MODERATE DEVIATIONS FOR DYNAMIC MODEL GOVERNED BY STATIONARY PROCESS

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Abstract. We give an example of the moderate deviations for the family $(X^n_t)_{t\ge 0}, n\ge 1$ with $\dot{X}^n_t = a(X^n_t) + b(X^n_t)\frac{n}{\sqrt{n^{k+1}}}\eta_{tn}$, where η_t is a stationary process obeying the Wold – decomposition: $\eta_t = \int_{-\infty}^t h(t-s)dN_s$ with respect to a process N_t with homogeneous independent square integrable increments and $1/2 < k < 1$.

1. Introduction and Main Result

1. It is well known (see [1]) that a family $X^n = (X^n_t)_{t\ge 0}, n>0$ of diffusion processes:

$$dX^n_t = a(X^n_t)dt + \frac{1}{\sqrt{n^k}} b(X^n_t)dW_t \tag{1.1}$$

subject to fixed initial point X_0, where W_t is a Wiener process and $\kappa > 0$, obeys the large deviation principle (LDP) in the space of continuous functions $C_{[0,\infty)}$, supplied by the locally uniform metric. This LDP is characterized by the rate of speed $\frac{1}{n^\kappa}$ and the rate function

$$I(\varphi) = \begin{cases} \frac{1}{2}\int_0^\infty \left(\frac{\dot{\varphi}_t - a(\varphi_t)}{b(\varphi_t)}\right)^2 dt, & \varphi_0 = X_0, \quad d\varphi \ll dt \\ \infty, & \text{otherwise,} \end{cases} \tag{1.2}$$

where notation "$\varphi_0 = X_0, \quad d\varphi \ll dt$" is used for the relation $\varphi_t = X_0 + \int_0^t \dot{\varphi}_s ds, t\ge 0$ ($\dot{\varphi}_t$ is the Radon-Nikodym derivative of φ_t).

Consider the same problem from more "realistic" model in which the Wiener process is replaced by its approximation

$$W^n_t = \frac{1}{\sqrt{n}}\int_0^{tn} \eta_s ds, \tag{1.3}$$

where $(\eta_t)_{t\in R}$ is a stationary ergodic process, $E\eta_0 = 0$, satisfying some weak dependence conditions. Thus, X^n_t is defined now by an ordinary differential equation

$$\dot{X}^n_t = a(X^n_t) + b(X^n_t)\frac{n}{\sqrt{n^{k+1}}}\eta_{tn}. \tag{1.4}$$

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Here, η_t is the stationary process obeying the Wold decomposition

$$\eta_t = \int_{-\infty}^{t} h(t-s)dN_s, \tag{1.5}$$

where $h(t)$ is a deterministic function, $\int_0^\infty h^2(t)dt < \infty$ and $N = (N_t)_{t\in R}$ is right-continuous having limits to the left homogeneous process with zero mean independent increments, $N_0 = 0$. The continuous (Gaussian) component of N^c_t has diffusion parameter σ^2, the jumps $\Delta N_t = N_t - N_{t-}$ of N_t is characterized by the Levy measure $K(dz)dt$ with the kernel $K(dz)$ such that $\int_{R\setminus\{0\}} z^2K(dz) < \infty$. Consequently, for any $s,t \in R$, $E|N_t - N_s|^2 = |t-s|\left(\sigma^2 + \int_{R\setminus\{0\}} z^2K(dz\right)$ and so the integral in (1.5) is understood as the Itô stochastic integral.

For $\kappa = 1$, the LDP with the of speed $\frac{1}{n}$ and rate function different (!) from given in (1.2) was obtained in [3]. Here, under the same assumptions as for $\kappa = 1$ have been used in [3], we show that the LDP for the case

$$\frac{1}{2} < \kappa < 1 \tag{1.6}$$

holds as well, with the rate of speed $\frac{1}{n^\kappa}$ and, in a contrast to the case $\kappa = 1$, with the rate function of given above diffusion type (1.2) with an artificial diffusion parameter $\nu^2\Sigma^2b^2(x)$, where

$$\begin{aligned} \nu^2 &= \sigma^2 + \int_{R\setminus\{0\}} z^2K(dz) \\ \Sigma^2 &= \left(\int_0^\infty h(t)dt\right)^2. \end{aligned} \tag{1.7}$$

As in [3], we assume that weak dependence conditions (WD) and Cramer's condition (CR), expressed in the terms of function $h(t)$ and the kernel $K(dz)$, are satisfied:

$$\begin{aligned} \int_0^\infty |h(t)|dt < \infty \\ \int_0^\infty \sqrt{\int_t^\infty h^2(s)dsdt} < \infty \\ \int_0^\infty \left(\int_t^\infty h(s)ds\right)^2 dt < \infty; \end{aligned} \tag{WD}$$

(CR) for any $\lambda \in R$

$$\begin{aligned} \int_{R\setminus\{0\}} (e^{\lambda z} - 1 - \lambda z)K(dz) < \infty \\ \int_0^\infty \int_{R\setminus\{0\}} (e^{h(t)\lambda z} - 1 - h(t)\lambda z)K(dz)dt < \infty. \end{aligned}$$

Instead of "LDP" we use the abbreviation "MDP" (moderate deviation principle) to emphasize the fact that the rate function is of a diffusion type (see (1.2)) and is independent of κ

The MDP for $X^n = \frac{1}{\sqrt{n^{\kappa+1}}} \sum_{j=1}^n \xi_j, n \geq 1$ of zero mean i.i.d. random variables $(\xi_j)_{j\geq 1}$, satisfying the Cramer condition is well known. The straightforward generalization of this example for a functional type MDP can be found in Puhalskii [5] (Example 7.2) for a family $X^n_t = \frac{1}{\sqrt{n^{\kappa+1}}} \sum_{j=1}^{[nt]} \xi_j$, where $\xi_k, k \geq 1$ is the same family of i.i.d. random variables and $[z]$ is the integer part of z. Other types of examples for families of random processes, obeying the MDP, are known from Bayer and Freidlin [6], Wentzell [7], Mogulskii [8], and Zajic [9].[1] Assuming $\nu^2 > 0$, $b^2 > 0$, and $\Sigma^2 > 0$, introduce

$$J(\varphi) = \begin{cases} \frac{1}{2} \int_0^\infty \frac{[\dot{\varphi}_t - a(\varphi_t)]^2}{[\nu \Sigma b(\varphi_t)]^2} dt, & \varphi_0 = X_0,\ d\varphi \ll dt \\ \infty, & \text{otherwise.} \end{cases} \tag{1.8}$$

The main result of the paper is formulated in

Theorem 1.1. *Assume the function $a(x)$ and $b(x)$ are Lipschitz continuous and the function $|b(x)|$ is uniformly positive and bounded. Assume $1/2 < \kappa < 1$, (WD) and (CR). Then the family (X^n_t), defined in (1.4), obeys the MDCP in the metric space (C, ρ) (ρ is the locally uniform metric) with the rate of speed $\frac{1}{n^\kappa}$ and the rate function $J(\varphi)$.*

The statement of this Theorem is derived, by using the contraction principle of Varadhan [4] (see also Freidlin's continuous mapping method [2]), from the theorem below, which has an independent interest (compare [8]).

Let N_t be the random process involved in Wold's decomposition (1.5).

Theorem 1.2. *Assume the first part of (CR). Then the family $\left(\frac{1}{\sqrt{n^{\kappa+1}}} N_{tn}\right)_{t\geq 0}$ obeys the MDP in the metric space (D, d) ($D = D_{[0,\infty)}$ is the Skorokhod space and d is the Skorokhod-Lindvall metric) with the rate of speed $\frac{1}{n^\kappa}$ and the rate function*

$$J_0(\varphi) = \begin{cases} \frac{1}{2} \int_0^\infty \frac{\dot{\varphi}_t^2}{\nu^2} dt, & \varphi_0 = 0,\ d\varphi \ll dt \\ \infty, & \textit{otherwise.} \end{cases} \tag{1.9}$$

The paper is organized as follows. In Section 2, we give the proof of Theorem 1.2. In Section 3, the particular case of equation (1.4) with $a(x) \equiv 0$ and $b(x) \equiv 1$ is considered. The proof of the main result (Theorem 1.1) is given in Section 4.

2. Proof of Theorem 1.2

2.1. Method. The process $(N_t)_{t\geq 0}$ has paths in the Skorokhod space D. Therefore, we formulate the MDP in the metric space (D, d) with the Skorokhod-Lindvall metric d (see [10] or, e.g., Ch.6 in [11]). The proof of the theorem is in a context of Theorem 1.2 from [12] which, in turn, uses the Pukhalskii theorem [13]. So, we have to check only two sets of conditions.

[1] The paper was written in 1995. Now there are a lot of results in MDP: for processes with independent increments Chen [15], Ledoux [16]; for the depended case Wu [17], Dembo [18] for martingales with bounded jumps, Dembo and Zajic [19] for functional empirical processes, Dembo and Zeitouni [20] for iterates of expanding maps, Liptser and Spokoiny [21] for smooth processes, Puhalskii [22] for queues in critical loading, Guillin [23]for empirical processes of Markov chains, etc.

(C.1) **C**-exponential tightness: for each $T > 0, \gamma > 0$

$$\lim_{c\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n^\kappa} \log P\left(\sup_{t\le T} \frac{1}{\sqrt{n^{\kappa+1}}} |N_{tn}| > c\right) = -\infty$$

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \sup \frac{1}{n^{\kappa+1}} \log P\left(\sup_{0\le t\le\delta} \frac{1}{\sqrt{n^{\kappa+1}}} |(N_{tn+\tau} - N_\tau| > \gamma\right) = -\infty,$$

where '*sup*' is taken over all stopping times $\tau \le T_n$ with respect to the filtration $(\mathcal{F}^N_{tn})_{t\ge 0}$ (satisfying the usual conditions) generated by the process N_t.

(C.2) **C**-local MDP: for each $T > 0$ and $\varphi \in C$

$$\begin{aligned}\hat{I}_T(\varphi) &= -\overline{\lim_{\delta\to 0}}\, \overline{\lim_{n\to\infty}} \frac{1}{n^\kappa} \log P\left(\sup_{t\le T}\left|\frac{1}{\sqrt{n^{\kappa+1}}} N_{tn} - \varphi\right| \le \delta\right) \\ &= -\underline{\lim_{\delta\to 0}}\, \underline{\lim_{n\to\infty}} \frac{1}{n^\kappa} \log P\left(\sup_{t\le T}\left|\frac{1}{\sqrt{n^{\kappa+1}}} N_{tn} - \varphi\right| \le \delta\right)\end{aligned}$$

with $\hat{I}_T(\varphi)$.

If (C.1) and (C.2) are fulfilled, then the MDP in (D, d) holds with rate function

$$\tilde{J}(\varphi) = \begin{cases} \sup_{T>0} \hat{I}_T(\varphi), & \varphi \in C \\ \infty, & \varphi \in D \setminus C. \end{cases}$$

2.2. Proof of (C.1). We apply the Chernoff, Jensen, Cauchy-Schwarz, and Doob inequalities:

$$\begin{aligned}
&P\left(\sup_{t\le T}\left|\frac{1}{\sqrt{n^{\kappa+1}}} N_{t/n}\right| > c\right) \\
&\le e^{-cn^\kappa} E \exp\left\{\sup_{t\le T} \sqrt{\frac{n^\kappa}{n}} |N_{tn}|\right\} \ \text{(Chernoff)} \\
&= e^{-cn^\kappa} E \sup_{t\le T} \exp\left\{\sqrt{\frac{n^\kappa}{n}} |N_{tn}|\right\} \\
&= e^{-cn^\kappa} E \sup_{t\le T} \exp\left\{\left|E\left(\sqrt{\frac{n^\kappa}{n}} |N_{Tn}| \mathcal{F}^N_{tn}\right)\right|\right\} \\
&\le e^{-cn^\kappa} E \sup_{t\le T} E\left(\exp\left\{\sqrt{\frac{n^\kappa}{n}} |N_{Tn}|\right\} | \mathcal{F}^N_{tn}\right) \ \text{(Jensen)} \\
&\le e^{-cn^\kappa} \sqrt{E\left[\sup_{t\le T} E\left(\exp\left\{\sqrt{\frac{n^\kappa}{n}} |N_{Tn}|\right\} \mid \mathcal{F}^N_{tn}\right)\right]^2} \ \text{(Cauchy-Schwarz)} \\
&\le 2e^{-cn^\kappa} \sqrt{E \exp\left\{2\sqrt{\frac{n^\kappa}{n}} |N_{Tn}|\right\}} \ \text{(Doob)} \\
&\le 2e^{-cn^\kappa} \sqrt{E\left\{e^{2\sqrt{\frac{n^\kappa}{n}} N_{Tn}} + e^{-2\sqrt{\frac{n^\kappa}{n}} N_{Tn}}\right\}}.
\end{aligned} \tag{2.1}$$

Since (N_t) is the process with independent increments we get (see Ch. II in [14])

$$Ee^{\pm 2N_{Tn}} = \exp\left\{Tn\left[4\sigma^2\frac{n^\kappa}{n} + \int_{R\setminus\{0\}}\left(e^{\pm 2\sqrt{\frac{n^\kappa}{n}}z} - 1 \mp 2\sqrt{\frac{n^\kappa}{n}}z\right)K(dz)\right]\right\}. \tag{2.2}$$

Therefore,

$$\begin{aligned}\frac{1}{n^\kappa}\log P\left(\sup_{t\le T}\left|\frac{N_{tn}}{\sqrt{n^{\kappa+1}}}\right| > c\right) &\le \frac{\log 2}{n^\kappa} - c + 2T\sigma^2\\ &+ \frac{T}{2}\int_{R\setminus\{0\}}\frac{e^{2\sqrt{\frac{n^\kappa}{n}}z} - 1 + 2\sqrt{\frac{n^\kappa}{n}}z}{\frac{n^\kappa}{n}}K(dz)\\ &+ \frac{T}{2}\int_{R\setminus\{0\}}\frac{e^{-2\sqrt{\frac{n^\kappa}{n}}z} - 1 - 2\sqrt{\frac{n^\kappa}{n}}z}{\frac{n^\kappa}{n}}K(dz).\end{aligned}$$

Noticing that $\frac{n^\kappa}{n} \to 0, n \to \infty$, we find that the right-hand side of the last inequality converges, as $n \to \infty$, to $-c + 2T\left(\sigma^2 + 2\int_{R\setminus\{0\}} z^2K(dz)\right)$ and so the first part of (C.1) holds.

To check the second part of (C.1), we use the strong Markovian property which holds for the process $(N_{nt}, \mathcal{F}^N_{nt})$ and the fact that N_t is the process with independent homogeneous increments which implies that the distributions of $(N_{tn+\tau} - N_\tau)_{t\ge 0}$ and $(N_{tn})_{t\ge 0}$ coincide. We apply the same method which has been used for the verification of the first part in (C.1). A distinction consists only in the part of the Chernoff inequality which is used with positive parameter λ :

$$P\left(\sup_{t\le\delta}\left|\frac{1}{\sqrt{n^{\kappa+1}}}N_{tn}\right| > \gamma\right) \le e^{-\gamma\lambda n^\kappa}E\exp\left\{\sup_{t\le\delta}\sqrt{\frac{n^\kappa}{n}}\lambda|N_{tn}|\right\}.$$

Therefore, in the same way as (2.1) and (2.2) have been obtained, we find

$$\overline{\lim_n}\frac{1}{n^\kappa}\log P\left(\sup_{0\le t\le\delta}\left|\frac{1}{\sqrt{n^{\kappa+1}}}(N_{tn+\tau} - N_\tau)\right| > \gamma\right) \le -\gamma\lambda + 2\delta\lambda^2\nu^2$$

and whereas any positive λ is available, taking $\lambda = \frac{\gamma}{\delta\nu^2}$, we arrive at an upper bound

$$\overline{\lim_n}\frac{1}{n^\kappa}\log P\left(\sup_{0\le t\le\delta}\left|\frac{1}{\sqrt{n^{\kappa+1}}}(N_{(t+\tau)n} - N_{\tau n})\right| > \gamma\right) \le -\frac{\gamma^2}{2\delta\nu^2}\ (\to -\infty, \delta \to 0),$$

that is the second part of (C.1) is valid.

2.3. Upper bound for C-local NDP.

Show that for any continuous function $\varphi(t)$ from $C_{[0,T]}$

$$\begin{aligned}&\overline{\lim_{\delta\to 0}}\ \overline{\lim_{n\to\infty}}\frac{1}{n^\kappa}\log P\left(\sup_{t\le T}\left|\frac{N_{tn}}{\sqrt{n^{\kappa+1}}} - \varphi_t\right| \le \delta\right)\\ &\le \begin{cases}-\frac{1}{2}\int_0^T\frac{\dot\varphi_t^2}{\nu^2}dt, & \varphi_0 = 0, d\varphi \ll dt\\ -\infty, & \text{otherwise.}\end{cases}\end{aligned} \tag{2.3}$$

Let $\lambda(t)$ be a piecewise constant function of the form $\lambda(t)=\sum_i \lambda_i I_{]t_{i-1},t_i]}(t)$. Put $\Phi(z)=e^z-1-z$ and introduce the cumulant function

$$G(\lambda)=\frac{\lambda^2\sigma^2}{2}+\int_{R\setminus\{0\}}\Phi(\lambda z)K(dz)$$

and

$$Z(\lambda)=\exp\left\{\int_0^T\lambda(t)\frac{dN_{tn}}{\sqrt{n^{\kappa+1}}}-\frac{1}{n}\int_0^T G\left(\frac{\lambda(t)}{\sqrt{n^{\kappa+1}}}\right)dt\right\}. \tag{2.4}$$

The straightforward calculation (see also Ch.II in [14]) shows that $EZ(\lambda)=1$ and $EZ(n^\kappa\lambda)=1$ as well. The latter equality provides an obvious inequality

$$1\ge EI\left(\sup_{t\le T}\left|\frac{N_{tn}}{\sqrt{n^{\kappa+1}}}-\varphi_t\right|\le\delta\right)Z(n^\kappa\lambda) \tag{2.5}$$

which can be sharpen by using a lower bound for $Z(n^\kappa\lambda)$ on the set $\Big\{\sup_{t\le T}\Big|\frac{N_{tn}}{\sqrt{n^{\kappa+1}}}-$ $-\varphi_t|\le\delta\Big\}$. Here and in the sequel, we use a notation $\int_0^T\lambda(t)d\varphi_t=\sum_i\lambda_i[\varphi_{T\wedge t_i}-\varphi_{T\wedge t_{i-1}}])$. Write

$$\begin{aligned}Z(n^\kappa\lambda)\ge&\left[\int_0^T n^\kappa\lambda(t)d\varphi_t-n\int_0^T G(n^\kappa\lambda(t))dt\right]\\&\times\exp\left\{-n^\kappa\left|\int_0^T\lambda(t)d\left[\frac{N_{tn}}{\sqrt{n^{\kappa+1}}}-\varphi_t\right]\right|\right\}.\end{aligned} \tag{2.6}$$

Taking into account now that on the above-mentioned set the following upper bound holds: $\left|\int_0^T\lambda(t)d\left[\frac{N_{tn}}{\sqrt{n^{\kappa+1}}}-\varphi_t\right]\right|\le$ const.$n^{-\kappa}\delta$, where const. depends only on $\lambda(t)$, we arrive at

$$\begin{aligned}\frac{1}{n^\kappa}\log P\left(\sup_{t\le T}\left|\frac{N_{tn}}{\sqrt{n^{\kappa+1}}}-\varphi_t\right|\le\delta\right)\le&-\text{const.}\delta\\&-\left[\int_0^T\lambda(t)d\varphi_t-\int_0^T\frac{G\left(\sqrt{\frac{n^\kappa}{n}}\lambda(t)\right)}{\frac{n^\kappa}{n}}dt\right].\end{aligned}$$

It is clear that the right-hand side of this inequality converges, as the limit $\lim\limits_{\delta\to0}\lim\limits_{n\to\infty}$ is taken, to a quadratic type functional with respect to $\lambda(t)$: $L(\lambda)=\int_0^T\Big[\lambda(t)d\varphi_t-\frac{\lambda^2(t)\nu^2}{2}dt\Big]$. Therefore, if $\varphi_0=0$, the upper bound is $-\sup L(\lambda)$, where '*sup*' is taken over all piecewise constant functions $\lambda(t)$. If φ_t is not absolutely continuous, then by Lemma 6 in [12], this upper bound is equal $-\infty$. Otherwise ($d\varphi_t=\dot\varphi_t dt$) due to the same lemma, it is given by the formula $-\int_0^T\sup_\lambda\left[\lambda\dot\varphi_t-\frac{\lambda^2\nu^2}{2}\right]dt$. Thus, for $\varphi_0=0$ the announced upper bound takes place. Obviously for $\varphi_0\ne0$, the upper bound is equal $-\infty$.

2.4. Lower bound for C-local MDP. We show that the following lower bound holds: for each absolutely continuous function $\varphi(t)$ from $C_{[0,T]}$ such that $\varphi_0 = 0$, $\int_0^T \dot{\varphi}_t^2 dt < \infty$ and each $\delta > 0$

$$\varliminf_{n\to\infty} \frac{1}{n^\kappa} \log P\left(\sup_{t\le T}\left|\frac{N_{tn}}{\sqrt{n^{\kappa+1}}} - \varphi_t\right| \le \delta\right) \ge -\frac{1}{2}\int_0^T \frac{\dot{\varphi}_t^2}{\nu^2} dt. \tag{2.7}$$

Let us define a bounded function

$$\lambda^n(t) = I(T \ge t)\frac{n^\kappa \dot{\varphi}_t}{\nu^2} \tag{2.8}$$

and a random process

$$\Lambda_t^n = \exp\left\{\int_0^t \frac{\lambda^n(s)}{\sqrt{n^{\kappa+1}}} dN_{sn} - n\int_0^t G\left(\frac{\lambda^n(s)}{\sqrt{n^\kappa+1}}\right) ds\right\}, \tag{2.9}$$

where $G(\lambda)$ is the defined above cumulant function. Since $G(0) = 0$, we have $\Lambda_t^n = \Lambda_{t\wedge T}^n$ and so $\Lambda_\infty^n = \Lambda_T^n$. It is easy to check that the process Λ_t^n is a square integrable martingale with respect to the filtration $(\mathcal{F}_{tn}^N)$ and so $E\Lambda_\infty^n = 1$. We introduce a new probability measure P^n with $dP^n = \Lambda_\infty^n dP$, where P is the original one. Noticing that $\Lambda_\infty^n > 0, P$-a.s., one can conclude that $dP = (\Lambda_\infty^n)^{-1} dP^n$. Denoting by $\mathcal{A}^{n\delta} = \left\{\sup_{t\le T}\left|\frac{N_{tn}}{\sqrt{n^{\kappa+1}}} - \varphi_t\right| \le \delta\right\}$ and $\mathcal{B}^{n\beta} = \left\{\left|\int_0^T \lambda^n(t)\left[\frac{dN_{sn}}{\sqrt{n^{\kappa+1}}} - \dot{\varphi}_t dt\right]\right| \le \beta n^\kappa\right\}$, we arrive at

$$\begin{aligned}
&P(\mathcal{A}^{n\delta})\\
&\ge P(\mathcal{A}^{n\delta} \cap \mathcal{B}^{n\beta})\\
&= \int_{\{\mathcal{A}^{n\delta}\cap\mathcal{B}^{n\beta}\}} (\Lambda_\infty^n)^{-1} dP^n\\
&= \int_{\{\mathcal{A}^{n\delta}\cap\mathcal{B}^{n\beta}\}} \exp\left\{-\int_0^T \lambda^n(t) d\left[\frac{N_{tn}}{\sqrt{n^{\kappa+1}}} - \dot{\varphi}_t dt\right]\right.\\
&\qquad \left. - \int_0^T \left[\lambda^n(t)\dot{\varphi}_t - nG\left(\frac{\lambda^n(t)}{\sqrt{n^\kappa+1}}\right)\right] dt\right\} dP^n\\
&\ge \exp\left\{-\beta n^\kappa - \int_0^T \left[\lambda^n(t)\dot{\varphi}_t - nG\left(\frac{\lambda^n(t)}{\sqrt{n^\kappa+1}}\right)\right] dt\right\} P^n(\mathcal{A}^{n\delta} \cap \mathcal{B}^{n\beta}).
\end{aligned} \tag{2.10}$$

Also note that

$$\begin{aligned}
\frac{1}{n^\kappa}\int_0^T \left[\lambda^n(t)\dot{\varphi}_t - nG\left(\frac{\lambda^n(t)}{\sqrt{n^\kappa+1}}\right)\right] dt &= \frac{1}{\nu^2}\int_0^T \left[\dot{\varphi}_t^2 - \nu^2 \frac{G\left(\sqrt{\frac{n^\kappa}{n}}\frac{\dot{\varphi}_t}{\nu^2}\right)}{\frac{n^\kappa}{n}}\right] dt\\
&\to \frac{\dot{\varphi}_t^2}{2\nu^2},\ n \to \infty.
\end{aligned} \tag{2.11}$$

(2.10) and (2.11) imply a lower bound: for each $\delta > 0$ and $\beta > 0$

$$\varliminf_{n\to\infty} \frac{1}{n^\kappa} \log P(\mathcal{A}^{n\delta}) \ge -\beta - \frac{1}{2}\int_0^T \frac{\dot{\varphi}_t^2}{\nu^2} dt - \varlimsup_{n\to\infty} \frac{1}{n^\kappa} \log P^n(\mathcal{A}^{n\delta} \cap \mathcal{B}^{n\beta}). \tag{2.12}$$

To get now the desired statement, it suffices to show that for each $\delta > 0$ and $\beta > 0$, $\lim_{n\to\infty} P^n(\mathcal{A}^{n\delta} \cap \mathcal{B}^{n\beta}) = 1$ or, what is equivalent,

$$\begin{aligned}\lim_{n\to\infty} P^n(\Omega \setminus \mathcal{A}^{n\delta}) = 0 \\ \lim_{n\to\infty} P^n(\Omega \setminus \mathcal{B}^{n\beta}) = 0.\end{aligned} \tag{2.13}$$

Since both sets $\mathcal{A}^{n\delta}$ and $\mathcal{B}^{n\beta}$ are defined via the random process N_{tn}, the proof of (2.13) requires a characterization of this process with respect to the measure P^n.

The process N_{tn} is the square integrable martingale with respect to $(\mathcal{F}^N_{tn}, P)$, so that with respect to P^n the process N_{tn} is a semimartingale $N_{tn} = A^n_t + M^n_t$ with the predictable drift A^n_t and the square integrable martingale M^n_t (see Ch 4, §5 in [11]). In additional, the drift A^n_t, Levy's measure $K^n(ds, dz)$, and predictable quadratic variation $\langle M^n\rangle_t$ are defined as:

$$\begin{aligned} A^n_t &= \int_0^t (\Lambda^n_{s-})^{-1} d\langle N_{\cdot n}, \Lambda^n\rangle_s \\ K^n(dt, dz) &= ne^{\frac{\lambda^n(t)}{\sqrt{n^{\kappa+1}}}} K(dz)dt \\ \langle M^n\rangle_t &= n\left[\sigma^2 t + \int_0^t \int_{R\setminus\{0\}} z^2 K^n(ds, dz)\right]. \end{aligned}$$

These relations (with respect to P^n) follow from Theorems 4.5.1 and 4.5.2 in [11] and from the fact that jumps of N_{tn} and M^n_t coincide. Below, we give more detailed descriptions for all above-mentioned objects. By the Itô formula we find

$$\begin{aligned} \Lambda^n_t = &\int_0^t \Lambda^n_{s-} \frac{\lambda^n(s)}{\sqrt{n^{\kappa+1}}} dN_{sn} \\ &- n\int_0^t \Lambda^n_s \int_{R\setminus\{0\}} \Phi\left(\frac{z\lambda^n(s)}{\sqrt{n^{\kappa+1}}}\right) K(dz)ds \\ &+ \sum_{s\le t} \Lambda^n_{s-}\Phi\left(\frac{\lambda^n(s)\Delta N_{sn}}{\sqrt{n^{\kappa+1}}}\right) \end{aligned}$$

and so the mutual quadratic covariation $[\Lambda^n, N_{\cdot n}]_t$ is defined as:

$$\begin{aligned} [\Lambda^n, N_{\cdot n}]_t = &\int_0^t \Lambda^n_s \frac{n\lambda^n(s)\sigma^2}{\sqrt{n^{\kappa+1}}} ds \\ &+ \sum_{s\le t} \Lambda^n_{s-}\Delta N_{sn}\left[\exp\left\{\frac{\lambda^n(s)\Delta N_{sn}}{\sqrt{n^{\kappa+1}}}\right\} - 1\right]. \end{aligned}$$

Hence, the mutual predictable quadratic covariation $\langle \Lambda^n, N_{\cdot n}\rangle_t$, being the compensator of $[\Lambda^n, N_{\dot n}]_t$, is given by the formula

$$\begin{aligned} \langle \Lambda^n, N_{\cdot n}\rangle_t = &\int_0^t \Lambda^n_s \frac{n\lambda^n(s)\sigma^2}{\sqrt{n^{\kappa+1}}} ds \\ &+ \int_0^t \Lambda^n_s \int_{R\setminus\{0\}} nz\left[\exp\left\{\frac{\lambda^n(s)z}{\sqrt{n^{\kappa+1}}}\right\} - 1\right] K(dz)ds. \end{aligned}$$

Thereby
(2.14)
$$\begin{aligned}A_t^n &= \sqrt{n^{\kappa+1}}\frac{\sigma^2}{\nu^2}\varphi_t + \int_0^t\int_{R\setminus\{0\}} nz\left[\exp\left\{\sqrt{\frac{n^\kappa}{n}}\frac{1}{\nu^2}\dot\varphi_t z\right\}-1\right]K(dz)ds\\ &= \sqrt{n^{\kappa+1}}\varphi_t + \int_0^t\int_{R\setminus\{0\}} nz\left[\exp\left\{\sqrt{\frac{n^\kappa}{n}}\frac{1}{\nu^2}\dot\varphi_t z\right\}-\sqrt{\frac{n^\kappa}{n}}\dot\varphi_t\frac{1}{\nu^2}z-1\right]K(dz)ds.\end{aligned}$$

With $B_t^n = \frac{1}{\sqrt{n^{\kappa+1}}}\int_0^t\int_{R\setminus\{0\}} nz\left[\exp\left\{\sqrt{\frac{n^\kappa}{n}}\frac{1}{\nu^2}\dot\varphi_t z\right\}-\sqrt{\frac{n^\kappa}{n}}\dot\varphi_t\frac{1}{\nu^2}z-1\right]K(dz)ds$ we arrive at the following decomposition (with respect to the measure P^n):

$$\frac{N_{tn}}{\sqrt{n^{\kappa+1}}} = \varphi_t + B_t^n + \frac{M_t^n}{\sqrt{n^{\kappa+1}}}. \tag{2.15}$$

The deterministic function B_t^n has a bounded variation

$$\operatorname*{Var}_t(B^n) = t\sqrt{\frac{n^\kappa}{n}}\int_{R\setminus\{0\}}|z|\frac{\left[\exp\left\{\sqrt{\frac{n^\kappa}{n}}\frac{1}{\nu^2}\dot\varphi_t z\right\}-\sqrt{\frac{n^\kappa}{n}}\dot\varphi_t\frac{1}{\nu^2}z-1\right]}{\frac{n^\kappa}{n}}K(dz)ds$$

and moreover

$$\lim_n \operatorname*{Var}_T(B^n) = 0. \tag{2.16}$$

On the other hand, $\frac{M_t^n}{\sqrt{n^{\kappa+1}}}$ has the predictable quadratic variation (for $t\le T$):

$$\begin{aligned}\left\langle\frac{M^n}{\sqrt{n^{\kappa+1}}}\right\rangle_t &= \frac{\langle M^n\rangle_t}{n^{\kappa+1}}\\ &= \frac{1}{n^{\kappa+1}}\left[t\sigma^2 + n\int_0^t\int_{R\setminus\{0\}} z^2\exp\left\{\frac{\lambda^n(s)z}{\sqrt{n^{\kappa+1}}}\right\}K(dz)ds\right]\\ &= \frac{1}{n^\kappa}\left[t\sigma^2 + \int_0^t\int_{R\setminus\{0\}} z^2\exp\left\{\sqrt{\frac{n^\kappa}{n}}\dot\varphi_s z\right\}K(dz)ds\right].\end{aligned}$$

Therefore $\lim_n\left\langle\frac{M^n}{\sqrt{n^{\kappa+1}}}\right\rangle_T = 0$ and due to Problem 1.9.2 in [11] we get

$$P^n\text{-}\lim_n \frac{\sup_{t\le T}|M_t^n|}{\sqrt{n^{\kappa+1}}} = 0. \tag{2.17}$$

Show now that (2.13) holds. In fact, taking into account (2.15), we get

$$\begin{aligned}\Omega\setminus\mathcal{A}^{n\delta} &= \left\{\sup_{t\le T}\left|\frac{N_{tn}}{\sqrt{n^{\kappa+1}}}-\varphi_t\right|>\delta\right\}\\ &\subseteq \left\{\operatorname*{Var}_T(B^n)\ge\frac{\delta}{2}\right\}\cup\left\{\frac{\sup_{t\le T}|M_t^n|}{\sqrt{n^{\kappa+1}}}>\frac{\delta}{2}\right\}\end{aligned}$$

and so the first part of (2.13) holds by virtue of proved above (2.16) and (2.17). The validity for the second part of (2.13) is proved in the same way whereas

$$\begin{aligned}\Omega \setminus \mathcal{B}^{n\beta} &= \left\{\left|\int_0^T \frac{\dot\varphi_t}{\nu^2} d\left[\frac{N_{tn}}{n^{\kappa+1}} - \varphi_t\right]\right| > \beta\right\} \\ &\subseteq \left\{\frac{N}{\nu^2}\operatorname*{Var}_T(B^n) > \frac{\beta}{2}\right\} \cup \left\{\left|\int_0^T \frac{\dot\varphi_t}{\nu^2}\frac{dM^n_t}{\sqrt{n^{\kappa+1}}}\right| > \frac{\beta}{2}\right\}.\end{aligned}$$

By virtue of (2.16) the first set on the right-hand side of this inclusion is empty, if n is large enough, while the probability of the second set goes to zero as $n \to \infty$ by virtue of Problem 1.9.2 in [11] and the fact that $\int_0^T \frac{\dot\varphi_t^2}{\nu^4}\frac{1}{n^{\kappa+1}} d\langle M^n\rangle_t \le \frac{C^2}{\nu^4}\frac{1}{n^{\kappa+1}}\langle M^n\rangle_T \to 0, n \to \infty$. Thus, for uniformly bounded $\dot\varphi_t$ and any positive δ we get

$$\underline{\lim}_n \frac{1}{n^\kappa}\log P(\mathcal{A}^{n\delta}) \ge -\frac{1}{2}\int_0^T \frac{\dot\varphi_t^2}{\nu^2}dt. \tag{2.18}$$

Evidently, (2.18) implies the desired lower bound (2.7). Also (2.18) is used to establish (2.7) for a general case, i.e. for absolutely continuous function φ_t with $\int_0^t \dot\varphi_t^2 dt < \infty$. If $\dot\varphi_t$ is not a bounded function, with fixed $C > 0$ set $\varphi_t^C = \int_0^t \dot\varphi_s I(|\dot\varphi_s| \le C)dt$. Since $|\dot\varphi_t^C| \le C$, (2.18) is applicable and so

$$\begin{aligned}&\underline{\lim}_n \frac{1}{n^\kappa}\log P\left(\sup_{t\le T}\left|\frac{N_{tn}}{\sqrt{n^{\kappa+1}}} - \varphi_t^C\right| \le \delta\right) \\ &\ge -\frac{1}{2}\int_0^T \frac{\dot\varphi_t^2}{\nu^2} I(|\dot\varphi_t| \le C)dt \\ &\ge -\frac{1}{2}\int_0^T \frac{\dot\varphi_t^2}{\nu^2}dt.\end{aligned} \tag{2.19}$$

On the other hand, since the total variation for the function $\varphi_t - \varphi_t^C$ on the interval $[0,T]$ is given by the formula $\|\varphi - \varphi^C\|_T = \int_0^T |\dot\varphi_t| I(|\dot\varphi_t| > C)dt$ we have $\lim_{C\to\infty}\|\varphi - \varphi^C\|_T = 0$. Consequently, taking C so large that $\|\varphi - \varphi^C\|_T \le \delta/2$, we find

$$\begin{aligned}&\underline{\lim}_n \frac{1}{n^\kappa}\log P\left(\sup_{t\le T}\left|\frac{N_{tn}}{\sqrt{n^{\kappa+1}}} - \varphi_t\right| \le \delta\right) \\ &\ge \underline{\lim}_n \frac{1}{n^\kappa}\log P\left(\sup_{t\le T}\left|\frac{N_{tn}}{\sqrt{n^{\kappa+1}}} - \varphi_t^C\right| \le \delta - \|\varphi - \varphi^C\|\right) \\ &\ge -\frac{1}{2}\int_0^T \frac{\dot\varphi_t^2}{\nu^2}dt\end{aligned}$$

which in turn implies (2.7).

2.5. Proof of (C.2). If the upper bound in the local MDP is finite, it coincides with the lower one, otherwise the upper bound, being equal to $-\infty$, coincides with lower bound as well. Consequently for any continuous function φ_t from $D, \tilde J(\varphi) = J(\varphi)$. Hence, $\tilde J(\varphi) \equiv J(\varphi)$ and the statement of the theorem holds.

3. Particular case of Theorem 1.1

In this section, we formulate the MDP for the family $Y_t^n = \frac{1}{\sqrt{n^{\kappa+1}}}\int_0^{tn}\eta_s ds$.

Lemma 3.1. *Assume (WD) and (CR). Then the family (Y_t^n) obeys the MDP in the metric space (C,ρ) (ρ is the locally uniform metric) with the rate of speed $\frac{1}{n^\kappa}$ and the rate function*

$$J(\varphi) = \begin{cases} \frac{1}{2}\int_0^\infty \frac{\dot\varphi_t^2}{\nu^2\Sigma^2}dt, & \varphi_0 = 0, \quad d\varphi \ll dt \\ \infty, & \textit{otherwise.} \end{cases}$$

Proof. Taking into account (1.5) and well-known property of the Itô integral, we find that $E(\eta_t|\eta_s, s\le 0) = \int_{-\infty}^0 h(t-s)dN_s$ a.s. On the other hand, by (WD) $\int_0^\infty \sqrt{\int_t^\infty h^2(s)ds}dt$ $< \infty$ and therefore

$$\int_0^\infty \sqrt{E(E(\eta_t|\eta_s, s\le 0))^2}dt < \infty.$$

The latter property and Lemma 9.2.1 in [11] guarantee the existence of a semimartingale V_t adapted to the filtration $\mathcal{F}_t^\eta$ (satisfying the general conditions) generated by the process η_t such that $V_t = V_0 + \int_0^t \eta_s ds - M_t$, where (M_t) is a square integrable martingale and V_t is in the restricted sense stationary process, where (see [3] and for more details Ch.9, Sec.2, Ex.2 in [11])

$$\begin{aligned} M_t &= \Sigma N_t \\ V_t &= \int_t^\infty \left(\int_{-\infty}^t h(u-s)dN_s\right)du. \end{aligned}$$

Therefore

$$Y_t^n = \frac{\Sigma}{\sqrt{n^{\kappa+1}}}N_{tn} + \frac{1}{\sqrt{n^{\kappa+1}}}[V_{tn} - V_0]. \tag{3.1}$$

By Theorem 1.2 the family $\frac{\Sigma}{\sqrt{n^{\kappa+1}}}N_{tn}$ obeys the MDP in the metric space (D,d) with the rate of speed $\frac{1}{n^\kappa}$ and the rate function given in the formulation of Lemma 3.1. Moreover, since this family is **C**-exponential tight, it obeys the MDP in the metric space (C,ρ).

Thus, to establish the statement of the lemma, it suffices to show that the process $\frac{1}{\sqrt{n^{\kappa+1}}}[V_{tn} - V_0]$ is locally **C**-exponential negligible, that is for each $T > 0$ and $\gamma > 0$, $\lim_{n\to\infty}\frac{1}{n^\kappa}\log P\left(\frac{1}{\sqrt{n^{\kappa+1}}}\sup_{t\le T}|V_{tn} - V_0| > \gamma\right) = -\infty$. We derive that property from

$$\lim_{n\to\infty}\frac{1}{n^\kappa}\log P\left(\frac{\sup_{t\le T}|V_{tn}|}{\sqrt{n^{\kappa+1}}} > \gamma\right) = -\infty. \tag{3.2}$$

To verify (3.2), set $\alpha_k = \sup_{T(k-1)\le t\le Tk}|V_t|, k\ge 1$. Noticing that $(\alpha_k)_{k\ge 1}$ forms in the restricted sense stationary sequence and using the Chernoff inequality, we find

$$
\begin{aligned}
P\left(\frac{1}{\sqrt{n^{\kappa+1}}}\sup_{t\le T}|V_{tn}|\ge\gamma\right) &= P\left(\sup_{t\le Tn}|V_t|\ge\gamma\sqrt{n^{\kappa+1}}\right)\\
&\le P\left(\max_{k\le\sqrt{n^{\kappa+1}}}\alpha_k\ge\gamma\sqrt{n^{\kappa+1}}\right)\\
&\le \sum_{k\le\sqrt{n^{\kappa+1}}}P(\alpha_k\ge\gamma\sqrt{n^{\kappa+1}})\\
&\le \sqrt{n^{\kappa+1}}P(\alpha_1\ge\gamma\sqrt{n^{\kappa+1}})\\
&\le \sqrt{n^{\kappa+1}}e^{-\gamma\sqrt{n^{\kappa+1}}}Ee^{\alpha_1}\ \text{(Chernoff)}\\
&= \sqrt{n^{k+1}}e^{-\gamma\sqrt{n^{\kappa+1}}}Ee^{\sup_{t\le T}|V_t|}.
\end{aligned}
$$

Hence

$$
\frac{1}{\sqrt{n^{\kappa+1}}}\log P\left(\frac{1}{\sqrt{n^{\kappa+1}}}\sup_{t\le T}|V_{tn}|\ge\gamma\right)\le-\frac{1}{n^\kappa}\log\sqrt{n^{\kappa+1}}-\gamma\sqrt{\frac{n}{n^\kappa}}+\frac{1}{n^\kappa}\log Ee^{\sup_{t\le T}|V_t|}
$$

and so (3.2) holds provided that

$$
E\exp\left\{\sup_{t\le T}|V_t|\right\}<\infty. \tag{3.3}
$$

Thus, only (3.3) remains to be verified. Assume for a moment that there exists a positive martingale L_t (with respect to the filtration $\mathcal{F}^\eta_t$) such that

$$
|V_t|\le L_t\quad\text{and}\quad Ee^{2L_T}<\infty \tag{3.4}
$$

and show that (3.3) is implied by (3.4). In fact, using the Jensen, Cauchy-Schwarz, and Doob inequalities we get

$$
\begin{aligned}
E\exp\{\sup_{t\le T}|V_t|\} &\le E\exp\{\sup_{t\le T}L_t\}\\
&= E\sup_{t\le T}\exp\{L_t\}\\
&= E\sup_{t\le T}\exp\{E(L_T|\mathcal{F}^\eta_t)\}\\
&\le E\sup_{t\le T}E\left(\exp\{L_T\}|\mathcal{F}^\eta_t\right)\ \text{(Jensen)}\\
&\le \sqrt{E\sup_{t\le T}E\left(\exp\{L_T\}|\mathcal{F}^\eta_t\right)^2}\ \text{(Cauchy-Schwarz)}\\
&\le 2\sqrt{E\exp\{2L_T\}}\ \text{(Doob)}\\
&<\infty.
\end{aligned}
$$

To find the required martingale L_t, we use an obvious representation $V_t=V_T-\int_0^t\eta_s ds+(N_T-N_t)\Sigma, t\le T$, the martingale property of N_t, and the fact that V_t is measurable with respect to $\mathcal{F}^\eta_t$. So, we get $V_t=E(V_T-\int_0^t\eta_s ds|\mathcal{F}^\eta_t)$ and therefore $|V_t|\le$

$E(|V_T| + \int_0^T |\eta_s| ds \mathcal{F}_t^\eta|)$, that is $L_t = E(|V_T| + \int_0^T |\eta_s| ds|\mathcal{F}_t^\eta)$ can be taken as the required martingale, provided that for each $\lambda \geq 2$

$$E \exp\left\{\lambda \left[|V_T| + \int_0^T |\eta_s| ds\right]\right\} < \infty. \tag{3.5}$$

We examine (3.5) by using the Cauchy-Schwarz and Jensen inequalities:

$$\begin{aligned}
&\left(E \exp\left\{\lambda \left[|V_T| + \int_0^T |\eta_s| ds\right]\right\}\right)^2 \\
&\leq E \exp\{2\lambda |V_T|\} E \exp\left\{2\lambda \int_0^T |\eta_s| ds\right\} \text{ (Cauchy-Schwarz)} \\
&= E \exp\{2\lambda |V_0|\} E \exp\left\{\frac{1}{T} \int_0^T 2T\lambda |\eta_s| ds\right\} \\
&\leq E \exp\{2\lambda |V_0|\} E \frac{1}{T} \int_0^T \exp\{2T\lambda |\eta_s|\} ds \text{ (Jensen)} \\
&= E \exp\{2\lambda |V_0|\} E \exp\{2T\lambda |\eta_0|\}.
\end{aligned}$$

Thus (3.5) holds if for any positive λ we have $E \exp\{\lambda |V_0|\} < \infty$ and $E \exp\{\lambda |\eta_0|\} < \infty$. The direct proof for the validity of these inequalities would be difficult. It is more convenient, instead of V_0 and η_0, to take $V_0' = \int_0^\infty \int_s^\infty h(u) du dN_s$ and $\eta_0' = \int_0^\infty h(s) dN_s$ having the same distributions as V_0 and η_0 respectively. Moreover, taking into account an obvious inequality $e^{|x|} \leq e^x + e^{-x}$, for any $\lambda \in R$ it suffices to verify that

$$\begin{aligned}
&E \exp\{\lambda V_0'\} < \infty \\
&E \exp\{\lambda \eta_0'\} < \infty.
\end{aligned}$$

To this end, with $H(t)$ being any of functions $h(t)$ or $\int_t^\infty h(u) du$, we get

$$\left.\begin{matrix} V_0' \\ \eta_0' \end{matrix}\right\} = \int_0^\infty H(t) dN_t \ (:= Z_\infty).$$

Introduce a square integrable martingale $Z_t = \int_0^t H(s) dN_s$ having Z_∞ as the limit point as $t \to \infty$. By the Fatou lemma

$$E \exp\{\lambda Z_\infty\} \leq \varliminf_{t \to \infty} E \exp\{\lambda Z_t\}.$$

To estimate the right-hand side of the latter inequality, we use the fact that (Z_t) is the process with independent increments and find (see Ch. II in [14])

$$E e^{\lambda Z_t} = \exp\left\{\frac{\lambda^2 \sigma^2}{2} \int_0^t H^2(s) ds + \int_0^t \int_{R^0} (e^{\lambda H(s) z} - 1 - \lambda H(s) z) K(dz) ds\right\}.$$

The right-hand side of this inequality increases, as $t \to \infty$, to

$$C(\lambda) = \exp\left\{\frac{\lambda^2 \sigma^2}{2} \int_0^\infty H^2(s) ds + \int_0^\infty \int_{R \setminus \{0\}} (e^{\lambda H(s) z} - 1 - \lambda H(s) z) K(dz) ds\right\},$$

where $C(\lambda)$ is finite for any $\lambda \in R$. In fact, for $H(t) \equiv h(t)$ it holds due to (CR) while for $H(t) \equiv \int_t^\infty h(s)ds$ it holds due to (WD) and (CR) whereas, with $\lambda_0 = |\lambda| \sup_{t\ge 0} |H(t)|$,

$$\begin{aligned} e^{\lambda H(t) z} - 1 - \lambda H(t) z &\le \lambda^2 H^2(t) e^{\lambda_0 |z|} \\ &\le \lambda^2 H^2(t)(e^{\lambda_0 z} + e^{-\lambda_0 z}) \end{aligned}$$

and in turn $\int_{R\setminus\{0\}} z^2 e^{\lambda_0 z} K(dz) < \infty$.

4. Proof of Theorem 1.1

We apply the contraction principle of Varadhan [4] (the continuous mapping method of Freidlin [2]). To this end, we have to find a continuous mapping in the locally uniform metric for

$$\left(\frac{1}{\sqrt{n^{\kappa+1}}} \int_0^{tn} \eta_s ds \right)_{t\ge 0} \Longrightarrow (X_t^n)_{t\ge 0}.$$

If Y_t is absolutely continuous function from C, then the required mapping is defined, due to (1.4), by a differential equation ($\dot{Y}_t$ is the Radon-Nykodim derivative)

$$\dot{X}_t = a(X_t) + b(X_t)\dot{Y}_t \tag{4.1}$$

subject to the initial condition X_0. We show that the mapping $(Y_t)_{t\ge 0} \Longrightarrow (X_t)_{t\ge 0}$ can be extended for any function Y_t from C.

Lemma 4.1. *Let functions $a(x)$ and $b(x)$ be Lipschitz continuous and there exist constants c and C such that $0 < c \le |b(x)| \le C$. Then for fixed X_0 there exists continuous in the metric ρ mapping $(Y_t)_{t\ge 0} \Longrightarrow (X_t)_{t\ge 0}, (X_t), (Y_t) \in C$ such that for any absolutely continuous function Y_t this mapping is defined by the differential equation (4.1)).*

Despite this lemma is proved in [3] (Lemma 5.1), for reader convenience we give here a sketch of its proof.

Sketch of the proof. Put $F(x) = \int_0^x \frac{dy}{b(y)}$. The function $F(x)$ is continuously differentiable having the inverse $F^{-1}(x)$ which is continuously differentiable as well. Moreover, $F(x)$ and $F^{-1}(x)$ satisfy the linear growth condition. Put $g(x) = \frac{a(F^{-1}(x))}{b(F^{-1}(x))}$. For X_t being a solution of differential equation (4.1) corresponding absolutely continuous function Y_t, put $\theta_t = F(X_t)$. It is easy to check that θ_t is a unique solution of a differential equation $\dot{\theta}_t = g(\theta_t) + \dot{Y}_t$ subject to the initial condition $\theta_0 = F(X_0)$. Therefore we arrive at the desired mapping

$$\begin{aligned} X_t &= F^{-1}(\theta_t) \\ \theta_t &= F(X_0) + \int_0^t g(\theta_s)ds + [Y_t - Y_0]. \end{aligned} \tag{4.2}$$

Thus it remains to check only that this mapping is continuous in the metric ρ or, what is equivalent, for any $T > 0$

$$\begin{cases} \lim_m \sup_{t\le T} |Y_t^{(m)} - Y_t| = 0 \\ (Y_t^{(m)}), (Y_t) \in C, m \ge 1 \end{cases} \Longrightarrow \begin{cases} \lim_m \sup_{t\le T} |X_t^{(m)} - X_t| = 0 \\ (X_t^{(m)}), (X_t) \in C, m \ge 1, \end{cases} \tag{4.3}$$

where $X_t^{(m)}, m \geq 1$ and X_t are solutions of (4.2) corresponding to $Y_t^{(m)}, m \geq 1$ and Y_t respectively. It is clear that (4.3) is valid due to the local Lipschitz condition for the function $g(x)$.

Due to the contraction principle the family X_t^n obeys the MDP in the metric space (C, ρ) with the rate of speed $\frac{1}{n^\kappa}$ and the rate function $J(\varphi)$, defined for absolutely continuous function φ_t with $\varphi_0 = X_0$, due to (4.1), as

$$J(\varphi) = \frac{1}{2} \int_0^\infty \frac{[\dot{\varphi}_t - a(\varphi_t)]^2}{[\nu \Sigma b(\varphi_t)]^2} dt$$

and otherwise is equal to ∞.

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