3. WIENER PROCESS. STOCHASTIC ITÔ INTEGRAL

3.1. Wiener process. Main properties. A Wiener process (notation $W = (W_t)_{t \geq 0}$) is named in the honor of Prof. Norbert Wiener; other name is the Brownian motion (notation $B = (B_t)_{t \geq 0}$). Wiener process is Gaussian process. As any Gaussian process, Wiener process is completely described by its expectation and correlation functions.

We give below a description of main properties of $W = (W_t)_{t \geq 0}$:

1. $W_0 \equiv 0$;
2. paths (trajectories) of Wiener process are continuous functions of $t \in [0, \infty)$;
3. expectation $E W_t \equiv 0$;
4. correlation function $E(W_t W_s) = t \wedge s$, $(a \wedge b = \min(a, b))$;
5. for any $t_1, ..., t_n$ the random vector $(W_{t_1}, ..., W_{t_n})$ is Gaussian;
6. For any $s, t$
   \[
   EW_t^2 \equiv t
   
   E[W_t - W_s] \equiv 0,
   
   E[(W_t - W_s)^2] = |t - s|;
   \]
7. Increments of Wiener process on non overlapping intervals are independent, i.e. for $(s_1, t_1) \cap (s_2, t_2) = \emptyset$ the random variables $W_{t_2} - W_{s_2}, W_{t_1} - W_{s_1}$ are independent;
8. paths of Wiener process are not differentiable functions;
9. martingale property (notation $W^s_0 = \{W_u, 0 \leq u \leq s\}$)
   \[
   E(W_t | W^s_0) = W_s
   
   E\{(W_t - W_s)^2 | W^s_0\} = t - s.
   \]

Proofs. 1-5 is nothing but the definition of Wiener process. 6. is implied by 3. and 4.

4. provides the orthogonality of increments for non overlapping intervals, that is for $s_1 < s_2 < s_3 < s_4$
   \[
   E(W_{s_4} - W_{s_3})(W_{s_2} - W_{s_1}) = (s_2 - s_1) - (s_2 - s_1) = 0.
   \]

The required independence property for these random variables follows from well known fact:

*orthogonal Gaussian random variables are independent.*

To verify the validity of 8., with $h > 0$ let define $\Delta(h) = \frac{W_{s+h} - W_s}{h}$ and show that

\[
\lim_{h \to 0} \Delta(h) \text{ "does not exists".}
\]
Assume that this limit exists. Then the limit for the Fourier transform (here $i = \sqrt{-1}$)
\[ \lim_{h \to 0} E e^{i \lambda \Delta(h)} \] “exists and is a continuous function of $\lambda$”.
Hence, since the random variable $\Delta(h)$ is zero mean Gaussian with the variance
\[ E \frac{(W_{s+h} - W_s)^2}{h^2} = \frac{1}{h} , \] we find
\[ E e^{i \lambda \Delta(h)} = e^{-\frac{\lambda^2}{2h}} \to_{h \to 0} \begin{cases} 1 & \lambda = 0 , \\ 0 & \lambda \neq 0 \end{cases} : = U(\lambda) . \]
Since $U(\lambda)$ is discontinuous function the assumed differentiability is not valid.
9. Both follow from the property for the increments of Wiener process to be
independent for non overlapping intervals:
\[ E(W_t | W_0) = E(W_t - W_s + W_s | W_0) \]
\[ = W_s + E(W_t - W_s | W_0) \]
\[ = W_s \]
and
\[ E((W_t - W_s)^2 | W_0) = E(W_t - W_s)^2 = t - s . \]

There exists the alternative definition of Wiener process based on the martingale
property. We formulate this result without of proof.

Levy’s Theorem: The random process $(W_t)_{t \geq 0}$ is Wiener process if $W_0 = 0$, the
trajectories of $W_t$ are continuous and the martingale property hold.

3.2. One more property of Wiener process.
If $\xi$ is zero mean Gaussian random variable with the variance $\sigma^2 = E\xi^2$, then
$E|\xi| = \sqrt{\frac{2}{\pi}}\sigma$. Therefore, we have
\[ E|W_{t_{j+1}} - W_{t_j}| = \sqrt{\frac{\pi}{2}} \sqrt{t_{j+1} - t_j} \]
and thus the series $\sum_j E|W_{t_{j+1}} - W_{t_j}|$ diverges with $t_{j+1} - t_j \to 0$, where $0 < t_1 < t_2 < \ldots < t_n \equiv t$.
However the increments of Wiener process obey very important property exposed in

Lemma 3.1. Let $0 \equiv t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)}$ be the subdivision of the interval $[0, t]$ with
$\max_j [t_j^{(n)} - t_j] \to 0, n \to \infty$. Then (here l.i.m. denotes the limit in $L^2$ sense)
\[ \text{l.i.m.}_{n \to \infty} \sum_{j=0}^{n-1} [W_{t_{j+1}}^{(n)} - W_{t_j}^{(n)}]^2 = t . \]
Proof: Note that \( E[W_{j+1}^{(n)} - W_j^{(n)}]^2 = [t_{j+1} - t_j] \) that is \( E \sum_{j=0}^{n-1} [W_{j+1}^{(n)} - W_j^{(n)}]^2 = t \).

Consequently, it is sufficient to show only that \( \lim \frac{1}{n} E \left( \sum_{j=0}^{n-1} [W_{j+1}^{(n)} - W_j^{(n)}]^2 - t \right)^2 = 0 \).

The latter holds since

\[
\begin{align*}
\sum_{j=0}^{n-1} [W_{j+1}^{(n)} - W_j^{(n)}]^2 - t &= \sum_{j=0}^{n-1} \left( [W_{j+1}^{(n)} - W_j^{(n)}]^2 - [t_{j+1} - t_j] \right) \\
\end{align*}
\]

and so

\[
\begin{align*}
\lim_n E \left( \sum_{j=0}^{n-1} [W_{j+1}^{(n)} - W_j^{(n)}]^2 - t \right)^2 &= \sum_{j=0}^{n-1} E \left( [W_{j+1}^{(n)} - W_j^{(n)}]^2 - [t_{j+1} - t_j] \right)^2 \\
&= \sum_{j=0}^{n-1} \left( E[W_{j+1}^{(n)} - W_j^{(n)}]^4 + [t_{j+1} - t_j]^2 \\
&\quad - 2[t_{j+1} - t_j] E[W_{j+1}^{(n)} - W_j^{(n)}]^2 \right) \\
&= \sum_{j=0}^{n-1} \left( 3E[W_{j+1}^{(n)} - W_j^{(n)}]^2 \right)^2 - [t_{j+1} - t_j]^2 \\
&= \sum_{j=0}^{n-1} \left( 3[t_{j+1}^{(n)} - t_j^{(n)}]^2 - [t_{j+1}^{(n)} - t_j^{(n)}]^2 \right)^2 \\
&= \sum_{j=0}^{n-1} 2[t_{j+1}^{(n)} - t_j^{(n)}]^2 \\
&\leq 2 \max_j [t_{j+1}^{(n)} - t_j^{(n)}] \sum_{j=0}^{n-1} [t_{j+1}^{(n)} - t_j^{(n)}] \\
&= 2t \max_j [t_{j+1}^{(n)} - t_j^{(n)}] \\
&\to 0, \ n \to \infty.
\end{align*}
\]

\( \square \)

3.3. The Itô Integral.

For a pair \((W_t, f(t))\) of a Wiener process \(W_t\) a random process \(f(t)\), we define the Itô integral

\[
I(f) = \int_0^\infty f(t)dW_t.
\]
Since paths of $W_t$ are not differentiable and sums $\sum E|W_{t_{j+1}} - W_{t_j}|$ diverge, the Itô integral is not “classical” integral.

We give below conditions under which the Itô integral might be defined.

(A.1.) $\mathbb{E} \int_0^\infty f^2(t)dt < \infty$;

(A.2.) for every fixed time $t$ and any $h > 0$ the random variables $f(s), s \leq t$ and increments $W_{t+h} - W_t$ are independent.

We start with the considerations of the particular case. Assume

$$f(t) = \sum_k \alpha_k I(t_k \leq t < t_{k+1}), \quad (3.1)$$

where $0 = t_0 < t_1 < t_2, ..., < t_n, ...$ is deterministic sequences of time values and $\alpha_k, k = 0, 1, ...$ are random variable such that for fixed $k$ and $h > 0$

$$\{\alpha_0, ..., \alpha_k\} \text{ and } W_{t_{k+h}} - W_{t_k} \text{ are independent.}$$

Due the assumption (A.1) $\sum_k \mathbb{E} \alpha_k^2 [t_{k+1} - t_k] < \infty$.

Set

$$I(f) := \sum_k \alpha_k [W_{t_{k+1}} - W_{t_k}]. \quad (3.2)$$

The sum in the right side of (3.2) converges in the mean square sense. In fact, $\alpha_k [W_{t_{k+1}} - W_{t_k}], k \geq 1$ forms the sequence of orthogonal zero mean random variables (in the third line of (3.3) $k > \ell$):

$$\mathbb{E} \alpha_k [W_{t_{k+1}} - W_{t_k}] = \mathbb{E} \alpha_k \mathbb{E} \left(W_{t_{k+1}} - W_{t_k}\right) = 0$$

$$\mathbb{E} \left(\alpha_k [W_{t_{k+1}} - W_{t_k}]\right)^2 = \mathbb{E} \alpha_k^2 \mathbb{E} \left(W_{t_{k+1}} - W_{t_k}\right)^2 = \mathbb{E} \alpha_k^2 (t_{k+1} - t_k)$$

$$\mathbb{E} \alpha_k [W_{t_{k+1}} - W_{t_k}] \alpha_\ell [W_{t_{\ell+1}} - W_{t_{\ell+1}}] = \mathbb{E} \left\{\alpha_k \alpha_\ell [W_{t_{k+1}} - W_{t_{\ell+1}}] \mathbb{E} \left(W_{t_{k+1}} - W_{t_k}\right)\right\} = 0. \quad (3.3)$$

Particularly, (3.3) provides

$$\mathbb{E} I^2(f) = \mathbb{E} \left(\sum_k \alpha_k [W_{t_{k+1}} - W_{t_k}]\right)^2$$

$$= \sum_k \mathbb{E} \alpha_k^2 [W_{t_{k+1}} - W_{t_k}]^2$$

$$= \sum_k \mathbb{E} \alpha_k^2 (t_{k+1} - t_k)$$

$$= \int_0^\infty \mathbb{E} f^2(t)dt.$$
So, the integral \( I(f) \) is a linear function in \( f \), i.e.

\[
I(c_1f_1 + c_2f_2) = c_1I(f_1) + c_2I(f_2)
\]

(3.4)

for any constants \( c_1, c_2 \) and random processes \( f_1(t), f_2(t) \) of (3.1) type.

To define now the Itô integral for a random process \( f(t) \) satisfying only (A.1) and (A.2) we will some additional fact given below without proof.

**Lemma 3.2.** Let the random process \( f(t), t \geq 0 \) be satisfied (A.1), (A.2). Then there exists a sequence \( f_n(t), t \geq 0, n \geq 1 \) of piece-wise constant random processes

\[
f_n(t) = \sum_k \alpha^*_n I(t_k^n \leq t < t_{k+1}^n),
\]

where \( t_k^n, k = 0, 1, 2... \) is a condensing sequence of deterministic time values and for every \( k \) the random variables \( \{\alpha_1, \ldots \alpha_k\} \) are independent of \( W_{t_k^n + h} - W_{t_k^n}, h > 0 \), moreover for every \( n, f_n \) satisfies (A.1), (A.2) and

\[
\lim_{n \to \infty} \int_0^\infty E(f(t) - f_n(t))^2 \, dt = 0.
\]

3.3.1. **Proof of existence** \( I(f) \).

For fixed \( n \), \( I(f_n) \) is well defined. By linear property of \( I(f_n) \) we have

\[
I(f_n) - I(f_m) = I(f_n - f_m).
\]

Hence

\[
E\left(I(f_n) - I(f_m)\right)^2 = E\left(I(f_n - f_m)\right)^2 = \int_0^\infty E(f_n(t) - f_m(t))^2 \, dt \leq 2 \int_0^\infty E(f(t) - f_n(t))^2 \, dt + 2 \int_0^\infty E(f(t) - f_m(t))^2 \, dt \to 0, \ n, m \to \infty.
\]

Consequently, \( I(f_n), n \geq 1 \) is the fundamental sequence and so that by the Cauchy criteria this sequence converges the mean square sense to some limit, which we denote by \( I(f) \). In other words we get

\[
\lim_{n \to \infty} E\left(I(f) - I(f_n)\right)^2 = 0.
\]

The random variable \( I(f) \) is unique in the following sense. If \( f'_n(t), n \geq 1 \) is another approximating sequence with a limit \( \tilde{I}(f) \), then

\[
E\left(I(f) - \tilde{I}(f)\right)^2 = 0.
\]

In fact
\[
E(I(f) - \tilde{I}(f))^2 = E(I(f) - I(f_n) + I(f_n) - I(\tilde{f}_n) + I(\tilde{f}_n) - \tilde{I}(f))^2 \\
\leq 3 \left\{ E(I(f) - I(f_n))^2 + EI^2(f_n) - \tilde{f}_n + E(I(\tilde{f}_n) - \tilde{I}(f))^2 \right\} \\
= 3 \left( E(I(f) - I(f_n))^2 + E(I(\tilde{f}_n) - \tilde{I}(f))^2 \right) \\
+ \int_{0}^{\infty} E(f_n(t) - \tilde{f}_n(t))^2 dt
\]

and the first and second terms in the right side of this inequality tend to zero by the
definition while the third

\[
\int_{0}^{\infty} E(f_n(t) - \tilde{f}_n(t))^2 dt \\
\leq 2 \left( \int_{0}^{\infty} E((t-f_n(t))^2 dt + \int_{0}^{\infty} E(f(t) - \tilde{f}_n(t))^2 dt \right) \rightarrow 0, \rightarrow \infty.
\]

The random variable \( I(f) \) is named the Itô integral.

### 3.3.2. Properties of \( I(f) \).

**(P.1.)** For \( f_i(t), i = 1, 2 \), satisfying (A.1.) and (A.2.), and any constants \( c_i, i = 1, 2 \)

\[ I(c_1f_1 + c_2f_2) = c_1I(f_1) + c_2I(f_2). \]

**(P.2.)** \( EI^2(f) = \int_{0}^{\infty} Ef^2(t) dt. \)

**Proofs:** For piece-wise constant processes (P.1), (P.2.) are obviously valid. They
remain valid under passing to limit in the mean square sense (see Home work 5.).

** Remark 1.** Instead of (A.1) assume:

**(A’.1.)** \( E \int_{0}^{T} f^2(t) dt < \infty, \ T > 0. \) Then the Itô integral \( I_T(f) = \int_{0}^{T} f(t) dW_t \) is
defined as well by setting \( I_T(f) = I(f_T) \), with \( f_T(t) = f(t)I(T > t). \)

**(A”.1.)** \( P(\int_{0}^{T} f^2(t) dt < \infty) = 1. \) then \( I_T(f) \) is well defined as well.

**Sketch Proof:** Set \( \tau_n = \min\{t \leq T : \int_{0}^{t} f^2(s) ds \geq n\}, n \geq 1 \) and put \( f_n(t) = f(t)I(\tau_n \geq t). \) Then \( I_T(f_n) \) is well defined (see Problem 4 from Home work). Set

\[ I_T(f) = I_T(f_1) + \sum_{n=1}^{\infty} [I_T(f_{n+1}) - I_T(f_n)]. \]
1. Home work: Wiener Process and Itô integral

1. Prove the following statement: let $\xi, \eta$ be Gaussian vector with zero mean and orthogonal components, i.e. $E\xi = 0$, $E\eta = 0$ and

$$E\xi\eta = 0.$$ 

Show that $\xi$ and $\eta$ are independent random variables.

2. Let $\xi_1, \ldots, \xi_n, \ldots$ be a sequence of Gaussian random variables. Assume that for any $z \in \mathbb{R}$ and $i = \sqrt{-1}$

$$\lim_{k \to \infty} E e^{iz\xi_k} \text{ (exists)}.$$ 

Show that

$$\lim_{k \to \infty} E\xi_k \quad \text{and} \quad \lim_{k \to \infty} E\xi_k^2 \text{ (exist)}.$$

3. Prove (P.1.) and (P.2).

4. With $f(t)$, satisfying (A.1) and (A.2), to prove that $f_n(t) = f(t)I(\tau_n \geq t)$, where $\tau_n = \inf\{t : \int_0^t f^2(s)ds \geq n\}$, satisfies (A.1) and (A.2) as well.