EXAMPLE OF LARGE DEVIATIONS FOR STATIONARY PROCESS

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Abstract. We give an example of the large deviations for the family $(X_t^\varepsilon)_{t\ge 0}, \varepsilon > 0$ with $\dot X_t^\varepsilon = a(X_t^\varepsilon) + b(X_t^\varepsilon)\eta_{t/\varepsilon}$, where η_t is stationary process obeying the Wold decomposition: $\eta_t = \int_{-\infty}^t h(t-s)dN_s$ w.r.t. homogeneous process N_t with independent square integrable increments.

Key words: Large deviation, Skorokhod space, Wold decomposition

1. Introduction and main result

1. It is well known (see [3]) that a family $X^\varepsilon = (X_t^\varepsilon)_{t\ge 0}, \varepsilon > 0$ of diffusion processes:

$$dX_t^\varepsilon = a(X_t^\varepsilon)dt + \sqrt{\varepsilon} b(X_t^\varepsilon)dW_t \tag{1.1}$$

subject to fixed X_0, where W_t is a Wiener process and $a(x), b(x)$ are Lipschitz continuous and $b^2(x) > 0$, obeys the large deviation principle (l.d.p.) in the space of continuous function $\mathbf{C}_{[0,T]}$ and the corresponding rate function is given by the formula:

$$I_T(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \left(\frac{\dot\varphi_t - a(\varphi_t)}{b(\varphi_t)}\right)^2 dt, & \varphi_0 = X_0,\ d\varphi \ll dt \\ \infty, & \text{otherwise,} \end{cases} \tag{1.2}$$

where the notation "$\varphi_0 = X_0,\ d\varphi \ll dt$" is used for designating $\varphi_t = X_0 + \int_0^t \dot\varphi_s ds, t \le T$, and where $\dot\varphi_t$ is the Radon-Nikodym derivative of φ_t. Evidently the l.d.p. for (X_t^ε) also holds in the metric space $(\mathbf{C}, \rho)$ with $\mathbf{C} = \mathbf{C}_{[0,\infty)}$,

$$\rho(X, Y) = \sum_{n\ge 1} 2^{-n} \frac{\sup_{t\le n}|X_t - Y_t|}{1 + \sup_{t\le n}|X_t - Y_t|}$$

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and the rate function $I(\varphi) = \sup_T I_T(\varphi)$ (see, e.g. [9]).

In the contrast to (1.1), in this paper we formulate the l.d.p. for family of processes $(X^\varepsilon_t)_{t\geq 0}, \varepsilon > 0$ defined by an ordinary differential equation

$$\dot{X}^\varepsilon_t = a(X^\varepsilon_t) + b(X^\varepsilon_t)\eta_{t/\varepsilon}, \tag{1.3}$$

subject to fixed X_0, where $(\eta_t)_{t\in R}$ is the restricted sense stationary and ergodic process with $\mathbf{E}\eta_0 = 0$.

Since the Markovian property for (η_t) is not to be assumed the general approach consists in checking the l.d.p. for the occupation measures of (η_t) for, so called, "third level" of the Donsker and Varadhan theory [2] and applying the contraction principle of Varadhan [11] to get the l.d.p. for X^ε. To avoid an application of the "third level" we restrict ourselves by the consideration of (η_t) obeying the Wold decomposition[1]

$$\eta_t = \int_{-\infty}^t h(t-s)dN_s, \tag{1.4}$$

w.r.t. right continuous having limits from the left homogeneous process with independent increments $N = (N_t)_{t\in R}$ such that $N_0 = 0$ and for any $s, t \in R$ $\mathbf{E}|N_t - N_s| = k^2|t-s|$, $\mathbf{E}(N_t - N_s) = 0$ and $\int_R h^2(t)dt < \infty$ (so the integral in (1.4) is understood as Ito's stochastic integral).

For such process (η_t) a simple proof of the l.d.p. for X^ε is found, the explicit formula for the rate function takes place and what is more this model serves different applications.

2. Formally, letting

$$W^\varepsilon_t = 1/\sqrt{\varepsilon}\int_0^t \eta_{s/\varepsilon}ds, \tag{1.5}$$

(1.3) can be rewritten to the similar form as (1.1): $dX^\varepsilon_t = a(X^\varepsilon_t)dt + \sqrt{\varepsilon}b(X^\varepsilon_t)dW^\varepsilon_t$. But the l.d.p. even for $(\sqrt{\varepsilon}W^\varepsilon_t)$ would be difficult to obtain by the method from [3]. Nevertheless, it is possible by different method and by virtue of (1.4). Two reasons are for $(\sqrt{\varepsilon}W^\varepsilon_t)$ to obey the l.d.p.:

1. By virtue of the Birkhoff - Khinchin theorem for any fixed $T > 0$ ($\mathbf{P} - a.s.$) $\lim_{\varepsilon\to 0}\sup_{t\leq T}|\sqrt{\varepsilon}W^\varepsilon_t| = 0$.

2. Under weak dependence assumptions

$$\int_0^\infty |h(t)|dt < \infty$$

$$\int_0^\infty \sqrt{\int_t^\infty h^2(s)ds}dt < \infty. \tag{1.6}$$

the functional central limit theorem for (W^ε_t) holds, i.e. $\mathcal{L} - \lim_{\varepsilon\to 0}(W^\varepsilon_t) = \Sigma(W_t)$, where (W_t) is Wiener process, $\mathcal{L}$ means convergence in the distribution sense , and

$$\Sigma = \int_0^\infty h(t)dt \tag{1.7}$$

[1] for discrete time case the corresponding result can be found in [4]

(see [8, Ch.9,Sec.2,Ex. 2]).

Along with (1.6) assume that

$$\int_0^\infty \Big(\int_t^\infty h(s)ds\Big)^2 dt < \infty. \tag{1.8}$$

Also the Cramer type conditions for jumps ($\Delta N_t = N_t - \lim_{s\uparrow t} N_s$) of N_t are required for getting the l.d.p. of $\sqrt{\varepsilon}W^\varepsilon_{t/\varepsilon}$. As any process with square integrable and independent increments N is characterized by its continuous Gaussian component $N^c = (N^c_t)_{t\in R}$ with $\mathbf{E}(Y^c_t)^2 = \sigma^2|t|$ and the Levy measure $K(dx)dt, x\in R^0 = R\backslash\{0\}, t\in R$, $\int_{R^0} x^2K(dx) < \infty$. Taking into account that for any $\lambda \in R$

$$\mathbf{E}\sum_{t\le 1}\big(e^{\lambda\Delta N_t} - 1 - \lambda\Delta N_t\big) = \int_{R^0}\big(e^{\lambda x} - 1 - \lambda x\big)K(dx)$$
$$\mathbf{E}\sum_{t\ge 0}\big(e^{\lambda h(t)\Delta N_t} - 1 - \lambda h(t)\Delta N_t\big) = \int_0^\infty\int_{R^0}\big(e^{\lambda h(t)x} - 1 - \lambda h(t)x\big)K(dx)dt$$

assume

(A.1) for any $\lambda \in R$

$$\int_{R^0}\big(e^{\lambda x} - 1 - \lambda x\big)K(dx) < \infty;$$

(A.2) for any $\lambda \in R$

$$\int_0^\infty\int_{R^0}[e^{\lambda h(t)x} - 1 - \lambda h(t)x]K(dx)dt < \infty.$$

Under (A.1) the cumulant function is defined as:

$$G(\lambda) = \frac{\lambda^2\sigma^2}{2} + \int_{R^0}(e^{\lambda x} - 1 - \lambda x)K(dx),\ \lambda\in R. \tag{1.9}$$

Due to (A.1) $G(\lambda)$ is twice continuous differentiable with

$$G'(\lambda) = \lambda\sigma^2 + \int_{R^0} x(e^{\lambda x} - 1)K(dx)$$
$$G''(\lambda) = \sigma^2 + \int_{R^0} x^2 e^{\lambda x}K(dx). \tag{1.10}$$

and so it is nonnegative ($G(0) = 0$) and convex function. The Legendre-Fenchel transformation of $G(\lambda)$ is defined in the usual way (see [10]):

$$H_{\sigma^2}(y) = \sup_{\lambda\in R}\big[\lambda y - G(\lambda)\big],\ y\in R. \tag{1.11}$$

As $G(\lambda)$ is continuous sup in (1.11) may be taken over rational λ and so $H_{\sigma^2}(y)$ is measurable function in y. In the notation $H_{\sigma^2}(y)$ a dependence on σ^2 is emphasized. In particularly, property $H_{\sigma^2}(y)\uparrow H_0(y),\ \sigma^2\to 0$ is used in proving the l.d.p. below.

3. For any $\varphi \in \mathbf{C}$, letting $\frac{0}{0} = 0$, put

$$I(\varphi) = \begin{cases} \int_0^\infty H_{\sigma^2}(\frac{\dot{\varphi}_t}{\Sigma})dt, & \varphi_0 = 0, d\varphi \ll dt \\ \infty, & \text{otherwise}, \end{cases} \tag{1.12}$$

where "$\varphi_0 = 0,\ d\varphi \ll dt$ is used for designating $\varphi_t = \int_0^t \dot{\varphi}_s ds,\ t \ge 0$, and where $\dot{\varphi}_t$ is the Radon-Nikodym derivative of φ_t. Evidently, if $\Sigma = 0$ then

$$I(\varphi) = \begin{cases} 0, & \varphi_t \equiv 0 \\ \infty, & \text{otherwise}. \end{cases}$$

Now, we are in the position to formulate the l.d.p. for the family $(\sqrt{\varepsilon} W_t^\varepsilon)$.

Theorem 1.1. *1) Let assumptions (1.6), (1.8), and (A.1), (A.2) be fulfilled. Then the family $(\sqrt{\varepsilon} W_t^\varepsilon)$ obeys the l.d.p. in $(\mathbf{C}, \rho)$ with the rate function $I(\varphi)$ is given by (1.12).*

The main result is implied by Theorem 1.1 and the contraction principle of Varadhan [11].

Theorem 1.2. *Let functions $a(x)$ and $b(x)$ be Lipschitz continuous and there exist constants c and C such that*

$$0 < c \le |b(x)| \le C.$$

Then under assumptions of Theorem 1.1 the family (X_t^ε) defined by (1.3) obeys the l.d.p. in $(\mathbf{C}, \rho)$ with the rate function $J(\varphi), \varphi \in \mathbf{C}$ defined as:

$$J(\varphi) = \begin{cases} \int_0^\infty H_{\sigma^2}\big(\frac{\dot{\varphi}_t - a(\varphi_t)}{\Sigma b(\varphi_t)}\big)dt, & \varphi_0 = X_0,\ d\varphi \ll dt \\ \infty, & \textit{otherwise}. \end{cases}$$

The proof of these theorems are situated in Sections 4 and 5. In Section 3 we formulate the l.d.p. for $\varepsilon N_{t/\varepsilon}$. Section 2 contains an auxiliary result. In the last Section an example is considered having an independent interest.

2. Properties of $\int_0^t \eta_s ds$

1. Taking into account (1.4) and well known property of the stochastic Ito integral we find $\mathbf{E}(\eta_t | \eta_s, s \le 0) = \int_\infty^0 h(t-s) dN_s$ $\mathbf{P}$-a.s. On the other hand, by (1.6) $\int_0^\infty \sqrt{\int_t^\infty h^2(s) ds dt} < \infty$. Both facts imply

$$\int_0^\infty \sqrt{\mathbf{E}\big(\mathbf{E}(\eta_t | \eta_s, s \le 0)\big)^2} dt < \infty.$$

Then by [8, Lemma 9.2.1] there exists a semimartingale $(V_t)_{t \ge 0}$, w.r.t. the filtration $\mathbf{F} = (\mathcal{F}_t)_{t \in R}$ generated by (η_t), satisfying the general conditions, such that

$$V_t = V_0 + \int_0^t \eta_s ds - M_t, \tag{2.1}$$

where (M_t) is a square integrable martingale with in the restricted sense ergodic stationary increments and $(V_t, \eta_t)_{t\ge 0}$ forms in the restricted sense stationary process. For η_t obeying the Wold decomposition (1.4), we have (with Σ given at (1.7))

$$\begin{aligned} M_t &= \Sigma N_t \\ V_t &= \int_t^\infty \Big(\int_{-\infty}^t h(u-s) dN_s \Big) du \end{aligned} \tag{2.2}$$

(for more details see [8, Ch.9, Sec.2, Ex.2].

The aim of this Section is to show an exponential integrability of $\sup_{t\le T} |V_t|$.

Lemma 2.1. *Let (1.6), (1.8), and (A.1), (A.2) be fulfilled. Then for any $\lambda > 0$ and $T > 0$*

$$\mathbf{E} \exp\big\{\lambda \sup_{t\le T} |V_t|\big\} < \infty.$$

Proof. Assume there exists some a positive martingale (L_t) w.r.t. $\mathbf{F}$ such that

$$\begin{aligned} &|V_t| \le L_t, \ \forall t \le T \\ &\mathbf{E} \exp\big\{\lambda L_T\big\} < \infty, \ \forall \lambda > 0. \end{aligned} \tag{2.3}$$

Then the result holds. In fact, using Jensen's, Cauchy-Schwartz's, and Doob's inequalities we get

$$\begin{aligned} \mathbf{E} \exp\big\{\lambda \sup_{t\le T} |V_t|\big\} &\le \mathbf{E} \exp\big\{\lambda \sup_{t\le T} L_t\big\} \\ &= \mathbf{E} \sup_{t\le T} \exp\big\{\lambda L_t\big\} \\ &= \mathbf{E} \sup_{t\le T} \exp\big\{\mathbf{E}(\lambda L_T|\mathcal{F}_t)\big\}. \\ &\le \mathbf{E} \sup_{t\le T} \mathbf{E}\Big(\exp\big\{\lambda L_T\big\}|\mathcal{F}_t\Big) \text{ (Jensen)} \\ &\le \sqrt{\mathbf{E} \sup_{t\le T} \mathbf{E}\Big(\exp\big\{\lambda L_T\big\}|\mathcal{F}_t\Big)^2} \text{ (Cauchy-Schwartz)} \\ &\le 2\sqrt{\mathbf{E} \exp\big\{2\lambda L_T\big\}} \text{ (Doob)} < \infty. \end{aligned}$$

Thus, it remains to find (L_t) satisfying (2.3). Two facts are used here: decomposition (2.1) and $\mathcal{F}_t$-measurability of V_t. From (2.1) we find $V_t = V_T - \int_0^t \eta_s ds + \Sigma(N_T - N_t), t \le T$ and what follows from martingale property of (N_t) that is $V_t = \mathbf{E}\big(V_T - \int_0^t \eta_s ds|\mathcal{F}_t\big)$. So $|V_t| \le \mathbf{E}\big(|V_T| + \int_0^T |\eta_s|ds|\mathcal{F}_t\big)$, i.e. $L_t = \mathbf{E}\big(|V_T| + \int_0^T |\eta_s|ds|\mathcal{F}_t\big)$.

Thereby, it remains to show only that

$$\mathbf{E} \exp\big\{\lambda[|V_T| + \int_0^T |\eta_s|ds]\big\} < \infty. \tag{2.4}$$

We examine (2.4) by using Cauchy-Schwartz's and Jensen's inequalities:

$$\begin{aligned}\Big(\mathbf{E}\exp\big\{\lambda[|V_T|+\int_0^T|\eta_s|ds]\big\}\Big)^2\\ &\le \mathbf{E}\exp\big\{2\lambda|V_T|\big\}\mathbf{E}\exp\big\{2\lambda\int_0^T|\eta_s|ds\big\}\ \text{(Cauchy-Schwartz)}\\ &= \mathbf{E}\exp\big\{2\lambda|V_0|\big\}\mathbf{E}\exp\big\{\frac{1}{T}\int_0^T 2T\lambda|\eta_s|ds\big\}\\ &\le \mathbf{E}\exp\big\{2\lambda|V_0|\big\}\mathbf{E}\frac{1}{T}\int_0^T\exp\big\{2T\lambda|\eta_s|\big\}ds\ \text{(Jensen)}\\ &= \mathbf{E}\exp\big\{2\lambda|V_0|\big\}\mathbf{E}\exp\big\{2T\lambda|\eta_0|\big\}.\end{aligned}$$

Thus (2.4) holds if for any positive λ $\mathbf{E}\exp\big\{\lambda|V_0|\big\}<\infty$ and $\mathbf{E}\exp\big\{\lambda|\eta_0|\big\}<\infty$. The direct proof for the validity of these inequalities would be difficult. It is more convenient to use instead of V_0 and η_0 random values $V_0'=\int_0^\infty\int_s^\infty h(u)dudN_s$ (V_0' is defined by virtue of (1.8)) and $\eta_0'=\int_0^\infty h(s)dN_s$ which coincide with V_0 and η_0 in the distributions and furthermore, using the estimate $e^{|x|}\le e^x+e^{-x}$, to examine only

$$\begin{aligned}&\mathbf{E}\exp\big\{\lambda V_0'\big\}<\infty\\ &\mathbf{E}\exp\big\{\lambda\eta_0'\big\}<\infty,\ \forall\lambda\in R.\end{aligned}\tag{2.5}$$

Denote by $H(t)$ any of functions $h(t)$ or $\int_t^\infty h(u)du$. From the definition o V_0' and η_0' it follows that

$$\left\{\begin{array}{l}V_0'\\ \eta_0'\end{array}\right.=\int_0^\infty H(t)dN_t\ (\equiv Z_\infty).$$

Thereby, the validity of

$$\mathbf{E}\exp\big\{\lambda Z_\infty\big\}<\infty,\ \forall\lambda\in R\tag{2.6}$$

has to be checked. To this end define a square integrable martingale $Z_t=\int_0^t H(s)dN_s$ having as the limit point $Z_\infty=\lim\limits_{t\to\infty}Z_t$. By the Fatou lemma

$$\mathbf{E}\exp\big\{\lambda Z_\infty\big\}\le\limsup_{t\to\infty}\mathbf{E}\exp\big\{\lambda Z_t\big\}$$

and so it is sufficient to show that for any $\lambda\in R$ there exists constant $C(\lambda)$ depending on λ only such that

$$\mathbf{E}\exp\big\{\lambda Z_t\big\}\le C(\lambda).\tag{2.7}$$

For finding $C(\lambda)$ we use the fact that (Z_t) is the process with independent increments. Namely, the Levy measure $K(dx)dt$ is a compensator for the measure $\mu(dt,dx)$ of jumps of (N_t) w.r.t. a filtration $\mathbf{F}^N=(\mathcal{F}_t^N)_{t\ge0}$ generated by (N_t). Then the pour discontinuous part (N_t^d) of (N_t) can be represented as Ito's integral w.r.t. the martingale measure $\mu(dt,dx)-K(dx)dt$:

$$N_t^d=\int_0^t\int_{R^0}x[\mu(ds,dx)-K(dx)ds]$$

and so

$$N_t = N_t^c + \int_0^t \int_{R^0} x[\mu(ds, dx) - K(dx)ds]. \tag{2.8}$$

Then

$$\begin{aligned} Z_t &= \int_0^t H(s)dN_s \\ &= \int_0^t H(s)dN_s^c + \int_0^t \int_{R^0} H(s)x[\mu(ds, dx) - K(dx)ds] \end{aligned}$$

and by [5,Ch.II] for any $\lambda \in R$ we find

$$\mathbf{E}e^{\lambda Z_t} = \exp\Big\{\frac{\lambda^2\sigma^2}{2}\int_0^t H^2(s)ds + \int_0^t \int_{R^0} \big(e^{\lambda H(s)x} - 1 - \lambda H(s)x\big)K(dx)ds\Big\}. \tag{2.9}$$

The right hand side of this inequality growth in $t \to \infty$ to

$$C(\lambda) = \exp\Big\{\frac{\lambda^2\sigma^2}{2}\int_0^\infty H^2(s)ds + \int_0^\infty \int_{R^0} \big(e^{\lambda H(s)x} - 1 - \lambda H(s)x\big)K(dx)ds\Big\} \tag{2.10}$$

and only it remains to show that this $C(\lambda)$ is finite. In case $H(t) \equiv h(t)$ it holds by (A.2). In case $H(t) \equiv \int_t^\infty h(s)ds$ it it is implied by (1.8) and (A.1) since ($\lambda_0 = |\lambda| \sup_{t\ge 0} |H(t)|$)

$$\begin{aligned} e^{\lambda H(t)x} - 1 - \lambda H(t)x &\le \lambda^2 H^2(t) e^{\lambda_0|x|} \\ &\le \lambda^2 H^2(t)\big(e^{\lambda_0 x} + e^{-\lambda_0 x}\big) \end{aligned}$$

and for any $\lambda \in R$ $\int_{R^0} x^2 e^{\lambda x} K(dx) < \infty$.

3. The l.d.p. for $\varepsilon N_{t/\varepsilon}$

In this Section the l.d.p. is established for family of homogeneous processes with independent increments (h.i.i) what kind is $\varepsilon N_{t/\varepsilon}$. Since N_t has paths in the Skorokhod space $\mathbf{D} = \mathbf{D}_{[0,\infty)}$ of the right continuous having limits from the left functions we formulate the l.d.p. in the metric space $(\mathbf{D}, d)$ with the Skorokhod-Lindvall metric d (see [6] or e.g. [8, Ch. 6]). The metric d is equivalent to some sense to the metric ρ (see Section 1) and is defined in the following way. For each $X = (X_t)_{t\ge 0}$ put $X_t^n = X_t \min(1, n-t), 0 \le t \le n$. If $X, Y \in \mathbf{D}$ and $X^n = (X_t^n)_{0\le t\le n}, Y^n = (Y_t^n)_{0\le t\le n}$ defined as it was mentioned above we have

$$d(X, Y) = \sum_{n\ge 1} 2^{-n} \frac{d_n(X^n, Y^n)}{1 + d_n(X^n, Y^n)},$$

where $d_n(X^n, Y^n)$ is the Skorokhod distance between X^n, Y^n (see [1]).

Theorem 3.1. *Let assumption (A.1) be fulfilled. The family* $\varepsilon N_{t/\varepsilon}$ *obeys the l.d.p. in the metric space* $(\mathbf{D}, d)$ *with rate function*

$$I^{hii}(\varphi) = \begin{cases} \int_0^\infty H_{\sigma^2}(\dot{\varphi}_t)dt, & \varphi_0 = 0, d\varphi << dt \\ \infty, & otherwise \end{cases}$$

with $H_{\sigma^2}(y)$ *from (1.11).*

Formally Theorem 3.1 follows from general result of the l.d.p. for Markovian processes [12] or semimartingales [7]. We give here the direct proof. The reason for this is that the process N_t has simpler structure than Markovian process or semimartingale considered in [12] and [7] and so suggested proof is simpler and could be interested by itself. Nevertheless, we use the method of proving from [7] which is based on the such notions as the exponential tightness, the partial l.d.p., and the Pukhalskii theorem [9]. Following it only two sets of conditions have to be checked.

(C.1) **C**-exponential tightness: for any $T > 0, \gamma > 0$

$$\lim_{c\to\infty} \overline{\lim}_{\varepsilon\to 0}\varepsilon \log \mathbf{P}\big(\sup_{t\le T} |\varepsilon N_{t/\varepsilon}| > c\big) = -\infty$$
$$\lim_{\delta\to 0} \overline{\lim}_{\varepsilon\to 0} \sup \varepsilon \log \mathbf{P}\big(\sup_{0\le t\le \delta} |\varepsilon(N_{(t+\tau)/\varepsilon} - N_{\tau/\varepsilon}| > \gamma\big) = -\infty,$$

where "sup″ is taken over all stopping times τ (w.r.t. $(\mathcal{F}^N_{t/\varepsilon})_{t\ge 0}$) which is bounded by T.

(C.2) **C**-local l.d.p.: for any $T > 0$ and $\varphi \in \mathbf{C}$

$$\begin{aligned} \hat{I}_T(\varphi) &= -\overline{\lim}_{\delta\to 0}\overline{\lim}_{\varepsilon\to 0}\varepsilon \log \mathbf{P}\big(\sup_{t\le T} |\varepsilon N_{t/\varepsilon} - \varphi_t| \le \delta\big) \\ &= -\underline{\lim}_{\delta\to 0}\underline{\lim}_{\varepsilon\to 0}\varepsilon \log \mathbf{P}\big(\sup_{t\le T} |\varepsilon N_{t/\varepsilon} - \varphi_t| \le \delta\big). \end{aligned}$$

If (C.1) and (C.2) are fulfilled then the l.d.p. in $(\mathbf{D}, d)$ holds with rate function

$$I^{hii}(\varphi) = \begin{cases} \sup_{T>0} \hat{I}_T(\varphi), & \varphi \in \mathbf{C} \\ \infty, & \varphi \in \mathbf{D}\setminus\mathbf{C}. \end{cases} \tag{3.1}$$

Conditions (C.1) and (C.2) are examined below in three lemmas.

Lamma 3.1. *Let assumption (A.1) be fulfilled. Then (C.1) holds.*

Proof. For checking the first condition in (C.1) Chernoff's, Jensen's, Cauchy-Schwartz's, and Doob's inequalities are applied:

$$\begin{aligned}\mathbf{P}\Big(\sup_{t\le T}|\varepsilon N_{t/\varepsilon}|>c\Big) &\le e^{-c/\varepsilon}\mathbf{E}\exp\big\{\sup_{t\le T}|N_{t/\varepsilon}|\big\}(\text{Chernoff})\\ &= e^{-c/\varepsilon}\mathbf{E}\sup_{t\le T}\exp\big\{|N_{t/\varepsilon}|\big\}\\ &= e^{-c/\varepsilon}\mathbf{E}\sup_{t\le T}\exp\big\{|\mathbf{E}(N_{T/\varepsilon}|\mathcal{F}^N_{t/\varepsilon}|\big\}\\ &\le e^{-c/\varepsilon}\mathbf{E}\sup_{t\le T}\mathbf{E}\Big(\exp\big\{|N_{T/\varepsilon}|\big\}|\mathcal{F}^N_{t/\varepsilon}\Big)(\text{Jensen})\\ &\le e^{-c/\varepsilon}\sqrt{\mathbf{E}\Big[\sup_{t\le T}\mathbf{E}\Big(\exp\big\{|N_{T/\varepsilon}|\big\}|\mathcal{F}^N_{t/\varepsilon}\Big)\Big]^2}(\text{Cauchy-Schwartz})\\ &\le 2e^{-c/\varepsilon}\sqrt{\mathbf{E}\exp\big\{2|N_{T/\varepsilon}|\big\}}(\text{Doob})\\ &\le 2e^{-c/\varepsilon}\sqrt{\mathbf{E}\big\{e^{2N_{T/\varepsilon}}+e^{-2N_{T/\varepsilon}}\big\}}. \qquad (3.2)\end{aligned}$$

Since (N_t) is the martingale with independent increments by virtue of (2.8) and [5, Ch.II] we get

$$\begin{aligned}\mathbf{E}e^{\pm 2N_{T/\varepsilon}} &= \exp\Big\{\frac{T}{\varepsilon}\big[2\sigma^2+\int_{R^0}\big(e^{\pm 2x}-1\mp 2x\big)K(dx)\Big\}\\ &\le \exp\big\{\frac{T\ell}{\varepsilon}\big\}, \qquad (3.3)\end{aligned}$$

where $\ell=2\sigma^2+\int_{R^0}\big(e^{2x}-1-2x\big)K(dx)+\int_{R^0}\big(e^{-2x}-1+2x\big)K(dx)(<\infty)$ (see (Cr.1)).

Then, as it follows from (3.2) and (3.3),

$$\varepsilon\log\mathbf{P}\big(\sup_{t\le T}|\varepsilon N_{t/\varepsilon}|>c\big)\le\varepsilon\log 2-c+\text{const}.T\to-\infty,\ \varepsilon\to 0,\ c\to\infty.$$

For checking the second condition in (C.1) note that by virtue of strong Markovian property $(N_{(t+\tau)/\varepsilon}-N_{\tau/\varepsilon})_{t\ge 0}$ coinsides in the distribution with $(N_{t/\varepsilon})_{t\ge 0}$. So by the same way as (3.2) and (3.3) have been obtained, we find for any $\lambda>0$

$$\begin{aligned}\mathbf{P}\big(\sup_{0\le t\le\delta}|\varepsilon(N_{(t+\tau)/\varepsilon}-N_{\tau/\varepsilon}|>\gamma\big) &\le 2e^{-\lambda\gamma/\varepsilon}\sqrt{\mathbf{E}\Big[e^{2\lambda N_{\delta/\varepsilon}}+e^{-2\lambda N_{\delta/\varepsilon}}\Big]}\\ &\le 2\exp\{-\lambda\gamma/\varepsilon+\frac{1}{2}\delta\ell(\lambda)/\varepsilon\},\end{aligned}$$

where

$$\ell(\lambda)=2\lambda^2\sigma^2+\int_{R^0}e^{2\lambda x}-1-2\lambda x)K(dx)+\int_{R^0}e^{-2\lambda x}-1+2\lambda x)K(dx).$$

Then

$$\overline{\lim}_{\varepsilon\to 0}\varepsilon\log\mathbf{P}\big(\sup_{0\le t\le\delta}|\varepsilon(N_{(t+\tau)/\varepsilon}-N_{\tau/\varepsilon}|>\gamma\big)\le-\lambda\gamma+\frac{1}{2}\delta\ell(\lambda). \qquad (3.4)$$

The function $\ell(\lambda),\lambda\ge 0$ is nonnegative, continuous, and increasing. If it is bounded then the result evidently holds. If it is increasing to ∞ then taking λ_δ such that $\ell(\lambda_\delta)=2/\delta$ we arrive to upper bound in (3.4): $-\lambda_\delta\gamma+1$ which decreases to $-\infty$ in $\delta\to 0$.

Lemma 3.2. *Let assumption (A.1) be fulfilled. Then for any $T>0$ and $\varphi\in\mathbf{C}$*

$$\overline{\lim}_{\delta\to 0}\overline{\lim}_{\varepsilon\to 0}\varepsilon\log\mathbf{P}\big(\sup_{t\le T}|\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta\big) \le -\begin{cases}\int_0^T H_{\sigma^2}(\dot\varphi_t)dt, & \varphi_0=0, d\varphi << dt\\ \infty, & otherwise,\end{cases}$$

where $H_{\sigma^2}(\dot\varphi_t$ is defined in (1.11).

Proof. Let $\nu(t)$ be a simple function of the form $\nu(t)=\sum_i \nu_i I_{]t_{i-1},t_i]}(t)$. Put

$$Z^\varepsilon_T(\nu)=\exp\Big\{\int_0^T\nu(t)dN_{t/\varepsilon}-\frac{1}{\varepsilon}\int_0^T G(\nu(t))dt\Big\}, \tag{3.5}$$

where $G(\lambda)$ is the cumulant given at (1.9). By [5,Ch.II] we have

$$\mathbf{E}Z^\varepsilon_T(\nu)=1 \tag{3.6}$$

which implies

$$1\ge\mathbf{E}I(\sup_{t\le T}|\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta)Z^\varepsilon_T(\nu). \tag{3.7}$$

Inequality (3.7) is the general tool in proving the upper bound. It is naturally to evaluate $Z^\varepsilon_T(\nu)$ from below on the set $\{\sup_{t\le T}|\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta\}$. Put by definition $\int_0^T\nu(t)d\varphi_t=\sum_i\nu_i[\varphi_{T\wedge t_i}-\varphi_{T\wedge t_{i-1}}]$. Then

$$\begin{aligned}Z^\varepsilon_T(\nu)\ge&\exp\Big\{\frac{1}{\varepsilon}\Big[\int_0^T\nu(t)d\varphi_t-\int_0^T G(\nu(t))dt\Big]\Big\}\\ &\times\exp\Big\{-\Big|\int_0^T\nu(t)[dN_{t/\varepsilon}-\frac{1}{\varepsilon}d\varphi_t]\Big|\Big\}.\end{aligned} \tag{3.8}$$

Taking into account that on the set $\{\sup_{t\le T}|\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta\}$ the following estimate holds: $|\int_0^T\nu(t)[dN_{t/\varepsilon}-\frac{1}{\varepsilon}d\varphi_t]|\le\text{const.}\frac{\delta}{\varepsilon}$, where const. depends only on ν_i, we derive from (3.7) and (3.8) the following upper bound:

$$\overline{\lim}_{\delta\to 0}\overline{\lim}_{\varepsilon\to 0}\varepsilon\log\mathbf{P}\big(\sup_{t\le T}|\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta\big)\le-\Big[\int_0^T\nu(t)d\varphi_t-\int_0^T G(\nu(t))dt\Big].$$

The equality

$$\sup\Big[\int_0^T\nu(t)d\varphi_t-\int_0^T G(\nu(t))dt\Big]=\begin{cases}\int_0^T H_{\sigma^2}(\dot\varphi_t)dt, & \varphi_0=0, d\varphi\ll dt\\ \infty, & \text{otherwise},\end{cases}$$

where sup is taken over all simple functions $\nu(t)$, follows from [7, Lemma 6].

Lemma 3.3. *Let assumption (A.1) be fulfilled. Then for any $T>0$ and $\varphi\in\mathbf{C}$*

$$\underline{\lim}_{\delta\to0}\underline{\lim}_{\varepsilon\to0}\varepsilon\log\mathbf{P}\big(\sup_{t\le T}|\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta\big)$$
$$\ge-\begin{cases}\int_0^T H_{\sigma^2}(\dot\varphi_t)dt, & \varphi_0=0, d\varphi\ll dt\\ \infty, & otherwise,\end{cases}$$

where $H_{\sigma^2}(\dot\varphi_t$ is defined in (1.11).

Proof. Since the announced lower bound coincides with the upper one only the validity of finite bound have to be checked, or it is sufficient to consider only the case "$\varphi_0=0, d\varphi\ll dt$".

Assume at first that

$$\begin{aligned}&\sigma^2>0\\&|\dot\varphi_t|\le N.\end{aligned}\tag{3.9}$$

Under these assumptions there exists a measurable bounded function $\nu(t)$ satisfying for a.s. (w.r.t. Lebesgue measure) $t\ge0$ the equality: $G'(\nu(t))=0$ (see (1.9)). Then we have

$$H_{\sigma^2}(\dot\varphi_t)=\nu(t)\dot\varphi_t-G(\nu(t)),\ \text{a.s.}\tag{3.10}$$

For fixed $T>0$ put $\nu_T(t)=I(T\ge t)\nu(t)$ and define the process

$$Z^\varepsilon_t(\nu_T)=\exp\Big\{\int_0^t\nu_T(s)dN_{s/\varepsilon}-\frac{1}{\varepsilon}\int_0^t G(\nu_T(t))dt\Big\}.\tag{3.11}$$

Since $G(0)=0$ $\lim\limits_{t\to\infty}Z^\varepsilon_t(\nu_T)=Z^\varepsilon_T(\nu_T)$ and

$$\mathbf{E}Z^\varepsilon_T(\nu_T)=1.\tag{3.12}$$

Noticing that $Z^\varepsilon_T(\nu_T)>0$ $\mathbf{P}$-a.s. and letting $d\mathbf{P}^\varepsilon_T=Z^\varepsilon_T(\nu_T)d\mathbf{P}$ we get the probability measure $\mathbf{P}^\varepsilon_T$, which is equivalent to $\mathbf{P}$ and $\frac{d\mathbf{P}}{d\mathbf{P}^\varepsilon_T}=\big(Z^\varepsilon_T(\nu_T)\big)^{-1}$ (the expectation w.r.t. $\mathbf{P}^\varepsilon_T$ is denoted by $\mathbf{E}^\varepsilon_T$). From identity

$$\begin{aligned}\mathbf{P}\big(\sup_{t\le T}|\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta\big)&=\mathbf{E}I(\sup_{t\le T}|\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta)Z^\varepsilon_T(\nu_T)\big(Z^\varepsilon_T(\nu_T)\big)^{-1}\\&=\mathbf{E}^\varepsilon_T I(\sup_{t\le T}|\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta)\big(Z^\varepsilon_T(\nu_T)\big)^{-1}\end{aligned}$$

with (see (3.10) and (3.11))

$$\big(Z^\varepsilon_T(\nu_T)\big)^{-1}=\exp\Big\{-\int_0^T\nu(t)\big[dN_{t/\varepsilon}-\frac{1}{\varepsilon}\dot\varphi_t dt\big]-\int_0^T H_{\sigma^2}(\dot\varphi_t)dt\Big\}$$

we find

$$\underline{\lim}_{\varepsilon\to 0}\varepsilon\log\mathbf{P}\big(\sup_{t\le T}|\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta\big)\ge-\int_0^T H_{\sigma^2}(\dot\varphi_t)dt$$
$$-\overline{\lim}_{\varepsilon\to 0}\Big|\varepsilon\log\mathbf{E}^\varepsilon_T\Big(I(\sup_{t\le T}|\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta)\exp\Big\{-\int_0^T\nu(t)\big[dN_{t/\varepsilon}-\frac{1}{\varepsilon}\dot\varphi_t dt\big]\Big\}\Big)\Big|$$
$$\ge-\int_0^T H_{\sigma^2}(\dot\varphi_t)dt$$
$$-\gamma-\overline{\lim}_{\varepsilon\to 0}\Big|\varepsilon\log\mathbf{P}^\varepsilon_T\Big(I\big(\sup_{t\le T}|\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta,\ \big|\int_0^T\nu(t)\big[dN_{t/\varepsilon}-\frac{1}{\varepsilon}\dot\varphi_t dt\big]\big|\le\frac{\gamma}{\varepsilon}\Big)\Big|.$$

Thereby by the arbitrariness of γ the desired lower bound holds if for any $\zeta>0$

$$\lim_{\varepsilon\to 0}\varepsilon\log\mathbf{P}^\varepsilon_T\Big(I\big(\sup_{t\le T}|\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta,\ \big|\int_0^T\nu(t)\big[\varepsilon dN_{t/\varepsilon}-\dot\varphi_t dt\big]\big|\le\zeta\Big)=0. \tag{3.13}$$

To this end we examine some properties of the process $Z^\varepsilon_t(\nu_T)$. Taking into account (2.8) and applying Ito's formula to the right hand side of (3.11) we find that

$$dZ^\varepsilon_t(\nu_T)=Z^\varepsilon_{t-}(\nu_T)\nu_T(t)dN_{t/\varepsilon}$$
$$+Z^\varepsilon_{t-}(\nu_T)\int_{R^0}\big(e^{\nu_T(t)x}-1-\nu_T(t)x\big)[\mu(d(t/\varepsilon),dx)-K(dx)d(t/\varepsilon)]$$

and so $Z^\varepsilon_t(\nu_T)$ is a martingale w.r.t. $(\mathbf{P},(\mathcal{F}^N_{t/\varepsilon})_{t\ge 0})$. Furthermore, it is a square integrable martingale since $\mathbf{E}\big(Z^\varepsilon_t(\nu_T)\big)^2=\exp\Big\{1/\varepsilon\int_0^T\big[G(2\nu(t))-2G(\nu(t))\big]dt\Big\}$. Also note that the mutually quadratic variation for pair of martingales $Z^\varepsilon_t(\nu_T)$ and $N_{t/\varepsilon}$ is defined as:

$$\big[Z^\varepsilon(\nu_T),N_{\cdot/\varepsilon}\big]_t=\frac{1}{\varepsilon}\int_0^t Z^\varepsilon_s(\nu_T)\sigma^2\nu_T(s)ds$$
$$+\int_0^t Z^\varepsilon_s(\nu_T)\int_{R^0}x\big[e^{\nu_T(s)x}-1\big]\mu(d(s/\varepsilon),dx)$$

and consequently, the mutually predictable quadratic variation, being the compensator for it, is given by the formula:

$$\big\langle Z^\varepsilon(\nu_T),N_{\cdot/\varepsilon}\big\rangle_t=\frac{1}{\varepsilon}\Big[\int_0^t Z^\varepsilon_s(\nu_T)\sigma^2\nu_T(s)ds$$
$$+\int_0^t Z^\varepsilon_s(\nu_T)\int_{R^0}x\big[e^{\nu_T(s)x}-1\big]K(dx)ds\Big].$$

Define new process $N^{T,\varepsilon}_t=N_{t/\varepsilon}-\frac{1}{\varepsilon}\varphi_{t\wedge T}$. It is easy to check that

$$N^{T,\varepsilon}_t=N_{t/\varepsilon}-\frac{1}{\varepsilon}\int_0^t\big(Z^\varepsilon_s(\nu_T)\big)^{-1}\big\langle Z^\varepsilon(\nu_T),N_{\cdot/\varepsilon}\big\rangle_s.$$

Then by [8, Ch.4] process $N_t^{T,\varepsilon}$ is a martingale w.r.t. $(\mathbf{P}_T^\varepsilon, (\mathcal{F}_{t/\varepsilon}^N)_{t\geq 0})$ and what is more continuous component $N_t^{T,\varepsilon,c}$ of it is Gaussian, $\mathbf{E}_T^\varepsilon\big(N_t^{T,\varepsilon,c}\big)^2 = \sigma^2 t/\varepsilon$, and the Levy measure of it is: $K_T^\varepsilon(dt,dx) = e^{\nu_T(t)x}K(dx)d(t/\varepsilon)$. So it is a square integrable martingale whose predictable quadratic variation is given by the formula:

$$\big\langle N^{T,\varepsilon,c}\big\rangle_t = \frac{1}{\varepsilon}\big[\sigma^2 t + \int_0^t \int_{R^0} x^2 e^{\nu_T(t)x}K(dx)dt\big].$$

Consequently, by Doob's inequality (see e.g. [8, Ch.I]) we have

$$\begin{aligned}\mathbf{E}_T^\varepsilon\Big(\sup_{t\leq T}\big|\varepsilon N_t^{T,\varepsilon,c}\big|^2\Big) &\leq 4\varepsilon\big[\sigma^2 T + \int_0^T \int_{R^0} x^2 e^{\nu_T(t)x}K(dx)dt\big].\\ &= \varepsilon \text{ const.}\end{aligned} \tag{3.14}$$

Analogously we get

$$\mathbf{E}_T^\varepsilon\Big(\sup_{t\leq T}\big|\int_0^t \nu_T(s)d(\varepsilon N_s^{T,\varepsilon,c})\big|^2\Big) = \varepsilon \text{ const.} \tag{3.15}$$

Since for $t\leq T$ we have $\varepsilon N_t^{T,\varepsilon} = \varepsilon N_{t/\varepsilon} - \varphi_t$ (3.13) is implied by (3.14) and (3.15).

In the next step we show the validity of the lower bound under weaker conditions than (3.9). Namely, taking into account that for $d\varphi \ll dt$ we have $\int_0^t |\dot\varphi_s|ds < \infty$ $\forall t>0$, only

$$\sigma^2 > 0 \tag{3.16}$$

has to be assumed.

For $N\geq 1$, taking $\dot\varphi_t^N = \dot\varphi_t I(|\dot\varphi_t|\leq N)$, put $\varphi_t^N = \int_0^t \dot\varphi_s^N ds$. Evidently φ_t^N satisfies (3.9) and so by the obtained above result for any $\delta>0$ we have

$$\limsup_{\varepsilon\to 0}\varepsilon\log\mathbf{P}\big(\sup_{t\leq T}|\varepsilon N_{t/\varepsilon} - \varphi_t^N|\leq\delta\big) \geq -\int_0^T H_{\sigma^2}(\dot\varphi_t^N)dt.$$

On the other hand $H_{\sigma^2}(0) = 0$ and so $H_{\sigma^2}(\dot\varphi_t^N) = H_{\sigma^2}(\dot\varphi_t)I(|\dot\varphi_t|\leq N)$ what implies inequality $\int_0^T H_{\sigma^2}(\dot\varphi_t^N)dt \leq \int_0^T H_{\sigma^2}(\dot\varphi_t)dt$.

Then

$$\begin{aligned}&\underline{\lim}_{\varepsilon\to 0}\varepsilon\log\mathbf{P}\big(\sup_{t\leq T}|\varepsilon N_{t/\varepsilon} - \varphi_t|\leq\delta - \sup_{t\leq T}|\varphi_t - \varphi_t^N|\big)\\ &\geq \underline{\lim}_{\varepsilon\to 0}\varepsilon\log\mathbf{P}\big(\sup_{t\leq T}|\varepsilon N_{t/\varepsilon} - \varphi_t^N|\leq\delta\big)\\ &\geq -\int_0^T H_{\sigma^2}(\dot\varphi_t)dt.\end{aligned}$$

The desired lower bound evidently holds since $|\varphi_t - \varphi_t^N| \leq \int_0^T |\dot\varphi_t I(|\dot\varphi_t|>N)dt \to 0,\ N\to\infty$.

Thus, it remains to check only the validity of the lower bound under

$$\sigma^2 = 0. \tag{3.17}$$

Due to $\sigma^2 = 0$ the Gaussian component of N_t equals zero. Put

$$N_t^\gamma = N_t + \gamma W_t,$$

where W_t is Wiener process which is to be assumed independent of N_t and $\gamma > 0$. By the obtained above result we get

$$\underline{\lim}_{\varepsilon\to 0}\varepsilon \log \mathbf{P}\big(\sup_{t\le T}|\varepsilon N^\gamma_{t/\varepsilon} - \varphi_t| \le \delta\big) \ge -\int_0^T H_{\gamma^2}(\dot{\varphi}_t)dt.$$

As it was mentioned in Section 1 function $H_{\sigma^2}(y)$ is increasing to $H_0(y)$ in $\sigma^2 \to 0$ and so $\int_0^T H_{\gamma^2}(\dot{\varphi}_t)dt \le \int_0^T H_0(\dot{\varphi}_t)dt$, i.e.

$$\underline{\lim}_{\varepsilon\to 0}\varepsilon \log \mathbf{P}\big(\sup_{t\le T}|\varepsilon N^\gamma_{t/\varepsilon} - \varphi_t| \le \delta\big) \ge -\int_0^T H_0(\dot{\varphi}_t)dt. \tag{3.18}$$

Now we use the following chain of the lower estimates:

$$\begin{aligned}
&2\max\Big\{\mathbf{P}\big(\sup_{t\le T}|\varepsilon N_{t/\varepsilon} - \varphi_t| \le 2\delta\big),\ \mathbf{P}\big(\gamma\sup_{t\le T}|\varepsilon W_{t/\varepsilon}| > \delta\big)\Big\} \\
&\ge \mathbf{P}\big(\sup_{t\le T}|\varepsilon N_{t/\varepsilon} - \varphi_t| \le 2\delta\big) + \mathbf{P}\big(\gamma\sup_{t\le T}|\varepsilon W_{t/\varepsilon}| > \delta\big) \\
&\ge \mathbf{P}\big(\sup_{t\le T}|\varepsilon N_{t/\varepsilon} - \varphi_t| \le \delta + \gamma\sup_{t\le T}|\varepsilon W_{t/\varepsilon}|\big) \\
&\ge \mathbf{P}\big(\sup_{t\le T}|\varepsilon N^\gamma_{t/\varepsilon} - \varphi_t| \le \delta\big)
\end{aligned} \tag{3.19}$$

From (3.19) and (3.18) it follows that for any $\delta > 0, \gamma > 0$

$$\begin{aligned}
&\max\begin{cases}\underline{\lim}_{\varepsilon\to 0}\varepsilon \log \mathbf{P}\big(\sup_{t\le T}|\varepsilon N_{t/\varepsilon} - \varphi_t| \le 2\delta\big) \\ \overline{\lim}_{\varepsilon\to 0}\varepsilon \log \mathbf{P}\big(\sup_{t\le T}|\varepsilon W_{t/\varepsilon}| > \frac{2\delta}{\gamma}\big)\end{cases} \\
&\ge -\int_0^T H_0(\dot{\varphi}_t)dt
\end{aligned}$$

and what follows from Lemma 3.1

$$\lim_{\gamma\to 0}\overline{\lim}_{\varepsilon\to 0}\varepsilon \log \mathbf{P}\big(\sup_{t\le T}|\varepsilon W_{t/\varepsilon}| > \frac{2\delta}{\gamma}\big) = -\infty.$$

Thus, the desired lower bound holds.

4. Proof of Theorem 1.1

Decomposition (2.1), inlet $M_t = \Sigma N_t$ (see (2.2)), implies

$$\sqrt{\varepsilon} W_t^{\varepsilon} = \varepsilon(V_{t/\varepsilon} - V_0) + \Sigma \varepsilon N_{t/\varepsilon}. \tag{4.1}$$

The first step consists in showing of exponential negligibility of the process $\varepsilon(V_{t/\varepsilon} - V_0)$ in a sense given below.

Lemma 4.1. *Let Cramer's conditions (A.1) and (A.2) be fulfilled. Then for any* $T > 0,\ \gamma > 0$

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbf{P}\big(\sup_{t \le T} |\varepsilon(V_{t/\varepsilon} - V_0| \ge \gamma\big) = -\infty.$$

Proof. Evidently, only

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbf{P}\big(\sup_{t \le T} |\varepsilon V_{t/\varepsilon}| \ge \gamma\big) = -\infty$$

has to be checked. Let $\alpha_k = \sup\limits_{T(k-1) \le t \le Tk} |V_t|, k \ge 1$. Noticing that $(\alpha_k)_{k \ge 1}$ forms in the restricted sense stationary sequence and using the Chernoff inequality (with parameter $\lambda > 0$), we find

$$\begin{aligned}
\mathbf{P}\big(\varepsilon \sup_{t \le T} |V_{t/\varepsilon}| \ge \gamma\big) &= \mathbf{P}\big(\sup_{t \le T/\varepsilon} |V_t| \ge \gamma/\varepsilon\big) \\
&\le \mathbf{P}\big(\max_{k \le 1/\varepsilon} \alpha_k \ge \gamma/\varepsilon\big) \\
&\le \sum_{k \le 1/\varepsilon} \mathbf{P}\big(\alpha_k \ge \gamma/\varepsilon\big) \\
&\le (1/\varepsilon)\mathbf{P}\big(\alpha_1 \ge \gamma/\varepsilon\big) \\
&\le (1/\varepsilon) e^{-\lambda\gamma/\varepsilon} \mathbf{E} e^{\lambda \alpha_1} \ \text{(Chernoff)} \\
&= (1/\varepsilon) e^{-\lambda\gamma/\varepsilon} \mathbf{E} e^{\lambda \sup_{t \le T} |V_t|}.
\end{aligned}$$

Then

$$\varepsilon \log \mathbf{P}\big(\varepsilon \sup_{t \le T} |V_{t/\varepsilon}| \ge \gamma\big) \le -\varepsilon \log \varepsilon - \lambda\gamma + \varepsilon \log \mathbf{E} e^{\lambda \sup_{t \le T} |V_t|}.$$

By virtue of Lemma 2.1 the right hand side of last inequality goes to $-\infty$ as limit " $\lim\limits_{\lambda \to \infty} \overline{\lim}_{\varepsilon \to 0}$" is taken.

In the second step the fact that the family $\varepsilon N_{t/\varepsilon}$, as well as $\Sigma \varepsilon N_{t/\varepsilon}$, is satisfied conditions (C.1) and (C.2) and Lemma 4.1 are used. Following them for any $\varphi \in \mathbf{C}$ and $T > 0$ we get

$$\begin{aligned}
&\overline{\lim}_{\delta \to 0} \overline{\lim}_{\varepsilon \to 0} \varepsilon \log \mathbf{P}\big(\sup_{t \le T} |\Sigma \varepsilon N_{t/\varepsilon} - \varphi_t| \le \delta\big) \\
&= \overline{\lim}_{\delta \to 0} \overline{\lim}_{\varepsilon \to 0} \varepsilon \log \mathbf{P}\big(\sup_{t \le T} |\sqrt{\varepsilon} W_{t/\varepsilon} - \varphi_t| \le \delta\big); \\
&= -\hat{I}_T(\varphi)
\end{aligned}$$

and

$$\begin{aligned}&\underline{\lim}_{\delta\to 0}\underline{\lim}_{\varepsilon\to 0}\varepsilon\log \mathbf{P}\big(\sup_{t\le T}|\Sigma\varepsilon N_{t/\varepsilon}-\varphi_t|\le\delta\big)\\ &=\underline{\lim}_{\delta\to 0}\underline{\lim}_{\varepsilon\to 0}\varepsilon\log \mathbf{P}\big(\sup_{t\le T}|\sqrt{\varepsilon}W_{t/\varepsilon}-\varphi_t|\le\delta\big).\\ &=-\hat{I}_T(\varphi),\end{aligned}$$

where, letting $\frac{0}{0}=0$,

$$\hat{I}_T(\varphi)=\begin{cases}\int_0^T H_{\sigma^2}(\frac{\dot{\varphi}_t}{\Sigma})dt, & \varphi_0=0\ d\varphi<<dt\\ \infty, & \text{otherwise.}\end{cases}\tag{4.2}$$

Thus by the method which has been used for proving Theorem 3.1 we get the l.d.p. for family $\sqrt{\varepsilon}W^\varepsilon_t$ in the metric space $(\mathbf{D},d)$ with the rate function

$$I(\varphi)=\sup_T \hat{I}_T(\varphi)$$

which coincides as such at (1.12).

On the other hand, since for each ε the paths of the process $\sqrt{\varepsilon}W^\varepsilon_t$ are continuous also the l.d.p. for this family holds in the metric space $(\mathbf{C},\rho)$ with the same rate function.

5. Proof of Theorem 1.2

1. To apply the contraction principle of Varadhan [11] a continuous mapping, serving $(\sqrt{\varepsilon}W^\varepsilon_s)_{t\ge 0}\Longrightarrow(X^\varepsilon_t)_{t\ge 0}$, has to be constructed.

Due to (1.3) consider differential equation

$$\dot{X}_t=a(X_t)+b(X_t)\dot{Y}_t,\tag{5.1}$$

subject to X_0, where $\dot{Y}_t$ is the Radon-Nikodym derivative of some absolutely continuous function Y_t from $\mathbf{C}$.

We show that the mapping $(Y_t)_{t\ge 0}\Longrightarrow(X_t)_{t\ge 0}$ defined by (5.1) can be extend for any function Y_t from $\mathbf{C}$.

Lemma 5.1. *Let functions $a(x)$ and $b(x)$ be Lipschitz continuous and there exist constant c and C such that*

$$0<c\le|b(x)|\le C.$$

Then for fixed X_0 there exists continuous in the metric ρ mapping

$$(Y_t)_{t\ge 0}\Longrightarrow(X_t)_{t\ge 0},\ (X_t),(Y_t)\in\mathbf{C}$$

such that for any absolutely continuous function (Y_t) this mapping is defined by differential equation (5.1).

Proof. Let

$$F(x) = \int_0^x \frac{dy}{b(y)}. \tag{5.2}$$

By virtue of assumptions making function $F(x)$ is continuous differentiable having inverse $F^{-1}(x)$ which is continuous differentiable too and both $F(x)$ and $F^{-1}(x)$ satisfy the linear growth condition: there exists positive constant ℓ such that

$$\left\{ \begin{array}{l} |F(x)| \\ |F^{-1}(x)| \end{array} \right. \leq \ell(1+|x|).$$

Put

$$g(x) = \frac{a(F^{-1}(x))}{b(F^{-1}(x))}. \tag{5.3}$$

Let Y_t be absolutely continuous function and X_t be a solution of (5.1). Put $\theta_t = F(X_t)$. It is easy to check that θ_t is a solution of a differential equation

$$\dot{\theta}_t = g(\theta_t) + \dot{Y}_t$$

subject to $\theta_0 = F(X_0)$ which is unique by virtue of the local lipschitzianity of $g(x)$. Then we arrive to a mapping defined by

$$\begin{aligned} X_t &= F^{-1}(\theta_t) \\ \theta_t &= F(X_0) + \int_0^t g(\theta_s)ds + [Y_t - Y_0]. \end{aligned} \tag{5.4}$$

Evidently this mapping takes place not only for absolutely continuous but also for continuous function Y_t. In fact $g(x)$, involving in the integral equation from (5.4), is satisfied the linear growth and the local Lipschitz conditions and so this integral equation obeys the unique solution also in case of continuous function Y_t.

Thus it remains to show that this mapping is continuous in the metric ρ. Note that the continuity in the metric ρ is equivalent to the following implication: for any $T > 0$

$$\left\{ \begin{array}{l} \lim_n \sup_{t\leq T} |Y^n_t - Y_t| = 0 \\ (Y^n_t), (Y_t) \in \mathbf{C}, n \geq 1 \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} \lim_n \sup_{t\leq T} |X^n_t - X_t| = 0 \\ (X^n_t), (X_t) \in \mathbf{C}, n \geq 1, \end{array} \right. \tag{5.5}$$

where $X^n_t, n \geq 1$ and X_t are solution of (5.4) corresponding to $Y^n_t, n \geq 1$ and Y_t respectively.

The validity of (5.5) also is implied by the local Lipschitz condition for function $g(x)$.

2. Taking $Y_t \equiv \sqrt{\varepsilon} W^\varepsilon_t$ we obtain $X_t \equiv X^\varepsilon_t$, i.e the contraction principle is applicable.

Then the family X^ε_t obeys the l.d.p. in the metric space $(\mathbf{C}, \rho)$ with rate function

$$J(\varphi) = \left\{ \begin{array}{ll} \inf I(\psi), & \varphi_0 = X_0, d\psi \ll dt \\ \infty, & \text{otherwise}, \end{array} \right.$$

where $I(\psi)$ is the rate function corresponding to the l.d.p. for $(\sqrt{\varepsilon}W_t^\varepsilon)$ and inf is taken over all absolutely continuous functions from $\mathbf{C}$ with $\varphi_0 = 0$ such that (see (5.1))

$$\dot{\varphi}_t = a(\varphi_t) + b(X_t)\dot{\psi}_t.$$

Since this equation has the unique solution the inf is attained on

$$\psi_t = \int_0^t \frac{\dot{\varphi}_s - a(\varphi_s)}{b(\varphi_s)} ds.$$

6. Application

From application point of view more realistic model than (1.4) for process η_t, involving in (1.3), is

$$\tilde{\eta}_t = \int_0^t h(t-s)dN_s, \tag{6.1}$$

where N_t and $h(t)$ are the same as in (1.4) (note that $\tilde{\eta}_t$ is not stationary process). Below we consider a such kind of process and show that nevertheless X^ε obeys the l.d.p. with the same rate function.

Let $\xi_t = (\xi_t^1, \xi_t^2, ..., \xi_t^n)$ be vector-column Ito's process

$$d\xi_t = A\xi_t dt + BdN_t, \ t \in R, \tag{6.2}$$

where A and B are matrices of dimensions $n \times n$ and $n \times 1$ respectively and also the eigenvalues of A belongs to the left half of the plane. Under assumption making

$$\begin{aligned}\xi_t &= \int_{-\infty}^t e^{A(t-s)} dN_s \\ &= e^{At}\xi_0 + \int_0^t E^{A(t-s)} dN_s \\ &= e^{At}\xi_0 + \tilde{\xi}_t. \end{aligned} \tag{6.3}$$

As a process $\tilde{\eta}_t$ take $\tilde{\xi}_t^1$ ($\tilde{\eta}_t \equiv \tilde{\xi}_t^1$). Following (6.3) we have for $\tilde{\eta}_t$ decomposition (6.1) type with

$$h(t) = \sum_{j=1}^n \left\{e^{At}\right\}_{1,j} B_j, \tag{6.4}$$

where $\left\{e^{At}\right\}_{1,j}, 1 \le j \le n$ are elements of the first row of matrix e^{At} and $B_j, 1 \le j \le n$ are elements of B.

Put

$$\tilde{W}_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}} \int_0^t \tilde{\eta}_{s/\varepsilon} ds \tag{6.5}$$

(comp. (1.5)). We show that the family $\sqrt{\varepsilon}\tilde{W}_t^\varepsilon$ obeys the l.d.p. in the metric space $(\mathbf{C},\rho)$ with the rate function $I(\varphi)$ given in (1.12) with $\Sigma=\int_0^\infty\sum_{j=1}^n\big\{e^{At}\big\}_{1,j}B_j$. To this end introduce

$$W_t^\varepsilon=\frac{1}{\sqrt{\varepsilon}}\int_0^t\xi_s^1ds. \tag{6.6}$$

Due to (6.3) $\xi_s^1=\int_{-\infty}^s h(s-u)dN_u$ with $h(t)$ from (6.4). Under assumptions making there exist positive constants c_1 and c_2 such that $|h(t)|\le c_1e^{c_2}$ and so Theorem 1.1 is applicable, i.e. $\sqrt{\varepsilon}W_t^\varepsilon$ obeys the l.d.p. with the mentioned above rate function.

Thus, the desired result for $\sqrt{\varepsilon}\tilde{W}_t^\varepsilon$ holds if for any $T>0$ and $\gamma>0$

$$\lim_{\varepsilon\to0}\log\mathbf{P}\big(\sup_{t\le T}|\sqrt{\varepsilon}(W_t^\varepsilon-\tilde{W}_t^\varepsilon)|>\gamma\big)=-\infty. \tag{6.7}$$

To check (6.7) note that under assumptions making $\int_0^\infty\sum_{j=1}^n\big|\big\{e^{As}\big\}_{1,j}\big|ds=c<\infty$. Then

$$\begin{aligned}|\sqrt{\varepsilon}(W_t^\varepsilon-\tilde{W}_t^\varepsilon)|&=\varepsilon\Big|\int_0^{t/\varepsilon}(\xi_s^1-\tilde{\xi}_s^1)ds\Big|\\&\le\varepsilon\int_0^\infty|\xi_s^1-\tilde{\xi}_s^1|ds\\&=\varepsilon\int_0^\infty\sum_{j=1}^n\big|\big\{e^{As}\big\}_{1,j}\xi_0^j\big|ds\\&\le c\varepsilon\sum_{j=1}^n|\xi_0^j|.\end{aligned} \tag{6.8}$$

The, using the Chernoff inequality, we find for any $\lambda>0$:

$$\begin{aligned}\mathbf{P}\big(\sup_{t\le T}|\sqrt{\varepsilon}(W_t^\varepsilon-\tilde{W}_t^\varepsilon)|>\gamma\big)&\le\mathbf{P}\Big(\sum_{j=1}^n|\xi_0^j|<\frac{\gamma}{c\varepsilon}\Big)\\&\le\sum_{j=1}^n\mathbf{P}\Big(|\xi_0^j|<\frac{\gamma}{nc\varepsilon}\Big)\\&\le n\max_{1\le j\le N}\Big\{\mathbf{P}\Big(|\xi_0^j|<\frac{\gamma}{nc\varepsilon}\Big)\Big\}\\&\le n\max_{1\le j\le N}\big\{e^{-\lambda\gamma/(nc\varepsilon)}\mathbf{E}e^{\lambda|\xi_0^j|}\big\}.\end{aligned}$$

In accordance to the last inequality the follwing upper estimate holds:

$$\begin{aligned}\varepsilon\log\mathbf{P}\big(\sup_{t\le T}|\sqrt{\varepsilon}(W_t^\varepsilon-\tilde{W}_t^\varepsilon)|>\gamma\big)&\le\varepsilon\log n-\lambda\gamma/(nc)+\max_{1\le j\le n}\big\{\varepsilon\log\mathbf{E}e^{\lambda|\xi_0^j|}\big\}\\&\le\varepsilon\log n-\lambda\gamma/(nc)+\sum_{j=1}^n\varepsilon\log\mathbf{E}e^{\lambda|\xi_0^j|}.\end{aligned} \tag{6.9}$$

Assume that

$$\mathbf{E}e^{\lambda\xi_0^j} < \infty, \ \forall\lambda \in R, \ j = 1, 2, ..., n. \tag{6.10}$$

Then, due to inequality $e^{|x|} \le e^x + e^{-x}$, we have for each $j = 1, 2, ..., n$ $\mathbf{E}e^{\lambda|\xi_0^j|} < \infty$ for all $\lambda > 0$. Consequently under assumption (6.10) the right hand side of (6.9) goes to $-\infty$ if limit " $\lim\limits_{\lambda\to\infty}\lim\limits_{\varepsilon\to 0}$ " is taken and so the desired property (6.7) takes place.

Therefore, only (6.10) has to be checked. From (6.3) it follows that $\xi_0 = \int_{-\infty}^0 e^{-As}BdN_s$ and by the homogeneity of increments of N_t random vector ξ_0 coinsides in the distribution with

$$\xi_0' = \int_0^\infty e^{As}BdN_s. \tag{6.11}$$

Let $\Lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ be a vector-row with $\lambda_j \in R, j = 1, 2, ..., n$. Putting

$$H(s) = \Lambda e^{As}B$$

analogously of proving (2.6) we get

$$\begin{aligned}\mathbf{E}e^{\Lambda\xi_0} &= \mathbf{E}e^{\Lambda\xi_0'} \\ &= \mathbf{E}\exp\Big\{\int_0^\infty H(s)dN_s\Big\} \\ &< \infty,\end{aligned}$$

i.e. (6.10) holds.

Thus, $\sqrt{\varepsilon}\tilde{W}_t^\varepsilon$ obeys the l.d.p. The l.d.p. for $\tilde{X}_t^\varepsilon$ follows from this result due to the contraction principle of Varadhan [11] and Lemma 5.1.

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