4. STOCHASTIC ITÔ INTEGRAL (continuation). ITÔ FORMULA

4.1. **The Itô integral as random process.**

The Itô integral $I(f)$ was defined in the previous lecture. Here, we define a random process $I_t(f), t \geq 0$ with

$$I_t(f) = I(f^t),$$

where $f^t(s) = I(t \geq s)f(s)$ and $f(t)$ is a random process with $\boldsymbol{E}\int_0^\infty f^2(s)ds < \infty$ such that for any s and $\Delta > 0$ random variables $f(s'), s' \leq s$ and increment $W_{s+\Delta} - W_s$ are independent.

Hereafter, we use "standard notation for integral":

$$I_t(f) = \int_0^t f(s)dW_s.$$

Assume τ is a random time. A natural questions: is $I_{t\wedge\tau}(f) = \int_0^{t\wedge\tau} f(s)dW_s$ well defined? The positive answer follows from the definition of the Itô integral: $I_{t\wedge\tau}(f) = I(f^{t\wedge\tau})$ with $f^{t\wedge\tau} = I(t\wedge\tau \geq s)f(s)$ if $I(\tau \geq s)$ and $W_{s+\Delta} - W_s$ are independent. The random time τ with the above-mentioned property is named *stopping time or Markov time* with respect to Wiener process W_t.

4.1.1. **Properties of $I_t(f)$.**

1.

$$1)\ \boldsymbol{E}I_t(f) = 0$$

$$2)\ \boldsymbol{E}I_t^2(f) = \int_0^t \boldsymbol{E}f^2(s)ds$$

$$3)\ \boldsymbol{E}I_{t''}(f)I_{t'}(f) = \int_0^{t''\wedge t'} \boldsymbol{E}f^2(s)ds.$$

$$4)\ I_t(c_1f_1 + c_2f_2) = c_1I_t(f_1) + c_2I_t(f_2),\ c_1, c_2 \text{ are constants.}$$

2. τ is a stopping time:

$$\begin{aligned} 1)\ \boldsymbol{E}I_{t\wedge\tau}(f) &= 0 \\ 2)\ \boldsymbol{E}I_{t\wedge\tau}^2(f) &= \boldsymbol{E}\int_0^{t\wedge\tau} f^2(s)ds. \end{aligned}$$

3. $I_t(f), t \geq 0$ is continuous random process at every in the mean square sense and with probability one.

4. For $T > t$ and any bounded function G

$$\boldsymbol{E}I_T(f)G(I_t(f)) = \boldsymbol{E}I_t(f)G(I_t(f))$$

$$\boldsymbol{E}I_T(f)G(I_\tau(f)) = \boldsymbol{E}I_{T\wedge\tau}(f)G(I_\tau(f)),\ \text{if } \tau \text{ is stopping time.}$$

and stopping time τ, $\boldsymbol{P}$-a.s.

$$\boldsymbol{E}\Big(I_T(f)\Big|I_{t'}(f), t'\le t\Big) = I_t(f) \quad \text{and} \quad \boldsymbol{E}\Big(I_T(f)\Big|I_{t'\wedge\tau}(f), t'\le t\Big) = I_{t\wedge\tau}(f)$$

.

Proofs: 1. 1) and 2) have been proved in Lecture 3. Property 3) is implied by 2) and an obvious formula $ab = \frac{1}{4}\big\{(a+b)^2 - (a-b)^2\big\}$:

$$\begin{aligned}
\boldsymbol{E}I_{t''}(f)I_{t'}(f) &= \frac{1}{4}\boldsymbol{E}\Big\{(I_{t''}(f)+I_{t'}(f))^2 - (I_{t''}(f)-I_{t'}(f))^2\Big\} \\
&= \frac{1}{4}E\Big\{(I(f_{t''}+f_{t'}))^2 - (I(f_{t''}-f_{t'})^2\Big\} \\
&= \boldsymbol{E}\int_0^\infty \frac{1}{4}\Big\{(f_{t''}(t)+f_{t'}(t))^2 - (f_{t''}(t)-f_{t'}(t))^2\Big\}dt \\
&= \boldsymbol{E}\int_0^\infty I(t''\ge t)I(t'\ge t)f^2(t)dt \\
&= \boldsymbol{E}\int_0^{t''\wedge t'} f^2(t)dt
\end{aligned}$$

2. The proofs of 1) and 2) are the same as for 1) and 2) in **1.**

3. The continuity in the mean square sense at any fixed point t follows from

$$\begin{aligned}
\boldsymbol{E}\Big(I_{t+h}(f)-I_t(f)\Big)^2 &= \boldsymbol{E}I^2(f_{t+h}-f_t) \\
&= \boldsymbol{E}\int_0^\infty \big(f_{t+h}(s)-f_t(s)\big)^2 ds \\
&= \boldsymbol{E}\int_t^{t+h} f^2(s)ds \\
&\to 0,\ h\to 0. \qquad (4.1)
\end{aligned}$$

The continuity in the mean square sense does not guarantee the continuity of trajectories as functions of the time t. In contrast, the continuity with probability one means that trajectories of random process are continuous function of t. If f is piece wise constant function, then the paths of $I_t(f)$ are continuous by virtue of the paths of Wiener process are continuous. In a general case the proof of the continuity with probability requires more fundamental facts and is omitted here.

4. We give here the proof only for piece wise constant process $f(s)$. Since

$$I_t(f) = \sum_{j:s_j\le t} f(s_j)[W_{s_{j+1}} - W_{s_j}],$$

we find

$$\begin{aligned}
&\boldsymbol{E}\Big(I_T(f)-I_t(f)\Big)G(I_t(f)) \\
= &\sum_{j:t\le s_j\le T}\boldsymbol{E}\Big(f(s_j)[W_{s_{j+1}}-W_{s_j}]\Big)G(I_t(f))
\end{aligned}$$

whereas $[W_{s_{j+1}}-W_{s_j}]$ and $f(s_j), I_t(f)$ are independent.

4.2. **The Doob inequality.**

The Doob inequalities assert: for any $T>0$

$$\begin{aligned}
\boldsymbol{P}(\sup_{t\le T}|I_t(f)|\ge x) &\le \frac{1}{x^2}\boldsymbol{E}\int_0^T f^2(t)dt,\ \forall x>0 \\
\boldsymbol{E}\sup_{t\le T}I_t^2(f) &\le 4\boldsymbol{E}\int_0^T f^2(t)dt
\end{aligned}$$

Proof: Introduce random time

$$\tau_x=\inf\{t:|I_t(f)|\ge x\}\wedge T$$

and notice that $I(\tau_x\le s), s\le t$ and $W_{t+\Delta}-W_t$, $\Delta>0$ are independent. Obviously,

$$\{\sup_{t\le T}|I_t(f)|\ge x\}=\{|I_{\tau_x}(f)|\ge x\}$$

and so

$$\boldsymbol{P}(\sup_{t\le T}|I_t(f)|\ge x)=\boldsymbol{P}(|I_{\tau_x}(f)|\ge x)+\boldsymbol{E}\int_0^{\tau_x}f^2(s)ds\le\boldsymbol{E}\int_0^T f^2(s)ds.$$

To prove the second Doob inequality, we use the following technical result: for a random variable ξ

$$\begin{aligned}
\boldsymbol{E}\xi^2 &= \boldsymbol{E}\int_0^{\xi^2}dx \\
&= \boldsymbol{E}\int_0^\infty \boldsymbol{I}(\xi^2\ge x)dx \\
&= \boldsymbol{E}\int_0^\infty \boldsymbol{I}(|\xi|\ge\sqrt{x})dx \\
&= \int_0^\infty \boldsymbol{P}(|\xi|\ge\sqrt{x})dx.
\end{aligned}\tag{4.2}$$

As previously, with $\tau_{\sqrt{x}}=\inf\{t:|I_t(f)|\ge\sqrt{x}\}\wedge T$, we use the equality

$$\boldsymbol{P}(\sup_{t\le T}|I_t(f)|\ge\sqrt{x})=\boldsymbol{P}(|I_{\tau_{\sqrt{x}}}(f)|\ge\sqrt{x}).$$

By (4.2), we get

$$\boldsymbol{E}\sup_{t\le T} I_t^2(f) = \int_0^\infty \boldsymbol{P}(|I_{\tau_{\sqrt{x}}}(f)| \ge \sqrt{x})dx. \tag{4.3}$$

It is clear that

$$\frac{|I_{\tau_{\sqrt{x}}}(f)|}{\sqrt{x}} \ge 1 \quad \text{on} \quad \{|I_{\tau_{\sqrt{x}}}(f)| \ge \sqrt{x}\}.$$

Hence,

$$\begin{aligned} \boldsymbol{P}(|I_{\tau_{\sqrt{x}}}(f)| \ge \sqrt{x}) &= \boldsymbol{E} I(|I_{\tau_{\sqrt{x}}}(f)| \ge \sqrt{x}) \\ &\le \boldsymbol{E}\Big\{\frac{|I_{\tau_{\sqrt{x}}}(f)|}{\sqrt{x}} I(|I_{\tau_{\sqrt{x}}}(f)| \ge \sqrt{x}\Big)\Big\} \end{aligned} \tag{4.4}$$

So, (4.3) is transformed to the inequality

$$\boldsymbol{E}\sup_{t\le T} I_t^2(f) \le \int_0^\infty \boldsymbol{E}\Big\{\frac{|I_{\tau_{\sqrt{x}}}(f)|}{\sqrt{x}} I(|I_{\tau_{\sqrt{x}}}(f)| \ge \sqrt{x}\Big)\Big\}dx.$$

The property **4.** is equivalent to

$$\boldsymbol{E}(I_T(f)|I_t(f)) = I_t(f) \quad \text{and} \quad \boldsymbol{E}(I_T(f)|I_{\tau_{\sqrt{x}}}(f)) = I_{\tau_{\sqrt{x}}}(f)).$$

Hence, $|I_{\tau_{\sqrt{x}}}(f))| = |\boldsymbol{E}(I_T(f)|I_{\tau_{\sqrt{x}}}(f))|$ and by the Jensen inequality for the conditional expectation

$$|I_{\tau_{\sqrt{x}}}(f))| = \boldsymbol{E}\Big(\big|I_T(f)\big||I_{\tau_{\sqrt{x}}}(f)\Big).$$

Then,

$$\begin{aligned} \boldsymbol{E}\sup_{t\le T} I_t^2(f) &\le \int_0^\infty \boldsymbol{E}\Big\{\frac{|I_T(f)|}{\sqrt{x}} I(|I_{\tau_{\sqrt{x}}}(f)| \ge \sqrt{x}\Big)\Big\}dx \\ &\le \int_0^\infty \boldsymbol{E}\Big\{\frac{|I_T(f)|}{\sqrt{x}} I(\sup_{t\le T}|I_t(f)| \ge \sqrt{x}\Big)\Big\}dx. \end{aligned}$$

So, with $\xi = \sup_{t\le T}|I_t(f)|$, we have

$$\boldsymbol{E}\sup_{t\le T} I_t^2(f) \le \boldsymbol{E} I_T(f)\int_0^{\xi^2}\frac{dx}{\sqrt{x}} = 2\boldsymbol{E} I_T(f)\sup_{t\le T} I_t(f)$$

and by the Chebyshev inequality

$$\boldsymbol{E}\sup_{t\le T} I_t^2(f) \le 2\sqrt{\boldsymbol{E} I_T^2(f)\boldsymbol{E}\sup_{t\le T} I_t^2(f)}.$$

Consequently,

$$\sqrt{\boldsymbol{E}\sup_{t\le T} I_t^2(f)} \le 2\sqrt{\boldsymbol{E} I_T^2(f)\boldsymbol{E}}$$

and the result is done. □

4.3. **The Itô formula.**

Example 1. Let $f(t) = W_t$. Since for every $t > 0$, $\int_0^t \boldsymbol{E} W_s^2 ds = \int_0^t s ds = \frac{t^2}{2}$ the Itô integral $I_t(W) = \int_0^t W_s dW_s$ is well defined.

If X_t is continuously differentiable function with $X_0 = 0$, then

$$\int_0^t X_s dX_s = \frac{1}{2} X_t^2. \tag{4.5}$$

The question is: remains this formula true for $I_t(W)$:

$$\int_0^t W_s dW_s = \frac{1}{2} W_t^2?$$

The answer is negative by virtue of the following contradiction

$$0 = \boldsymbol{E} \int_0^t W_s dW_s = \frac{1}{2} \boldsymbol{E} W_t^2 = \frac{t}{2} > 0.$$

Now, we explain why (4.5) takes place for continuously differentiable function X_t and fails for W_t. Set $H(x) = \frac{x^2}{2}$. Since X_t is assumed to be continuously differentiable function we get $\frac{d}{dt} H(X_t) = H'(X_t)\dot{X}_t = X_t \dot{X}_t$ and so

$$\frac{1}{2} X_t^2 = H(X_t) = \int_0^t X_s dX_s,$$

i.e. (4.5) is really correct for X_t.

For $H(W_t)$ and $0 = t_0^n < t_1^n < ... < t_n^n = t$, $n \geq 1$, write

$$\begin{aligned} H(W_t) &= \sum_{j=1}^n [H(W_{t_{j+1}^n}) - H(W_{t_j^n})] \\ &= \sum_{j=1}^n \frac{1}{2} \Big(W_{t_{j+1}^n}^2 - W_{t_j^n}^2 \Big) \\ &= \sum_{j=1}^n W_{t_j^n} \big(W_{t_{j+1}^n} - W_{t_j^n} \big) + \frac{1}{2} \sum_{j=1}^n \big(W_{t_{j+1}^n} - W_{t_j^n} \big)^2. \end{aligned} \tag{4.6}$$

Under the assumption $\lim_{n\to\infty} \sup_j [t_{j+1}^n - t_j^n] = 0$

$$\operatorname*{l.i.m.}_n \sum_{j=1}^n [H(W_{t_{j+1}^n}) - H(W_{t_j^n})] = \int_0^t W_s dW_s + \frac{t}{2}$$

since (see Lecture 3)

$$\underset{n}{\text{l.i.m.}}\sum_{j=1}^{n}\left(W_{t^n_{j+1}}-W_{t^n_j}\right)^2=t.$$

Thus

$$\frac{1}{2}W_t^2=\int_0^t W_s dW_s+\frac{t}{2} \tag{4.7}$$

or, what is equivalent (recall that $H(x)=\frac{1}{2}x^2$ and so $H'(x)=x$ and $H''(x)\equiv 1$)

$$H(W_t)=\int_0^t H'(W_s)dW_s+\frac{1}{2}\int_0^t H''(W_s)ds. \tag{4.8}$$

The formula given in (4.8) is famousItô formula.

A phenomena of the Itô formula arises from the property of trajectories of Wiener process which are continuous but not differentiable functions.

If $G(x)$ is any twice continuously differentiable function (not obligatory a quadratic form of x), the Itô formula for $G(W_t)$ is:

$$G(W_t)=G(0)+\int_0^t G'(W_s)dW_s+\frac{1}{2}\int_0^t G''(W_s)ds. \tag{4.9}$$

An heuristic proof of (4.9) is similar to (4.8). In fact, by the mean value theorem $(0\le\theta^n_j\le 1)$

$$\begin{aligned}
G(W_t) &= \sum_{j=1}^{n}G'(W_{t^n_j})[W_{t^n_{j+1}}-W_{t^n_j}] \\
&\quad +\sum_{j=1}^{n}\frac{1}{2}G''(W_{t^n_j})[W_{t^n_{j+1}}-W_{t^n_j}]^2 \\
&\quad +\sum_{j=1}^{n}\frac{1}{2}\Big\{G''(W_{t^n_j}+\theta^n_j\{W_{t^n_{j+1}}-W_{t^n_j}\}) \\
&\quad -G''(W_{t^n_j})\Big\}[W_{t^n_{j+1}}-W_{t^n_j}]^2
\end{aligned} \tag{4.10}$$

The first sum in the right side converges in the mean square sense to the Itô integral $\int_0^t G'(W_s)dW_s$, the second to $\int_0^t \frac{1}{2}G''(W_s)ds$ while the third to zero.

Thus, (4.9) holds.

4.4. **More about Itô formula.** Assume

$$X_t=X_0+\int_0^t\alpha(s)ds+\int_0^t\beta(s)dW_s,$$

where X_0 and (W_t) are independent, $\beta(s)$ and $W_{t+\delta} - W_t$ are independent for $t \geq s$ and $\delta > 0$, and $E \int_0^t \beta^2(s)ds < \infty, t > 0$. Then for smooth $G(t, x)$ we have

$$G(t, X_t) = G(0, X_0) + \int_0^t \big[G_t(s, X_s) + G_x(s, X_s)\alpha(s) + \frac{1}{2}G_{xx}(s, X_s)\big]dsds + \int_0^t G_x(s, X_s)\beta(s)dW_s. \qquad \textbf{(bonus)}$$

Home work: the Itô formula

1. Derive the Itô formula for $Z_t = \exp\Big(W_t - \frac{t}{2}\Big)$.
2. Derive the Itô formula for $Z_t = \exp\Big(I_t(f) - \frac{1}{2}\int_0^t f^2(s)ds\Big)$.
3. Derive **bonus** to get better grade.