

3. WIENER PROCESS. STOCHASTIC ITÔ INTEGRAL

3.1. Wiener process. Main properties. A Wiener process (notation $W = (W_t)_{t \geq 0}$) is named in the honor of Prof. Norbert Wiener; other name is the Brownian motion (notation $B = (B_t)_{t \geq 0}$). Wiener process is Gaussian process. As any Gaussian process, Wiener process is completely described by its expectation and correlation functions.

We give below a description of main properties of $W = (W_t)_{t \geq 0}$:

1. $W_0 \equiv 0$;
2. paths (trajectories) of Wiener process are continuous functions of $t \in [0, \infty)$;
3. expectation $\mathbf{E}W_t \equiv 0$;
4. correlation function $\mathbf{E}(W_t W_s) = t \wedge s$, ($a \wedge b = \min(a, b)$);
5. for any t_1, \dots, t_n the random vector $(W_{t_1}, \dots, W_{t_n})$ is Gaussian;
6. For any s, t

$$\begin{aligned} \mathbf{E}W_t^2 &\equiv t \\ \mathbf{E}[W_t - W_s] &\equiv 0, \\ \mathbf{E}[W_t - W_s]^2 &= |t - s|; \end{aligned}$$

7. Increments of Wiener process on non overlapping intervals are independent, i.e. for $(s_1, t_1) \cap (s_2, t_2) = \emptyset$ the random variables $W_{t_2} - W_{s_2}$, $W_{t_1} - W_{s_1}$ are independent;
8. paths of Wiener process are not differentiable functions;
9. martingale property (notation $W_0^s = \{W_u, 0 \leq u \leq s\}$)

$$\begin{aligned} \mathbf{E}(W_t | W_0^s) &= W_s \\ \mathbf{E}\{(W_t - W_s)^2 | W_0^s\} &= t - s. \end{aligned}$$

Proofs. 1-5 is nothing but the definition of Wiener process. 6. is implied by 3. and 4.

4. provides the orthogonality of increments for non overlapping intervals, that is for $s_1 < s_2 < s_3 < s_4$

$$\mathbf{E}(W_{s_4} - W_{s_3})(W_{s_2} - W_{s_1}) = (s_2 - s_1) - (s_2 - s_1) = 0.$$

The required independence property for these random variables follows from well known fact:

orthogonal Gaussian random variables are independent.

To verify the validity of 8., with $h > 0$ let define $\Delta(h) = \frac{W_{s+h} - W_s}{h}$ and show that

$$\lim_{h \rightarrow 0} \Delta(h) \text{ "does not exists"}. \quad 1$$

Assume that this limit exists. Then the limit for the Fourier transform (here $\mathbf{i} = \sqrt{-1}$)

$$\lim_{h \rightarrow 0} \mathbf{E} e^{\mathbf{i}\lambda\Delta(h)} \text{ "exists and is a continuous function of } \lambda \text{".}$$

Hence, since the random variable $\Delta(h)$ is zero mean Gaussian with the variance $\mathbf{E} \frac{(W_{s+h} - W_s)^2}{h^2} = \frac{1}{h}$, we find

$$\mathbf{E} e^{\mathbf{i}\lambda\Delta(h)} = e^{-\frac{\lambda^2}{2h}} \xrightarrow{h \rightarrow 0} \begin{cases} 1 & \lambda = 0, \\ 0 & \lambda \neq 0 \end{cases} := U(\lambda).$$

Since $U(\lambda)$ is discontinuous function the assumed differentiability is not valid.

9 . Both follow from the property for the increments of Wiener process to be independent for non overlapping intervals:

$$\begin{aligned} \mathbf{E}(W_t | W_0^s) &= \mathbf{E}(W_t - W_s + W_s | W_0^s) \\ &= W_s + \mathbf{E}(W_t - W_s | W_0^s) \\ &= W_s \end{aligned}$$

and

$$\mathbf{E}((W_t - W_s)^2 | W_0^s) = \mathbf{E}(W_t - W_s)^2 = t - s.$$

□

There exists the alternative definition of Wiener process based on the martingale property. We formulate this result without of proof.

Levy's Theorem: *The random process $(W_t)_{t \geq 0}$ is Wiener process if $W_0 = 0$, the trajectories of W_t are continuous and the martingale property hold.*

3.2. One more property of Wiener process.

If ξ is zero mean Gaussian random variable with the variance $\sigma^2 = \mathbf{E}\xi^2$, then $\mathbf{E}|\xi| = \sqrt{\frac{\pi}{2}}\sigma$. Therefore, we have

$$\mathbf{E}|W_{t_{j+1}} - W_{t_j}| = \sqrt{\frac{\pi}{2}} \sqrt{t_{j+1} - t_j}$$

and thus the series $\sum_j \mathbf{E}|W_{t_{j+1}} - W_{t_j}|$ diverges with $t_{j+1}^n - t_j^n \rightarrow 0$, where $0 < t_1^n < t_2^n < \dots < t_n^n \equiv t$.

However the increments of Wiener process obey very important property exposed in

Lemma 3.1. *Let $0 \equiv t_0^{(n)} < t_1^{(n)} < \dots < t_n^{(n)}$ be the subdivision of the interval $[0, t]$ with $\max_j [t_{j+1}^{(n)} - t_j^{(n)}] \rightarrow 0, n \rightarrow \infty$. Then (here l.i.m. denotes the limit in L^2 sense)*

$$l.i.m. \sum_{j=0}^{n-1} [W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}]^2 = t.$$

Proof: Note that $\mathbf{E}[W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}]^2 = [t_{j+1} - t_j]$ that is $\mathbf{E} \sum_{j=0}^{n-1} [W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}]^2 = t$.

Consequently, it is sufficient to show only that $\lim_n \mathbf{E} \left(\sum_{j=0}^{n-1} [W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}]^2 - t \right)^2 = 0$. The latter holds since

$$\sum_{j=0}^{n-1} [W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}]^2 - t = \sum_{j=0}^{n-1} \left([W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}]^2 - [t_{j+1} - t_j] \right)$$

and so

$$\begin{aligned} \lim_n \mathbf{E} \left(\sum_{j=0}^{n-1} [W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}]^2 - t \right)^2 &= \sum_{j=0}^{n-1} \mathbf{E} \left([W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}]^2 - [t_{j+1} - t_j] \right)^2 \\ &= \sum_{j=0}^{n-1} \left(\mathbf{E}[W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}]^4 + [t_{j+1} - t_j]^2 \right. \\ &\quad \left. - 2[t_{j+1} - t_j] \mathbf{E}[W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}]^2 \right) \\ &= \sum_{j=0}^{n-1} \left(\left\{ 3\mathbf{E}[W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}]^2 \right\}^2 - [t_{j+1}^{(n)} - t_j^{(n)}]^2 \right) \\ &= \sum_{j=0}^{n-1} \left(3[t_{j+1}^{(n)} - t_j^{(n)}]^2 - [t_{j+1}^{(n)} - t_j^{(n)}]^2 \right)^2 \\ &= \sum_{j=0}^{n-1} 2[t_{j+1}^{(n)} - t_j^{(n)}]^2 \\ &\leq 2 \max_j [t_{j+1}^{(n)} - t_j^{(n)}] \sum_{j=0}^{n-1} [t_{j+1}^{(n)} - t_j^{(n)}] \\ &= 2t \max_j [t_{j+1}^{(n)} - t_j^{(n)}] \\ &\rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

□

3.3. The Itô Integral.

For a pair $(W_t, f(t))$ of a Wiener process W_t a random process $f(t)$, we define the Itô integral

$$I(f) = \int_0^\infty f(t) dW_t.$$

Since paths of W_t are not differentiable and sums $\sum \mathbf{E}|W_{t_{j+1}} - W_{t_j}|$ diverge, the Itô integral is not “classical” integral.

We give below conditions under which the Itô integral might be defined.

(A.1.) $\mathbf{E} \int_0^\infty f^2(t)dt < \infty$;

(A.2.) for every fixed time t and any $h > 0$ the random variables $f(s), s \leq t$ and increments $W_{t+h} - W_t$ are independent.

We start with the considerations of the particular case. Assume

$$f(t) = \sum_k \alpha_k I(t_k \leq t < t_{k+1}), \quad (3.1)$$

where $0 = t_0 < t_1 < t_2, \dots, < t_n, \dots$ is deterministic sequences of time values and $\alpha_k, k = 0, 1, \dots$ are random variable such that for fixed k and $h > 0$

$$\{\alpha_0, \dots, \alpha_k\} \text{ and } W_{t_k+h} - W_{t_k} \text{ are independent.}$$

Due the assumption **(A.1)** $\sum_k \mathbf{E}\alpha_k^2[t_{k+1} - t_k] < \infty$.

Set

$$I(f) := \sum_k \alpha_k [W_{t_{k+1}} - W_{t_k}]. \quad (3.2)$$

The sum in the right side of (3.2) converges in the mean square sense. In fact, $\alpha_k [W_{t_{k+1}} - W_{t_k}], k \geq 1$ forms the sequence of orthogonal zero mean random variables (in the third line of (3.3) $k > \ell$):

$$\begin{aligned} \mathbf{E}\alpha_k [W_{t_{k+1}} - W_{t_k}] &= \mathbf{E}\alpha_k \mathbf{E}([W_{t_{k+1}} - W_{t_k}]) = 0 \\ \mathbf{E}(\alpha_k [W_{t_{k+1}} - W_{t_k}])^2 &= \mathbf{E}\alpha_k^2 \mathbf{E}([W_{t_{k+1}} - W_{t_k}]^2) = \mathbf{E}\alpha_k^2 (t_{k+1} - t_k) \\ \mathbf{E}\alpha_k [W_{t_{k+1}} - W_{t_k}] \alpha_\ell [W_{t_{\ell+1}} - W_{t_\ell}] &= \mathbf{E}\left\{ \alpha_k \alpha_\ell [W_{t_{\ell+1}} - W_{t_\ell}] \mathbf{E}([W_{t_{k+1}} - W_{t_k}]) \right\} = 0. \end{aligned} \quad (3.3)$$

Particularly, (3.3) provides

$$\begin{aligned} \mathbf{E}I^2(f) &= \mathbf{E}\left(\sum_k \alpha_k [W_{t_{k+1}} - W_{t_k}]\right)^2 \\ &= \sum_k \mathbf{E}\alpha_k^2 [W_{t_{k+1}} - W_{t_k}]^2 \\ &= \sum_k \mathbf{E}\alpha_k^2 [t_{k+1} - t_k] \\ &= \int_0^\infty \mathbf{E}f^2(t)dt. \end{aligned}$$

So, the integral $I(f)$ is a linear function in f , i.e.

$$I(c_1f_1 + c_2f_2) = c_1I(f_1) + c_2I(f_2) \quad (3.4)$$

for any constants c_1, c_2 and random processes $f_1(t), f_2(t)$ of (3.1) type.

To define now the Itô integral for a random process $f(t)$ satisfying only **(A.1)** and **(A.2)** we will use some additional fact given below without proof.

Lemma 3.2. *Let the random process $f(t), t \geq 0$ be satisfied **(A.1.)**, **(A.2)**. Then there exists a sequence $f_n(t), t \geq 0, n \geq 1$ of piece-wise constant random processes*

$$f_n(t) = \sum_k \alpha_k^n I(t_k^n \leq t < t_{k+1}^n),$$

where $t_k^n, k = 0, 1, 2, \dots$ is a condensing sequence of deterministic time values and for every k the random variables $\{\alpha_1, \dots, \alpha_k\}$ are independent of $W_{t_k^n+h} - W_{t_k^n}, h > 0$, moreover for every n , f_n satisfies **(A.1.)**, **(A.2.)** and

$$\lim_{n \rightarrow \infty} \int_0^\infty \mathbf{E}(f(t) - f_n(t))^2 dt = 0.$$

3.3.1. Proof of existence $I(f)$.

For fixed n , $I(f_n)$ is well defined. By linear property of $I(f_n)$ we have

$$I(f_n) - I(f_m) = I(f_n - f_m).$$

Hence

$$\begin{aligned} \mathbf{E}\left(I(f_n) - I(f_m)\right)^2 &= \mathbf{E}I^2(f_n - f_m) \\ &= \int_0^\infty \mathbf{E}(f_n(t) - f_m(t))^2 dt \\ &\leq 2 \int_0^\infty \mathbf{E}(f(t) - f_n(t))^2 dt + 2 \int_0^\infty \mathbf{E}(f(t) - f_m(t))^2 dt \\ &\rightarrow 0, \quad n, m \rightarrow \infty. \end{aligned}$$

Consequently, $I(f_n), n \geq 1$ is the fundamental sequence and so that by the Cauchy criteria this sequence converges the mean square sense to some limit, which we denote by $I(f)$. In other words we get

$$\lim_n \mathbf{E}(I(f) - I(f_n))^2 = 0.$$

The random variable $I(f)$ is unique in the following sense. If $\tilde{f}_n(t), n \geq 1$ is another approximating sequence with a limit $\tilde{I}(f)$, then

$$\mathbf{E}\left(I(f) - \tilde{I}(f)\right)^2 = 0.$$

In fact

$$\begin{aligned}
\mathbf{E}(I(f) - \tilde{I}(f))^2 &= \mathbf{E}(I(f) - I(f_n) + I(f_n) - I(\tilde{f}_n) + I(\tilde{f}_n) - \tilde{I}(f))^2 \\
&\leq 3 \left\{ \mathbf{E}(I(f) - I(f_n))^2 + \mathbf{E}I^2(f_n - \tilde{f}_n) + \mathbf{E}(I(\tilde{f}_n) - \tilde{I}(g))^2 \right\} \\
&= 3 \left(\mathbf{E}(I(f) - I(f_n))^2 + \mathbf{E}(\tilde{I}(f_n) - \tilde{I}(f))^2 \right. \\
&\qquad \qquad \qquad \left. + \int_0^\infty \mathbf{E}(f_n(t) - \tilde{f}_n(t))^2 dt \right)
\end{aligned}$$

and the first and second terms in the right side of this inequality tend to zero by the definition while the third

$$\begin{aligned}
&\int_0^\infty \mathbf{E}(f_n(t) - \tilde{f}_n(t))^2 dt \\
&\leq 2 \left(\int_0^\infty \mathbf{E}((t) - f_n(t))^2 dt + \int_0^\infty \mathbf{E}(f(t) - \tilde{f}_n(t))^2 dt \right) \rightarrow 0, \rightarrow \infty.
\end{aligned}$$

The random variable $I(f)$ is named the Itô integral.

3.3.2. Properties of $I(f)$.

(P.1.) For $f_i(t), i = 1, 2$, satisfying **(A.1.)** and **(A.2.)**, and any constants $c_i, i = 1, 2$

$$I(c_1 f_1 + c_2 f_2) = c_1 I(f_1) + c_2 I(f_2).$$

(P.2.) $\mathbf{E}I^2(f) = \int_0^\infty \mathbf{E}f^2(t)dt$.

Proofs: For piece-wise constant processes **(P.1.)**, **(P.2.)** are obviously valid. They remain valid under passing to limit in the mean square sense (see Home work 5). \square

Remark 1. Instead of **(A.1)** assume:

(A'.1.) $\mathbf{E} \int_0^T f^2(t)dt < \infty, T > 0$. Then the Itô integral $I_T(f) = \int_0^T f(t)dW_t$ is defined as well by setting $I_T(f) = I(f_T)$, with $f_T(t) = f(t)I(T > t)$.

(A''.1.) $\mathbf{P}(\int_0^T f^2(t)dt < \infty) = 1$. then $I_T(f)$ is well defined as well.

Sketch Proof: Set $\tau_n = \min\{t \leq T : \int_0^t f^2(s)ds \geq n\}$, $n \geq 1$ and put $f_n(t) = f(t)I(\tau_n \geq t)$. Then $I_T(f_n)$ is well defined (see Problem 4 from Home work). Set

$$I_T(f) = I_T(f_1) + \sum_{n=1}^{\infty} [I_T(f_{n+1}) - I_T(f_n)].$$

\square

1. Home work: Wiener Process and Itô integral

1. Prove the following statement: let ξ, η be Gaussian vector with zero mean and orthogonal components, i.e. $\mathbf{E}\xi = 0$, $\mathbf{E}\eta = 0$ and

$$\mathbf{E}\xi\eta = 0.$$

Show that ξ and η are independent random variables.

2. Let $\xi_1, \dots, \xi_n, \dots$ be a sequence of Gaussian random variables. Assume that for any $z \in \mathbb{R}$ and $\mathbf{i} = \sqrt{-1}$

$$\lim_{k \rightarrow \infty} \mathbf{E}e^{\mathbf{i}z\xi_k} \text{ (exists).}$$

Show that

$$\left. \begin{array}{l} \lim_{k \rightarrow \infty} \mathbf{E}\xi_k \\ \lim_{k \rightarrow \infty} \mathbf{E}\xi_k^2 \end{array} \right\} \text{ (exist).}$$

3. Prove **(P.1.)** and **(P.2.)**.

4. With $f(t)$, satisfying **(A.1)** and **(A.2)**, to prove that $f_n(t) = f(t)I(\tau_n \geq t)$, where $\tau_n = \inf\{t : \int_0^t f^2(s)ds \geq n\}$, satisfies **(A.1)** and **(A.2)** as well.