5. STOCHASTIC ITÔ'S DIFFERENTIAL EQUATION

5.1. **Differential equation with white noise.**

For an approximation of "derivative" $\dot{W}_t(h) = \frac{W_{t+h}-W_t}{h}$ for Wiener process W_t, let us define the correlation function (Home work):

$$K_h(t,s) = \begin{cases} \frac{1}{h}\Big(1 - \frac{|t-s|}{h}\Big), & |t-s| < h, \\ 0, & |t-s| \geq h. \end{cases} \tag{5.1}$$

We have $K_h(t,s) \equiv K_h(t-s)$, so that $\dot{W}_t(h)$ is a stationary Gaussian process. Moreover the function $K_h(\tau)$ for δ small enough reminds the δ-function; for example the area

$$\int_{-\infty}^{\infty} K_h(u)du \equiv 1$$

is independent of h.

A "formal" derivative, $\dot{W}_t$, of Wiener process is called *Gaussian white noise.* One can interpret $\dot{W}_t$ as a generalized derivative in a sense: for a smooth deterministic function $\varphi(t)$

$$\int_0^t \varphi(s)\dot{W}_s ds := \varphi(t)W_t - \int_0^t \dot{\varphi}(s)W_s ds.$$

To show a relation of this formula with the Itô integral. By the Itô formula (see the end of Lecture 4) we have

$$\phi(t)W_t = \int_0^t \phi(s)dW_s + \phi(t)W_t - \int_0^t \dot{\phi}(s)W_s ds. \tag{5.2}$$

In other words,

$$\int_0^t \phi(s)\dot{W}_s ds := \int_0^t \phi(s)dW_s,$$

that is the symbolic notion $\int_0^t \phi(s)\dot{W}_s ds$ coincides with the Itô integral $\int_0^t \phi(s)dW_s$. The generalized random process $n_\phi(t) := \phi(t)\dot{W}_t$ is named Gaussian white noise with the correlation function

$$K(t,s) = \phi^2(t)\delta(t-s). \tag{5.3}$$

Consider now a linear differential equation with this white noise

$$\dot{X}_t = a(t)X_t + n_\phi(t) \tag{5.4}$$

subject to the deterministic initial point X_0, where $a(t)$ is deterministic and bounded function. Due to (5.2), it makes sense to consider the integral form of (5.4) with the Itô intrgral:

$$X_t = x + \int_0^t a(s)X_s ds + \int_0^t \phi(s)dW_s. \tag{5.5}$$

In this case, the unique solution is the following (Heme work) Lets us check that

$$X_t = e^{\int_0^t a(s)ds} X_0 + e^{\int_0^t a(s)ds} \int_0^t e^{\int_0^s -a(u)du} n_\phi(s)ds. \tag{5.6}$$

The uniqueness is verified in standard way. If X'_t, X''_t are two solution, then the difference $\triangle_t = X'_t - X''_t$ solves the linear differential equation

$$\dot{\triangle}_t = a(t)\triangle_t$$

subject to $\triangle_0 = 0$ and so $\triangle_t \equiv 0$.

5.2. **Nonlinear Itô equation.**

Consider Itô equation

$$X_t = x + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s, \tag{5.7}$$

Theorem 5.1 *Let $a(s,x)$ and $b(s,x)$ be continuous functions in s, x and Lipschitz continuous in x:*

$$|a(s,x') - a(s,x'')| + |b(s,x') - b(s,x'')| \le L|x' - x''|,$$

where L is independent of s.

Then equation (5.7) has the unique solution.

If instead of the fixed initial condition x is used the random variable X_0, independent of Wiener process W_t, with $\boldsymbol{E}X_0^2 < \infty$, then for every t, $\boldsymbol{E}X_t^2 < \infty$.

Proof. 1. Uniqueness: Assume X_t and Y_t are solutions. Set $\Delta_t = X_t - Y_t$. We introduce

$$\alpha(t) = \begin{cases} \frac{a(t,X_t)-a(t,Y_t)}{X_t-Y_t} & X_t \neq Y_t \\ 0 & X_t = Y_t \end{cases}$$

and

$$\beta(t) = \begin{cases} \frac{b(t,X_t)-b(t,Y_t)}{X_t-Y_t} & X_t \neq Y_t \\ 0 & X_t = Y_t \end{cases}$$

and note that, due to the Lipschitz condition, $|\alpha(t)| \le L$ and $|\beta(t)| \le L$.

It is clear that

$$\Delta_t = \int_0^t \alpha(s)\Delta_s ds + \int_0^t \beta(s)\Delta_s dW_s.$$

Applying the Itô formula to $\Delta_t^2 \equiv 0$, we get

$$\Delta_t^2 = 2\int_0^t \alpha(s)\Delta_s^2 ds + 2\int_0^t \beta(s)\Delta_s^2 dW_s + \int_0^t \beta^2(s)\Delta^2 ds.$$

Taking into account that $|\alpha(s)| \leq L$ and $|\beta(s)| \leq L$ and taking the expectation we find

$$\begin{aligned}
\boldsymbol{E}\Delta_t^2 &= 2\int_0^t \boldsymbol{E}\alpha(s)\Delta_s^2 ds + + \int_0^t \boldsymbol{E}\beta^2(s)\Delta^2 ds \\
&\leq 2L\int_0^t \boldsymbol{E}\Delta_s^2 ds + L^2\int_0^t \boldsymbol{E}\Delta^2 ds \\
&\leq (2L+L^2)\int_0^t \boldsymbol{E}\Delta_s^2 ds.
\end{aligned}$$

Hence $\boldsymbol{E}\Delta_t^2 \equiv 0$.

Proof. Set $V_t = \int_0^t \boldsymbol{E}\triangle_s^2 ds$, $r = 2L + L^2$, and $v(t) = V_t - \int_0^t V_s ds$. Obviously $v(t) \leq 0$ and $\dot{V})t = rV_t + v(t)$. Since $V_0 = 0$, we have $V_t = \int_0^t e^{r(t-s)}v(s)ds \leq 0$. This $V_t \equiv 0$ and the desired statement is valid. ☐

2. Existence: Set $X_t^0 \equiv x$ and define the recursion

$$X_t^n = x + \int_0^t a(s, X_s^{n-1})ds + \int_0^t b(s, X_s^{n-1})dW_s. \tag{5.8}$$

Set

$$\alpha^n(t) = \begin{cases} \frac{a(t,X_t^n)-a(t,X_t^{n-1})}{X_t^n - X_t^{n-1}} & X_t^n \neq X_t^{n-1} \\ 0 & X_t^n = X_t^{n-1} \end{cases}$$

and

$$\beta^n(t) = \begin{cases} \frac{b(t,X_t^n)-b(t,X_t^{n-1})}{X_t^n - X_t^{n-1}} & X_t^n \neq X_t^{n-1} \\ 0 & X_t^n = X_t^{n-1}. \end{cases}$$

and note that $|\alpha^n(t)| \leq L$, $|\beta^n(t)| \leq L$. Define also $\Delta_t^0 \equiv x$ and $\Delta_t^n = X_t^n - X_t^{n-1}$ which obviously possess presentation:

$$\begin{aligned}
\Delta_t^1 &= \int_0^t a(s,x)ds + \int_0^t b(s,x)dW_s \\
\Delta_t^n &= \int_0^t \alpha^{n-1}(s)\Delta_s^{n-1}ds + \int_0^t \beta^{n-1}(s)\Delta_s^{n-1}dW_s, \text{ for } n \geq 2.
\end{aligned}$$

For $T>0$, estimate above the value $\boldsymbol{E}\sup\limits_{t\le T}(\Delta_t^n)^2$. Write

$$\begin{aligned}\boldsymbol{E}\sup_{t'\le t}(\Delta_{t'}^1)^2 &= \boldsymbol{E}\sup_{t'\le t}\Big(\int_0^{t'} a(s,x)ds+\int_0^{t'} b(s,x)dW_s\Big)^2\\ &\le 2\boldsymbol{E}\sup_{t'\le t}\Big(\int_0^{t'} a(s,x)ds\Big)^2+2\boldsymbol{E}\sup_{t'\le t}\Big(\int_0^{t'} b(s,x)dW_s\Big)^2\\ &\le 2L^2Tt+2\boldsymbol{E}\sup_{t'\le t}\Big(\int_0^{t'} b(s,x)dW_s\Big)^2. \qquad (5.9)\end{aligned}$$

By the Doob inequality $\boldsymbol{E}\sup\limits_{t'\le t}\Big(\int_0^t b(s,x)dW_s\Big)^2\le 4\int_0^t b^2(s,x)ds\le 4L^2t$. Hence

$$\boldsymbol{E}\sup_{t'\le t}(\Delta_{t'}^1)^2\le rt,\ r=L^2(2T+8).$$

Applying the induction method we show that for $n\ge 2$ an upper bound holds

$$\boldsymbol{E}\sup_{t'\le t}(\Delta_{t'}^n)^2\le\frac{r^nt^n}{n!}. \qquad (5.10)$$

Let us assume that for some $n\ge 2$ (5.10) holds and prove that (5.10) is valid for for $n+1$.

Below, we will exploit the Cauchy-Schwartz inequality

$$\Big(\int_0^{t'}\alpha^n(s)\Delta^n(s)ds\Big)^2\le\int_0^t(\alpha^n(s))^2ds\int_0^t(\Delta^n(s))^2ds$$

Write

$$\begin{aligned}\boldsymbol{E}\sup_{t'\le t}(\Delta_{t'}^{n+1})^2 &= \boldsymbol{E}\sup_{t'\le t}\Big(\int_0^{t'}\alpha^n(s)\Delta^n(s)ds+\int_0^{t'}\beta^n(s)\Delta^n(s)dW_s\Big)^2\\ &\le 2\boldsymbol{E}\sup_{t'\le t}\Big(\int_0^{t'}\alpha^n(s)\Delta^n(s)ds\Big)^2+2\boldsymbol{E}\sup_{t'\le t}\Big(\int_0^{t}\beta^n(s)\Delta^n(s)dW_s\Big)^2\\ &\le 2TL^2\boldsymbol{E}\int_0^t(\Delta^n(s))^2ds+2\boldsymbol{E}\sup_{t'\le t}\Big(\int_0^{t'}\beta^n(s)\Delta^n(s)dW_s\Big)^2.\end{aligned}$$

By the Doob inequality

$$\boldsymbol{E}\sup_{t'\le t}\Big(\int_0^t\beta^n(s)\Delta^n(s)dW_s\Big)^2\le 4L^2\boldsymbol{E}\int_0^t\Delta^n(s)ds$$

we arrive at

$$\begin{aligned}
\boldsymbol{E}\sup_{t'\leq t}(\Delta^{n+1}_{t'})^2 &\leq 2TL^2\int_0^t \boldsymbol{E}\sup_{s'\leq s}(\Delta^n_{s'})^2 ds + 8L^2\int_0^t \boldsymbol{E}\sup_{s'\leq s}(\Delta^n_{s'})^2 ds && (5.11)\\
&= (2T+8)L^2\int_0^t \boldsymbol{E}\sup_{s'\leq s}(\Delta^n_{s'})^2 ds. && (5.12)
\end{aligned}$$

Due to the assumption made $\boldsymbol{E}(\sup\limits_{s'\leq s}(\Delta^n_{s'})^2 \leq \frac{L^2 r^n s^n}{n!}$ and (5.12) it follows that

$$\begin{aligned}
\boldsymbol{E}\sup_{t'\leq t}(\Delta^{n+1}_{t'})^2 &\leq L^2(2T+8)\int_0^t \frac{r^n s^n}{n!}ds\\
&= L^2(2T+8)\frac{r^n}{n!}\int_0^t \frac{s^n}{n!}ds\\
&= L^2(2T+8)\frac{r^n}{n!}\frac{t^{n+1}}{(n+1)}\\
&= \frac{r^{n+1}t^{n+1}}{(n+1)!}.
\end{aligned}$$

We are now in the position to check the existence of a solution for the Itô equation given in (5.7). The above obtained upper bound allows to conclude that for every $t > 0$ we have $\sum\limits_{n=1}^{\infty}\boldsymbol{E}\sup\limits_{t'\leq t}|\Delta^n_{t'}| < \infty$, i.e. $\sum\limits_{n=1}^{\infty}\sup\limits_{t\leq t}|X^n_t - X^{n-1}_t|$. Consequently the random process

$$\widetilde{X}_t = x + \sum_{n=1}^{\infty}[X^n_t - X^{n-1}_t]$$

is well defined. On the other side, whereas $X^n_t = x + \sum_{j=1}^n [X^j_t - X^{j-1}_t]$, $n \geq 1$, for any $T > 0$ we find

$$\begin{aligned}
\boldsymbol{E}\sup_{t\leq T}\left(\widetilde{X}_t - X^n_t\right)^2 &\leq \boldsymbol{E}\Big(\sum_{j=n+1}^{\infty}\sup_{t'\leq T}|\Delta^j_{t'}|\Big)^2\\
&\leq \sum_{j=n+1}^{\infty}\sum_{i=n+1}^{\infty}\boldsymbol{E}\sup_{t'\leq T}|\Delta^j_{t'}|\sup_{t'\leq T}|\Delta^i_{t'}|\\
&= \sum_{j=n+1}^{\infty}\sum_{i=n+1}^{\infty}\Big(\boldsymbol{E}\sup_{t'\leq T}|\Delta^j_{t'}|^2\boldsymbol{E}\sup_{t'\leq T}|\Delta^i_{t'}|^2\Big)^{1/2}\\
&= \Big(\sum_{j=n+1}^{\infty}\sqrt{\frac{r^j T^j}{j!}}\Big)^2\\
&\to 0\ n\to\infty.
\end{aligned}$$

Therefore, for any time interval $[0,T]$ the sequence of random processes $(X_t^n)_{t\le T},\ n\ge 1$ converges in the mean square sense to the limit $(\widetilde{X}_t)_{t\le T}$ uniformly in $t\le T$ what enables us to choose a subsequence $(X_t^{n'})_{t\le T}$ which converges $\boldsymbol{P}$-a.s. to the same limit which is continuous process by virtue of the random processes $(X_t^n)_{t\le T},\ n\ge 1$ are continuous. Now, it remains now to show only that $\widetilde{X}_t$ is a solution of (5.7). Due to (5.8), write

$$\begin{aligned}
&\Big|\widetilde{X}_t - x - \int_0^t a(s,\widetilde{X}_s)ds - \int_0^t b(s,\widetilde{X}_s)dW_s\Big| \\
= & \Big|\widetilde{X}_t - X_t^n - \int_0^t [a(s,\widetilde{X}_s) - a(s,X_s^{n-1})]ds - \int_0^t [b(s,\widetilde{X}_s) - b(s,X_s^{n-1})]dW_s\Big| \\
\le & \sup_{t\le T}|\widetilde{X}_t - X_t^n| \\
& + \int_0^T |a(s,\widetilde{X}_s) - a(s,X_s^{n-1})|ds \\
& + \sup_{t\le T}\Big|\int_0^t [b(s,\widetilde{X}_s) - b(s,X_s^{n-1})]dW_s\Big| \\
\le & \sup_{t\le T}|\widetilde{X}_t - X_t^n| + LT\sup_{t\le T}|\widetilde{X}_t - X_t^{n-1}| + \sup_{t\le T}\Big|\int_0^t [b(s,\widetilde{X}_s) - b(s,X_s^{n-1})]dW_s\Big|.
\end{aligned}$$

Further, by the Doob inequality we have

$$\begin{aligned}
\boldsymbol{E}\sup_{t\le T}\Big(\int_0^t [b(s,\widetilde{X}_s) - b(s,X_s^{n-1})]dW_s\Big)^2 &\le 4\boldsymbol{E}\int_0^T [b(s,\widetilde{X}_s) - b(s,X_s^{n-1})]^2 ds \\
&\le 4L^2T\boldsymbol{E}\sup_{t\le T}\Big(\widetilde{X}_t - X_t^{n-1}\Big)^2.
\end{aligned}$$

Hence, there exists a constant $C>0$, so that

$$\begin{aligned}
\boldsymbol{E}\Big(\widetilde{X}_t - x - \int_0^t a(s,\widetilde{X}_s)ds - \int_0^t b(s,\widetilde{X}_s)dW_s\Big)^2 &\le C\Big\{\boldsymbol{E}\sup_{t\le T}\Big(\widetilde{X}_t - X_t^n\Big)^2 \\
&\quad + \boldsymbol{E}\sup_{t\le T}\Big(\widetilde{X}_t - X_t^{n-1}\Big)^2\Big\} \\
&\to 0,\ n\to\infty.
\end{aligned}$$

□

Home work:

1. Check (5.1).

2. Check (5.2). **Hint:** Use the rule: if $H(t,x)$ is continuously differentiable one in t and twice in x, then $dH(t,W_t) = H_t'(t,W_t)dt + H_x'(t,W_t)dW_t + \frac{1}{2}H_{xx}''(t,W_t)dt$.

3. Check (5.6).