FILTERING OF NONLINEAR STOCHASTIC FEEDBACK SYSTEMS

F. CARRAVETTA[1], A. GERMANI[1,2], R. LIPTSER[3], AND C. MANES[1,2]

Abstract. This paper concerns the filtering problem for a class of stochastic nonlinear systems where the drift term may depend either on some external function (*open-loop system*) or on the system output (*closed-loop system*), through a *controller*. Such systems are denoted *feedback systems*. The following result is proven: for feedback systems, the optimal filter in the open-loop case remains optimal when the feedback is closed.

1. Introduction

Consider the class of nonlinear stochastic systems described by the equations:

$$\begin{aligned} dX_t^\phi &= f(t, X_t^\phi, u(t, \phi_{[0,t]}))dt + b(t, X_t^\phi)dW_t', \\ dY_t^\phi &= h(t, X_t^\phi)dt + B(t)dW_t'', \end{aligned} \tag{1.1}$$

where $X_t^\phi \in \mathbb{R}^n$ is the system state, $Y_t^\phi \in \mathbb{R}^m$ is the observation process, $u(t, \phi_{[0,t]}) \in \mathbb{R}^p$ is the input function, generated by some driving function ϕ. f, h are vector functions of suitable dimensions. $W_t' \in \mathbb{R}^n$ and $W_t'' \in \mathbb{R}^m$ are independent Wiener processes (without loss of generality we consider square diffusion matrices b and B).

If in system (1.1) the driving function ϕ is *replaced* by the system output Y, we obtain the following system

$$\begin{aligned} dX_t &= f(t, X_t, u(t, Y_{[0,t]}))dt + b(t, X_t)dW_t', \\ dY_t &= h(t, X_t)dt + B(t)dW_t''. \end{aligned} \tag{1.2}$$

So, the term $u(t, Y_{[0,t]})$ represents a causal map of the observation process into the input, describing a behavior of some feedback control device (*the controller*). We will refer to system (1.1) as the *open loop* system, and to system (1.2) as the *closed loop* system.

Let $F : \mathbb{R}^n \mapsto \mathbb{R}^{n'}$ be a function of the system state that defines a signal to be estimated for the open and closed loop systems:

$$S_t^\phi = F\big(X_t^\phi\big), \tag{1.3}$$

$$S_t = F\big(X_t\big). \tag{1.4}$$

Assume for every fixed t there is a function $\Psi_t(y_{[0,t]}; \phi_{[0,t]})$, $(y_t, \phi(t)$, $t \geq 0$, are continuous vector functions valued in $\mathbb{R}^p$) such that

$$\Psi_t(Y_{[0,t]}^\phi; \phi_{[0,t]}) = E\big(S_t^\phi / Y_{[0,t]}^\phi\big). \tag{1.5}$$

1991 *Mathematics Subject Classification.* 93E03, 93E11.

Key words and phrases. Nonlinear filtering, closed-loop systems, Girsanov theorem, convergence of probability measures.

This is the *open-loop filter*, i.e. the optimal filter for the open-loop system (1.1), forced by the system output and by the forcing term ϕ. For every t assume also there exists a function $\Phi_t(y_{[0,t]})$ such that

$$\Phi_t(Y_{[0,t]}) = E\big(S_t/Y_{[0,t]}\big),\ \ P\text{-a.s.} \tag{1.6}$$

This is the *closed-loop filter*, i.e. the optimal filter for the closed-loop system (1.2), that is forced by the system output only.

The following question arises:

$$\Psi_t(Y_{[0,t]}; Y_{[0,t]}) \stackrel{?}{=} \Phi_t(Y_{[0,t]}),\ \ P\text{-a.s.}, \tag{1.7}$$

stated in other words: if we apply the *open-loop filter to the closed-loop system*, then does the estimate agree with the optimal state-estimate for the closed-loop system?

The question if (1.7) holds or not is not only interesting by itself, but is important in many applications. For instance, in all cases in which a finite-dimensional filter exists for the open loop system (see [3]), identity (1.7) proves that the filter remains optimal and finite-dimensional also when the feedback is closed. Another interesting application is when $\Phi_t(Y_{[0,t]})$ is computed by the Monte-Carlo method via $\Psi_t(Y^{\phi}_{[0,t]}; \phi_{[0,t]})$.

Up to now, the correctness of (1.7) has been proved only for particular cases of the problem, such as in the case of linear-Gaussian system under nonlinear feedback [5]) of the type

$$\begin{aligned} dX_t &= A(t, Y_{[0,t]})X_t dt + F(t, Y_{[0,t]})dW'_t, \\ dY_t &= C(t, Y_{[0,t]})X_t dt + G(t, Y_{[0,t]})dW''_t, \end{aligned}$$

important from an application point of view.

In this paper, we give affirmative answer to the question (1.7) for the nonlinear models (1.1), (1.2), under some not really restrictive assumptions.

The paper is organized as follows: section 2 reports the rigorous statement of the problem is given and section 3 presents the main theorem. Conclusions follow.

2. Problem Statement

On a probability space $\{\Omega, \mathcal{F}, P\}$, consider two independent Wiener processes W'_t and W''_t, $t \in [0, \infty)$, of dimension n and m, respectively, and a random vector $\mathcal{X} \in \mathbb{R}^n$. Let $\mathcal{F}^t$ be the nondecreasing family of σ-algebras generated by $\{(\mathcal{X}, W'_s, W''_s),\ 0 \le s \le t\}$. Throughout the paper $\mathcal{C}_{[0,\infty)}(\mathbb{R}^q)$ shall denote the space of $\mathbb{R}^q$-valued continuous functions over the interval $[0, \infty)$. On this space, let $\mathcal{B}^q_t$, $t \ge 0$, be the σ-algebra generated by cylinder sets of the form

$$\big\{\varphi \in \mathcal{C}_{[0,\infty)}(\mathbb{R}^q):\ \phi(t_k) \in B_k;\ t_k < t;\ k{=}1,\dots,\bar{k};\ \bar{k} \in \mathbb{N};\ B_k \in \mathcal{B}(\mathbb{R}^q)\big\}, \tag{2.1}$$

where $\mathcal{B}(\mathbb{R}^q)$ is the Borel σ-algebra of $\mathbb{R}^q$. Moreover, let $\mathcal{B}^q_\infty = \bigvee_{t\ge 0} \mathcal{B}^q_t$. Let $\mathcal{R}_+$ be the Borel σ-algebra on $\mathbb{R}_+$.

Given a process ξ_t, let $\sigma_t(\xi)$ be the σ-algebra generated by $\{\xi_s,\ 0 \le s \le t\}$.

For a given $\phi \in \mathcal{C}_{[0,\infty)}(\mathbb{R}^m)$, consider the *open loop* model:

$$\begin{aligned} dX_t^\phi &= f(t, X_t^\phi, u(t,\phi))dt + b(t, X_t^\phi)dW_t', \quad & X_0^\phi = \mathcal{X}, \\ dY_t^\phi &= h(t, X_t^\phi)dt + B(t)dW_t'', & Y_0^\phi = 0, \\ S_t^\phi &= F\big(X_t^\phi\big). \end{aligned} \tag{2.2}$$

Consider also the *closed loop* model:

$$\begin{aligned} dX_t &= f(t, X_t, u(t,Y))dt + b(t, X_t)dW_t', \quad & X_0 = \mathcal{X}, \\ dY_t &= h(t, X_t)dt + B(t)dW_t'', & Y_0 = 0, \\ S_t &= F\big(X_t\big). \end{aligned} \tag{2.3}$$

In both models the state space is $\mathbb{R}^n$, the observation space is $\mathbb{R}^m$ and the signal space is $\mathbb{R}^{n'}$. The independent Wiener processes W_t' and W_t'' are n and m dimensional, respectively.

For models (2.2) and (2.3) we make the following assumptions:

i) the function $u:\ \mathbb{R}_+\times\mathcal{C}_{[0,\infty]}(\mathbb{R}^m) \mapsto \mathbb{R}^p$ is $\mathcal{R}_+\otimes\mathcal{B}_\infty^m$-measurable and $\{\mathcal{B}_t^m\}_{t\geq 0}$-adapted.

ii) for any $t \in \mathbb{R}_+$ the functions $f(t,\cdot,\cdot)$, $h(t,\cdot,\cdot)$, $F(\cdot)$ have bounded components;

iii) there exist an increasing function $L(t)$ and a measure $\mu(dt)$ on $\mathbb{R}_+$, with $\int_0^t \mu(ds) < \infty$, $t > 0$, so that (here $\|\cdot\|$ is the Euclidean norm) ,

$$\begin{aligned} \big\|f(t,x',u(t,y')) - f(t,x'',u(t,y''))\big\| \\ \leq L(t)\Big(\|x'-x''\| + \int_0^t \|y_s'-y_s''\|\mu(ds)\Big), \\ \big\|h(t,x') - h(t,x''))\big\| \leq L(t)\Big(\|x'-x''\|\Big), \\ \big\|b(t,x') - b(t,x''))\big\| \leq L(t)\Big(\|x'-x''\|\Big); \end{aligned} \tag{2.4}$$

iv) matrices $\mathfrak{D}_t := BB^*(t)$ and $\mathfrak{d}_t := bb^*(t,x)$ (* is the transposition symbol) are uniformly nonsingular respectively in $\mathbb{R}_+$ and in $\mathbb{R}_+\times\mathbb{R}^n$, with bounded inverse;

v) (*Open loop filter.*) There exists a function $\Psi : \mathbb{R}_+\times\mathcal{C}_{[0,\infty]}(\mathbb{R}^m)\times\mathcal{C}_{[0,\infty]}(\mathbb{R}^m) \mapsto \mathbb{R}^{n'}$, $\mathcal{R}_+\otimes\mathcal{B}_\infty^m\otimes\mathcal{B}_\infty^m$-measurable and $\{\mathcal{B}_t^m\otimes\mathcal{B}_t^m\}_{t\geq 0}$-adapted, such that

$$\Psi_t(Y^\phi;\phi) = E\big(S_t^\phi/\sigma_t(Y^\phi)\big),\ P\text{-a.s.},\ \forall t \in \mathbb{R}_+. \tag{2.5}$$

vi) (*Closed loop filter.*) There exists a function $\Phi : \mathbb{R}_+\times\mathcal{C}_{[0,\infty]}(\mathbb{R}^m) \mapsto \mathbb{R}^{n'}$, $\mathcal{R}_+\otimes\mathcal{B}_\infty^m$-measurable and $\{\mathcal{B}_t^m\}_{t\geq 0}$-adapted, such that

$$\Phi_t(Y) = E\big(S_t/\sigma_t(Y)\big),\ P\text{-a.s.},\ \forall t \in \mathbb{R}_+. \tag{2.6}$$

Note that thanks to the assumption of $\{\mathcal{B}_t\}_{t\geq 0}$-measurability of the function u, the term $u(t,Y)$ performs a causal mapping of the observation process into the input. Moreover, note that condition (**iii**) guarantees existence and uniqueness of strong solutions of (2.2) and of (2.3), adapted to $\mathcal{F}^t$.

3. Main Result

The main result of this paper is given by the following theorem, that answers to the question (1.7).

Theorem 3.1. *Consider the open-loop and the closed-loop nonlinear stochastic models* (2.2) *and* (2.3). *Let the assumptions (**i-vi**) be satisfied. Then the functions* Ψ_t *and* Φ_t *defined in* (2.5) *and* (2.6) *are such that*

$$\Psi_t(Y;Y) = \Phi_t(Y), \;\; P\text{-}a.s., \;\; \forall t \in \mathbb{R}_+. \tag{3.1}$$

Before to give the proof of this theorem we need to state some preliminary results. Throughout the paper we will use the following notation

$$\big\|h(t,x)\big\|^2_{\mathfrak{D}_t^{-1}} = h^*(t,x)\left(BB^*\right)^{-1}(t)h(t,x). \tag{3.2}$$

Moreover, for a given process ξ taking values on $\mathcal{C}_{[0,\infty)}(\mathbb{R}^q)$, we shall denote with μ^t_ξ the measure induced by the process on $\big\{\mathcal{C}_{[0,\infty)}(\mathbb{R}^q), \mathcal{B}^q_t\big\}$.

Let $F_t : \mathcal{C}_{[0,\infty]}(\mathbb{R}^n) \mapsto \mathbb{R}^{n'}$ be the bounded function defined by the equality $F_t(z) = F(z(t))$, where F is the function defining the signals for systems (2.2), (2.3).

Lemma 3.1. **(Kallianpur-Striebel formula for $\Psi_t(Y^\phi;\phi)$)** *For any* $t \geq 0$ *the open-loop filter can be written as*

$$\Psi_t(Y^\phi;\phi) = \frac{\int_{\mathcal{C}_{[0,\infty)}(\mathbb{R}^n)} F_t(z)\Lambda_t(z,Y^\phi)\mu^t_{X^\phi}(dz)}{\int_{\mathcal{C}_{[0,\infty)}(\mathbb{R}^n)} \Lambda_t(z,Y^\phi)\mu^t_{X^\phi}(dz)}, \tag{3.3}$$

where

$$\Lambda_t(X^\phi,Y^\phi) = \exp\left(\int_0^t h^*(s,X^\phi_s)\mathfrak{D}_s^{-1}dY^\phi_s - \frac{1}{2}\int_0^t \big\|h(s,X^\phi_s)\big\|^2_{\mathfrak{D}_s^{-1}}ds\right). \tag{3.4}$$

Proof. Consider the process

$$d\zeta_t = B(t)dW''_t, \quad \zeta_0 = 0. \tag{3.5}$$

By Theorem 7.20 and comments from Subsection 7.6.4 after this theorem in [4], for any $t \geq 0$ the distributions of processes $(X^\phi_s, Y^\phi_s)_{s\leq t}$, $(X^\phi_s, \zeta_s)_{s\leq t}$ are equivalent. Moreover, it is

$$\begin{aligned} \Lambda_t(z,y) &= \frac{d\mu^t_{X^\phi,Y^\phi}}{d\mu^t_{X^\phi,\zeta}}(z,y) \\ (z,y) &\in \mathcal{C}_{[0,\infty)}(\mathbb{R}^n)\times\mathcal{C}_{[0,\infty)}(\mathbb{R}^m) \end{aligned} \tag{3.6}$$

From this, the following equation is obtained

$$\begin{aligned} &\int_{\mathcal{C}_{[0,\infty)}(\mathbb{R}^n)} \Lambda_t(z,y)\mu^t_{X^\phi}(dz) = \frac{d\mu^t_{Y^\phi}}{d\mu^t_\zeta}(y) \\ &y \in \mathcal{C}_{[0,\infty)}(\mathbb{R}^m). \end{aligned} \tag{3.7}$$

From Theorem 7.23 in [4], and its multi-dimensional analog Lemma 2.3 in [10], it is

$$\Psi_t(Y^\phi;\phi) = \int_{\mathcal{C}_{[0,\infty)}(\mathbb{R}^n)} F_t(z)\rho_t(z,Y^\phi)\mu^t_{X^\phi}(dz), \tag{3.8}$$

with

$$\rho_t(z,y) = \frac{d\mu^t_{X^\phi,Y^\phi}}{d\mu^t_{X^\phi,\zeta}}(z,y) \Big/ \frac{d\mu^t_{Y^\phi}}{d\mu^t_\zeta}(y). \tag{3.9}$$

From the expressions of the Radon-Nikodym derivatives it follows

$$\begin{aligned} \rho_t(z,y) &= \frac{\Lambda_t(z,y)}{\int_{\mathcal{C}_{[0,\infty)}(\mathbb{R}^n)} \Lambda_t(z,y)\mu^t_{X^\phi}(dz)} \\ (z,y) &\in \mathcal{C}_{[0,\infty)}(\mathbb{R}^n) \times \mathcal{C}_{[0,\infty)}(\mathbb{R}^m), \end{aligned} \tag{3.10}$$

and from this equation (3.3) follows. □

From assumptions **(i–iii)**, there exists a $Q : \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{C}_{[0,\infty)}(\mathbb{R}^n) \times \mathcal{C}_{[0,\infty)}(\mathbb{R}^m) \mapsto \mathbb{R}^n$, measurable and $\{\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}^n_t \otimes \mathcal{B}^m_t\}$-adapted, such that the closed-loop state process can be written as:

$$X_t = Q_t(\mathcal{X}, W', Y). \tag{3.11}$$

$Q(\mathcal{X}, W', Y)$ will denote the process $\{Q_s(\mathcal{X}, W', Y), s \in \mathbb{R}_+\}$.

Lemma 3.2. **(Kallianpur-Striebel formula for $\Phi_t(Y)$).** *For any $t \geq 0$ the closed-loop filter can be written as*

$$\Phi_t(Y) = \frac{\int_{\mathbb{R}^n \times \mathcal{C}_{[0,t]}(\mathbb{R}^n)} F\big(Q(x,w,Y)\big)\mathfrak{A}_t(x,w,Y)\mu_{\mathcal{X}}(dx)\mu^t_{W'}(dw)}{\int_{\mathbb{R}^n \times \mathcal{C}_{[0,t]}(\mathbb{R}^n)} \mathfrak{A}_t(x,w,Y)\mu_{\mathcal{X}}(dx)\mu^t_{W'}(dw)}, \tag{3.12}$$

where

$$\begin{aligned} \mathfrak{A}_t(x,w,Y) = \exp\Big\{ &\int_0^t h^*\big(s, Q_s(x,w,Y)\big)\mathfrak{D}_s^{-1} dY_s \\ &- \frac{1}{2}\int_0^t \big\|h\big(s, Q_s(x,w,Y)\big)\big\|^2_{\mathfrak{D}_s^{-1}} ds\Big\}. \end{aligned} \tag{3.13}$$

Proof. As in the proof of Lemma 3.1, apply Theorem 7.20 of [4] to the processes $\big(\mathcal{X}, W', Y\big)$ and $\big(\mathcal{X}, W', \zeta\big)$. One has that for all $t \geq 0$ the distributions $\mu^t_{\mathcal{X},W',Y}$ and $\mu^t_{\mathcal{X},W',\zeta}$ are equivalent, and the Radon-Nikodym derivative is

$$\begin{aligned} &\frac{d\mu^t_{\mathcal{X},W',Y}}{d\mu^t_{\mathcal{X},W',\zeta}}(x,w,y) = \mathfrak{A}_t(x,w,y) \\ &(x,w,y) \in \mathbb{R}^n \times \mathcal{C}_{[0,\infty)}(\mathbb{R}^n) \times \mathcal{C}_{[0,\infty)}(\mathbb{R}^m) \end{aligned} \tag{3.14}$$

where $\mathfrak{A}_t$ is defined in (3.13). The following equation can be verified

$$\begin{aligned} &\int_{\mathcal{C}_{[0,\infty)}(\mathbb{R}^n)} \mathfrak{A}_t(x,w,y)\mu_{\mathcal{X}}(dx)\mu^t_{W'}(dw) = \frac{d\mu^t_Y}{d\mu^t_\zeta}(y), \\ &y \in \mathcal{C}_{[0,\infty)}(\mathbb{R}^m). \end{aligned} \tag{3.15}$$

Using Theorem 7.23 in [4], and its multi-dimensional analog Lemma 2.3 in [10], it is

$$\Phi_t(Y) = \int_{\mathbb{R}^n \times \mathcal{C}_{[0,\infty)}(\mathbb{R}^n)} F\big(Q_s(x,w,Y)\big)\gamma_s(x,w,Y)\mu_{\mathcal{X}}(dx)\mu^t_{W'}(dw), \tag{3.16}$$

with

$$\gamma_t(x,w,y) = \frac{d\mu^t_{\mathcal{X},W',Y}}{d\mu^t_{\mathcal{X},,W',\zeta}}(x,w,y) \Big/ \frac{d\mu^t_Y}{d\mu^t_\zeta}(y). \tag{3.17}$$

From these one has

$$\gamma_t(x,w,y) = \frac{\mathfrak{A}_t(x,w,y)}{\int_{\mathbb{R}^n\times\mathcal{C}_{[0,\infty)}(\mathbb{R}^n)} \mathfrak{A}_t(x,w,y)\mu_{\mathcal{X}}(dx)\mu^t_{W'}(dw)} \tag{3.18}$$
$$(x,w,y) \in \mathbb{R}^n \times \mathcal{C}_{[0,\infty)}(\mathbb{R}^n) \times \mathcal{C}_{[0,\infty)}(\mathbb{R}^m).$$

Equation (3.12) follows. □

Let us define the process Υ as follows

$$\Upsilon_t(X,Y) = \exp\Big\{ \int_0^t h^*(s,X_s)\mathfrak{D}_s^{-1} dY_s - \frac{1}{2}\int_0^t \big\|h\big(s,X_s\big)\big\|^2_{\mathfrak{D}_s^{-1}} ds\Big\}. \tag{3.19}$$

Note that, from (3.11) and (3.13) it is

$$\Upsilon_t\big(Q(\mathcal{X},W',Y),Y\big) = \mathfrak{A}_t(\mathcal{X},W',Y). \tag{3.20}$$

In the following we need to rewrite the expressions of the open and closed loop filters, given by (3.3) and (3.12), respectively, in a more convenient form related to the underlying probability space. To this purpose, we introduce a copy $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{P})$ of the original probability space, so that all processes defined on it are independent copies of the original ones. We introduce also random variables and processes on the product probability space $\big(\Omega\times\widetilde{\Omega},\mathcal{F}\otimes\widetilde{\mathcal{F}},P\times\widetilde{P}\big)$.

Let $Z(\omega,\tilde{\omega})$ be a random variable defined on the product space. Let us define the operator $\widetilde{E}$ as follows

$$\widetilde{E}(Z)(\omega) = \int_{\widetilde{\Omega}} Z(\omega,\tilde{\omega})P(d\tilde{\omega}). \tag{3.21}$$

For a given process ξ defined on the original space, we shall denote by $\tilde{\xi}$ a process defined on the product space as $\tilde{\xi}(\omega,\tilde{\omega}) = \xi(\tilde{\omega})$. Whenever it does not cause confusion, we shall use the same symbol ξ to denote both the original process and its extension to the product space: $\xi(\omega,\tilde{\omega}) = \xi(\omega)$.

On the product space it is possible to define the process $\widetilde{X}^Y$ as follows

$$\widetilde{X}^Y_t = Q_t(\widetilde{\mathcal{X}},\widetilde{W}',Y) \tag{3.22}$$

With these positions, recalling also the definition of Υ given in (3.19), we can rewrite the expression (3.3) and (3.12) as follows:

$$\Psi_t(Y;\phi) = \frac{\widetilde{E}\big\{F_t(\widetilde{X}^\phi)\Lambda_t(\widetilde{X}^\phi,Y^\phi)\big\}}{\widetilde{E}\big\{\Lambda_t(\widetilde{X}^\phi,Y^\phi)\big\}}, \quad P\text{-a.s.} \tag{3.23}$$

$$\Phi_t(Y) = \frac{\widetilde{E}\big\{F_t(\widetilde{X}^Y)\Upsilon_t(\widetilde{X}^Y,Y)\big\}}{\widetilde{E}\big\{\Upsilon_t(\widetilde{X}^Y,Y)\big\}}, \quad P\text{-a.s.} \tag{3.24}$$

Now we are in a position to give the proof of Theorem 3.1.

Proof of Theorem 3.1. From expressions (3.23) and (3.24), Theorem 3.1 is proved as soon as it is shown that

$$\Lambda_t(\widetilde{X}^\phi, Y^\phi)\big|_{\phi=Y} = \Upsilon_t(\widetilde{X}^Y, Y), \quad P \times \widetilde{P}\text{-a.s.} \tag{3.25}$$

From definitions (3.4) and (3.19) we have

$$\Lambda_t(\widetilde{X}^\phi, Y^\phi) = \exp\left(\int_0^t h^*(s, \widetilde{X}_s^\phi)\mathfrak{D}_s^{-1} dY_s^\phi - \frac{1}{2}\int_0^t \big\|h(s, \widetilde{X}_s^\phi)\big\|^2_{\mathfrak{D}_s^{-1}} ds\right), \tag{3.26}$$

$$\Upsilon_t(\widetilde{X}^Y, Y) = \exp\left(\int_0^t h^*(s, \widetilde{X}_s^Y)\mathfrak{D}_s^{-1} dY_s - \frac{1}{2}\int_0^t \big\|h\big(s, \widetilde{X}_s^Y)\big)\big\|^2_{\mathfrak{D}_s^{-1}} ds\right). \tag{3.27}$$

Let us consider some $\sigma_t(\widetilde{X}^\phi, Y^\phi)$-measurable functions H_t and L_t such that

$$\begin{aligned} H_t(\widetilde{X}^\phi, Y^\phi) &= \int_0^t h^*(s, \widetilde{X}_s^\phi)\mathfrak{D}_s^{-1} dY_s^\phi, \\ L_t(\widetilde{X}^\phi, Y^\phi) &= \int_0^t \big\|h(s, \widetilde{X}_s^\phi)\big\|^2_{\mathfrak{D}_s^{-1}} ds, \end{aligned} \tag{3.28}$$

and $\sigma_t(\widetilde{X}^Y, Y)$-measurable functions H'_t and L'_t such that

$$\begin{aligned} H'_t(\widetilde{X}^Y, Y) &= \int_0^t h^*(s, \widetilde{X}_s^Y)\mathfrak{D}_s^{-1} dY_s, \\ L'_t(\widetilde{X}^Y, Y) &= \int_0^t \big\|h\big(s, \widetilde{X}_s^Y)\big)\big\|^2_{\mathfrak{D}_s^{-1}} ds. \end{aligned} \tag{3.29}$$

We can use Lemma 4.10 of [4] to prove that

$$H_t(\widetilde{X}^Y, Y) = H'_t(\widetilde{X}^Y, Y), \qquad L_t(\widetilde{X}^Y, Y) = L'_t(\widetilde{X}^Y, Y). \tag{3.30}$$

by showing that the measures $\mu^t_{\widetilde{X}^\phi, Y^\phi}$ and $\mu^t_{\widetilde{X}^Y, Y}$ are equivalent.

As a matter of fact, Theorem 7.19 of [4] guarantees the equivalence of the measures induced by the processes $(\widetilde{X}^\phi, X^\phi, Y^\phi)$ and $(\widetilde{X}^Y, X, Y)$, that are defined on $\big(\Omega \times \widetilde{\Omega}, \mathcal{F} \otimes \widetilde{\mathcal{F}}, P \times \widetilde{P}\big)$ as follows

$$\begin{aligned} d\widetilde{X}_t^\phi &= f(t, \widetilde{X}_t^\phi, u(t,\phi))dt + b(t, \widetilde{X}_t^\phi) d\widetilde{W}'_t, \quad \widetilde{X}_0^\phi = \widetilde{\mathcal{X}}, \\ dX_t^\phi &= f(t, X_t^\phi, u(t,\phi))dt + b(t, X_t^\phi) dW'_t, \quad X_0^\phi = \mathcal{X}, \\ dY_t^\phi &= h(t, X_t^\phi)dt + B(t)dW''_t, \quad Y_0^\phi = 0, \end{aligned} \tag{3.31}$$

$$\begin{aligned} d\widetilde{X}_t^Y &= f(t, \widetilde{X}_t^Y, u(t,Y))dt + b(t, \widetilde{X}_t^Y) d\widetilde{W}'_t, \quad \widetilde{X}_0^Y = \widetilde{\mathcal{X}}, \\ dX_t &= f(t, X_t, u(t,Y))dt + b(t, X_t) dW'_t, \quad X_0 = \mathcal{X}, \\ dY_t &= h(t, X_t)dt + B(t)dW''_t, \quad Y_0 = 0. \end{aligned} \tag{3.32}$$

Since $\mu^t_{\widetilde{X}^\phi, Y^\phi}$ and $\mu^t_{\widetilde{X}^Y, Y}$ are marginal distributions of $\mu^t_{\widetilde{X}^\phi, X^\phi, Y^\phi}$ and $\mu^t_{\widetilde{X}^Y, X, Y}$, respectively, their equivalence follows as well. □

4. Conclusions

The contribution of this paper is Theorem 3.1, that represents a general property of stochastic systems that can be informally expressed in these words: whenever the optimal filter is available for a given system in open-loop, the *same filter* will work optimally on the same system in closed-loop. Among the implications of Theorem 3.1 there is the result that any system that admits a finite dimensional filter in open-loop, admits a finite dimensional filter also in closed-loop.

Acknowledgments : The authors gratefully acknowledge the careful review of an anonymous referee, who also pointed out a mistake in the original version. Due to his comments the paper has been significantly improved.

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[1]Istituto di Analisi dei Sistemi e Informatica del CNR
Viale Manzoni 30, 00185 Roma, Italy. fax: +39 06 7716461, phone: +39 06 7716410
e-mail: carravetta@iasi.rm.cnr.it

[2]Dipartimento di Ingegneria Elettrica, Università dell'Aquila
67100 Monteluco di Roio, L'Aquila, Italy.
fax: +39 0862 434403, phone: +39 0862 434422, e-mail: {germani,manes}@ing.univaq.it

[3]Dept. Electrical Engineering-Systems, Tel Aviv University
69978 Tel Aviv, Israel. e-mail: liptser@eng.tau.ac.il