e^{-at} cc^* e^{-a^*t} dt\Big)^{-1}\Big] x \\ &= 0. \end{aligned}$$

The latter allows us to conclude that $\widetilde{X}_t \equiv 0$, $t \ge 1$, and at the same time

$$J(x, \widetilde{U}) = J_1(x, \widetilde{U}) < \infty.$$

So, it remains to prove that A is nonsingular matrix.

Theorem. *The matrix A is nonsingular, if the block-matrix (of size $k \times kr$)*

$$G(a, c) = \begin{pmatrix} c & ac & \dots & a^{k-1}c \end{pmatrix}. \tag{2.14}$$

is of the full rank k.

Proof. Assume A is singular. Since A is nonnegative definite, there is vector Z so that

$$0 = Z^* \int_0^1 e^{-at} cc^* e^{-a^*t} ds Z = \int_0^1 Z^* e^{-at} cc^* e^{-a^*t} Z dt.$$

On the other hand, $Z^* e^{-at} cc^* e^{-a^*t} Z$, being nonnegative and continuous in t, is equal zero for all $t \in [0, 1]$. Hence, $Z^* e^{-at} c \equiv 0$. Differentiating this identity in t and letting

$t = 0$, we get

$$\begin{aligned}
\Big(z^* e^{-at} c\Big)_{|t=0} &= z^* c = 0\\
-\Big(\frac{d}{dt}\big(z^* e^{-at} c\big)\Big)_{|t=0} &= z^* a c = 0\\
\dots\dots\dots\dots\dots\dots &\quad \dots\dots\dots\\
-\Big(\frac{d^j}{dt^j}\big(z^* e^{-at} c\big)\Big)_{|t=0} &= z^* a^j c = 0\\
\dots\dots\dots\dots\dots\dots &\quad \dots\dots\dots
\end{aligned}$$

Consequently, $Z^* G(a, c) G^*(a, c) Z = \sum_{j=0}^{k-1} Z^* c a^j (a^j)^* c^* Z = 0$, so that the matrix $G(a, c) G^*(a, c)$ is singular.

The contradiction obtained, finishing the proof. □

Definition. *The pair of matrices (a, c) is said to be controllable, if the matrix $G(a, c)$ is of rank k.*

Thus, if (a, c) is controllable, $\lim_{k\to\infty} \|x\|^2_{\Gamma^k(0)}$ exists for any x and is finite. Then, obviously, $\lim_{k\to\infty} \Gamma^k(0)$ exists as well and denote this limit by Γ. Since

$$-\dot{\Gamma}^k(t) = a^* \Gamma^k(t) + \Gamma^k(t) a + H - \Gamma^k(t) c R^{-1} c^* \Gamma^k(t)$$

subject to $\Gamma^k(T) = 0$, the limit Γ solves the algebraic Riccati equation

$$a^* \Gamma + \Gamma a + H - \Gamma c R^{-1} c^* \Gamma = 0. \tag{2.15}$$

The next fact we use (without proof) is that (2.15) possesses the unique positive definite solution provided that the matrix

$$g(H, c) = \begin{pmatrix} H \\ Hc \\ \vdots \\ Hc^{k-1} \end{pmatrix}$$

is of the full rank k. In particular, it holds if H is nonsingular.

Definition. *The pair of matrices (H, c) is said to be observable, if the matrix $g(H, c)$ is of rank k.*

2.3.1. **The optimal control for infinite horizon.** Since $\|x\Gamma^k(0)\|^2 \le J_{T_k}(x, U) \le J(x, U)$ for any k, and $\lim_{k\to\infty} \|x\|^2_{\Gamma^k(0)} = \|x\Gamma\|^2$, we find the lower bound

$$J(x, U) \ge \|x\|^2_{\Gamma}. \tag{2.16}$$

The optimal control regarding to $[0, T_k]$ is

$$U^\circ(t, X^\circ_t) = -R^{-1}(t) c^*(t) \Gamma^k(t) X^\circ_t.$$

Hence a control $U^\circ(t,X_t^\circ) = -R^{-1}(t)c^*(t)\Gamma X_t^\circ$. is a candidate to be the optimal control for the infinite horizon. The check that supposition, let us consider a quadratic form $\|X_t^\circ\|^2_\Gamma$. Write

$$\begin{aligned}
\frac{d}{dt}\|X_t^\circ\|^2_\Gamma &= \dot{X}_t^\circ \Gamma X_t^\circ + X_t^\circ \Gamma \dot{X}_t^\circ \\
&= \Big(X_t^\circ a^* + U^\circ(X_t)c^*\Big)\Gamma X_t^\circ + X_t^\circ\Gamma\Big(aX_t^\circ + cU^\circ(X_t^\circ)\Big) \\
&= \Big(X_t^\circ\Big[a^* - \Gamma c R^{-1}c^*\Big]\Big)\Gamma X_t^\circ + X_t^\circ\Gamma\Big(\Big[a - cR^{-1}c^\circ\Gamma\Big]X_t\Big) \\
&= X_t^\circ\Big(a^\circ\Gamma + \Gamma a - 2PcR^{-1}c^*\Gamma\Big)X_t^\circ \\
&= -X_t^\circ\Big(H + \Gamma c R^{-1}c^\circ\Gamma\Big)X_t^\circ \\
&= -X_t^\circ H X_t^\circ - X_t^\circ\Gamma c R^{-1}RR^{-1}c^\circ\Gamma X_t \\
&= -X_t^\circ H X_t^\circ - U^\circ(X_t^\circ)RU^\circ(X_t^\circ).
\end{aligned}$$

Hence, we find that for any $T > 0$

$$J_T(x,U^\circ) = \|x\|^2_\Gamma - \|X_T^\circ\|^2_\Gamma \le \|x\|^2_\Gamma.$$

Thereby, $J(x,U^\circ) = \lim_{T\to\infty} J_T(x,U^\circ) \le \|x\|^2_\Gamma$. So, by (2.16) we have

$$J(x,U^\circ) = \|x\|^2_\Gamma,$$

that is $U^\circ(X\circ_t)$ is the optimal control.

1. Home work: Vector case. Feedback control. Bellman equation

Let $X_t, t = 0,1,...,N$ be the controlled vector sequence sequence defined by linear vector-matrix recursion

$$X_{t+1} = aX_t + cU(t),$$

where U_t is the control. X_t and U_t are vectors of sizes $k$and r respectively. $a = a_{k\times k}$ and $c = c_{k\times r}$ are known matrices. $X_0 = x$ and x is known.

It is required to choose the optimal control $U^\circ(t)$ minimizing the cost functional ($h_{k\times k}$, $H_{k\times k}$ are nonnegative definite matrices, $R_{r\times r}$ is positive definite matrix)

$$J(x,U) = \|X_T\|^2_h + \sum_{t=1}^{T-1}\|X_t\|^2_H + \|U(t)\|^2_R. \tag{2.17}$$

1. Derive Bellman's equation for the Bellman function

$$V(x,t) = \min_{U_s: t\le s\le T}\Big[\|X_T\|^2_h + \sum_{s=t}^{T-1}\|X_s\|^2_H + \|U(s)\|^2_R\Big]$$

2. Find the optimal control.