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Mathematical modeling of observed natural behavior: a fuzzy logic approach

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Abstract

Fuzzy modeling is routinely used to transform the knowledge of an expert, be it a physician or a process operator, into a mathematical model. The emphasis is on constructing a fuzzy expert system that replaces the human expert. In this paper, we advocate a different application of fuzzy modeling, namely, as a tool that can assist human observers in the difficult task of transforming their observations into a mathematical model. In many fields of science, including biology, psychology and economy, human observers have provided linguistic descriptions and explanations of various systems. However, to study these phenomena in a systematic manner, there is a need to construct a suitable mathematical model, a process that usually requires subtle mathematical understanding. Fuzzy modeling is a simple, direct, and natural approach for transforming the linguistic description into a mathematical model. Furthermore, fuzzy modeling offers a unique advantage—the close relationship between the linguistic description and the mathematical model can be used to verify the validity of the verbal explanation suggested by the observer. We demonstrate this using an example of territorial behavior of fish.

Keywords: Linguistic modeling; Fuzzy system models; Ethology

1. Introduction

Formal models play a crucial role in all fields of science. Rosen [20] claims that they are based on the belief that

> … causal sequences in the world of phenomena can be faithfully imaged by implication in the formal world of propositions describing these phenomena.

Thus, inference in a formal model parallels causality in the natural world.

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Mathematical models, defined by Casti [4] as a program or algorithm that encapsulates the order present in a given natural system, play a fundamental role in science. The derivation of a mathematical model forces the researcher to formalize his theory in a logical and deductive way. Once derived, mathematical models allow us to analyze the system qualitatively and quantitatively using mathematical tools, and to relate the theory with empirical data. Furthermore, we can use such a model for various ends:

- *Simulation*—studying the system's behavior by simulating the mathematical model. In particular, it is sometimes possible to simulate beyond what is feasible in the real world, for example, testing a controller when it might be dangerous to apply it to the real system.
- *Parameter influence*—checking how various parameters of the system influence its behavior.
- *Prediction*—studying the model in scenarios that were not observed in the real system provides testable predictions on the system's behavior (e.g., a future state, many generations ahead, in population dynamics).
- *Verification*—checking whether the stated linguistic description indeed yields the behavior that was actually observed.
- *Stimulation of further study*—useful models allow testing various hypotheses, stimulate new experiments, and suggest the existence of new phenomena.

Traditional approaches to mathematical modeling require advanced mathematical skills. This is why many mathematical models are actually derived by mathematicians rather than the actual observers, be it the ethologist studying animal behavior or the sociologist studying human behavior. However, even when the observer that studies the system cannot produce a mathematical model for it directly, she might be able to describe the system and its behavior *linguistically*. Thus, the question is how we can transform linguistic observations into mathematical entities.

Fuzzy modeling is the most effective approach for transforming linguistic data into mathematical formulas and vice versa. Indeed, Dubois et al. [6] state that the real power of fuzzy logic lies in its ability to combine *modeling* (constructing a function that accurately mimics given data) and *abstracting* (articulating knowledge from the data). [1]

The close relationship between the linguistic information and the fuzzy model offers many important advantages. A fuzzy model represents the real system in a form that corresponds closely to the way humans perceive it. Thus, the model is easily understandable, even by a non-professional audience, and each parameter has a readily perceivable meaning. The model can be easily altered to incorporate new phenomena, and if its behavior is different than expected, it is usually easy to find which rule/term should be modified and how. Furthermore, being a mature field, the mathematical procedures used in fuzzy modeling have been tried and tested many times, and the standard techniques are well documented.

Thus, fuzzy modeling can allow researchers in various fields to build mathematical models quickly, focusing on the 'what' instead of the 'how'. The resulting mathematical model allows the observer to enjoy all the benefits of a mathematical model and, in particular, can be used to prove or refute the modeler's ideas as to how the natural system behaves and why. This approach eliminates the

[1] This is well demonstrated in the computing with words version of Lyapunov synthesis developed in [16] (see also [17]).

artificial distance between the observer who studied the actual system and the mathematician who creates a suitable mathematical model for it.

To date, most of the applications of fuzzy modeling are based not on modeling real-world phenomena, but rather on transforming the knowledge of a human expert (be it a physician or a skilled operator of some process) into a fuzzy expert system. Namely, the goal is to replace the human expert with a computer.

We propose to utilize fuzzy modeling as a tool for assisting human observers in the difficult task of transforming their observations into mathematical models. Thus, the main novelty of this paper is in advocating a new application of fuzzy modeling.

The remainder of this paper is organized as follows. Section 2 describes several approaches for transforming linguistic observations to a mathematical model. Section 3 describes the fuzzy modeling approach. Section 4 provides a detailed example from the field of ethology. Section 5 concludes.

2. From linguistic observations to a mathematical model

In this section we discuss several known methods for creating a mathematical model from linguistic data.

During the 1950s, Forrester and his colleagues developed *system dynamics* [2] as a method for modeling the dynamic behavior of complex systems. The basic idea is to represent the causation structure of the system using *elementary structures* that include positive, negative, or combined positive and negative feedback loops. These are depicted graphically, and then transformed into a set of differential equations. The method was applied successfully to numerous real-world applications in social, economic, and industrial sciences. However, the inherent fuzziness and vagueness of the linguistic description are ignored and exact terms and phrases (e.g., a temperature of 30°C) are modeled in precisely the same way as fuzzy terms (e.g., warm weather).

A more systematic approach to modeling physical systems is *qualitative reasoning* (QR) [13] which allows transforming qualitative descriptions of a system into *qualitative differential equations*. These are generalizations of differential equations that include two main ingredients: (1) functional relationships between variables can be represented by functions that are either monotonically increasing or decreasing, but do not have to be completely specified, and (2) the values of the variables are described using a set of *landmark values* rather than exact numerical values. QR was applied to simulate and analyze many real-world systems. However, its applications seem to indicate that it is suitable when modeling with accurate, yet incomplete, knowledge rather than with linguistic knowledge.

Zadeh laid down the foundations of fuzzy sets and fuzzy logic and linked them to human linguistics [31,32]. Mamdani designed the first fuzzy controller based on a linguistic control protocol [15]. His work demonstrated the applicability of fuzzy expert systems and led the way to numerous applications (see, e.g., [7,24,25,30,23]). However, most of these applications are based not on modeling real-world phenomena, but rather on transforming the knowledge of a human expert into a fuzzy expert system. Thus, the expert may be replaced by a computer.

[2] To date, more than 30 books on *system dynamics* appeared. A very readable one is [19]. The journal *Systems Dynamics Review* contains up-to-date papers.

The pioneering work of Wenstop [28,27] was aimed at building *verbal models* that can represent and process *linguistic* information. The basic ingredients of this model include (1) *generative grammar*—used for defining the semantics of the linguistic statements; (2) *fuzzy logic based inferencing*—used in the deductive process; and (3) *linguistic approximation*—used for attaching linguistic labels to the outputs. These ingredients were implemented using the APL computer language so the entire process was automated. Wenstop [28] and Kickert [10,11] developed *verbal models* of several interesting systems from the social sciences. However, *verbal models* are not standard mathematical models as their input and output are linguistic values rather than numerical values. As such, *verbal models* suffer from two drawbacks. First, there are no methods for analyzing the behavior of such a model so it can only be used as a simulation tool. Second, when modeling dynamic systems the amount of fuzziness (or uncertainty) increases with every iteration. This implies that the system can be simulated effectively only for relatively short time spans.

Kosko [12] suggested *fuzzy cognitive maps* (FCMs) as a tool for the representation of causal relationships between various linguistic concepts. FCMs were used to model several interesting real-world phenomena (see [1] and the references therein). However, the inferencing process used in FCMs yields a discrete-time *linear* system in the form $\mathbf{x}(k+1)=A\mathbf{x}(k)$, which is clearly too simplified to faithfully depict many real-world systems.

3. The fuzzy modeling approach

Fuzzy modeling is the most suitable approach for constructing the mathematical model. It offers a unique advantage—the close relationship between the linguistic description and the resulting mathematical model can be used to verify the validity of the verbal explanations suggested by the observer.

The starting point in the fuzzy modeling approach is a complete linguistic description of the system, which should include the following:

1. *The* "*players*": For example, in a model from the field of ethology describing territorial behavior, the players might be two animals of the same species.
2. *The* "*environment*": The surrounding factors that influence the players' behavior and interaction. For example, in a model of humans reacting to a fire alarm this should include the size of the room they are in and the location and size of the exit.
3. The behavior of each player, that is, its reaction to the other players and the environment.
4. The overall patterns observed in the natural system as a result of each player's behavior and the interaction between the players and their environment. For example, in a model describing foraging ants, an observed pattern might be that eventually all the ants follow a trail from their nest to the *nearest* food source. This information is vital because it allows us to validate the mathematical model, once derived.

The linguistic information is transformed into a mathematical model using the following steps (see the detailed example in the following section):

1. *Identify the variables*: These include the players and the parts of their environment that affect the behavior of the system. For example if the model describes animals that move in

a 3D world, then the position of each player is a variable described by a vector with three coordinates.
2. Transform the description of the players' behavior into fuzzy rules stating the relations between the variables.
3. Define the fuzzy terms (logical operators) in the rules using suitable fuzzy membership functions (mathematical operators).
4. Transform the fuzzy rules into a mathematical model using fuzzy inferencing.

At this point we can analyze and simulate the mathematical model. Its suitability is determined, among other factors, by how well it mimics the patterns that were actually observed in the natural system.

When creating a mathematical model from linguistic data there are always numerous degrees of freedom. For example, one can use ordinary differential equations, partial differential equations, a model with or without time-delays, neglect or include "small" effects and so on.

In the fuzzy modeling approach, this is manifested in the freedom in choosing the ingredients of the fuzzy model: the type of membership functions, logical operators, inferencing method, and the values of the different parameters.

We now discuss several guidelines that can assist in selecting the different ingredients of the fuzzy model (see also [22] for details on how the different elements in the fuzzy model influence its behavior).

It is important that the resulting mathematical model will have a simple (as possible) form, and that it will be amenable to analysis. Thus, for example, a Takagi–Sugeno model with singleton consequents might be preferred to a model based on Zadeh's compositional rule of inference.

When modeling real-worlds systems, the variables are physical quantities with dimensions (e.g., length, time). *Dimensional analysis* [2,21], that is, the process of introducing dimensionless variables, can many times simplify the resulting equations and decrease the number of parameters. For example, suppose that the fuzzy model includes two parameters c_1 and c_2 (e.g., the centers of two membership functions). Applying dimensional analysis to the mathematical equations, we might find that these parameters occur only in the combination c_1/c_2, thus, we need to specify a single value for $k = c_1/c_2$, rather than two values for c_1 and c_2.

Finally, sometimes the linguistic description of the system is accompanied by qualitative data such as measurements of various quantities in the system. In this case, methods such as fuzzy clustering, neural learning, or least squares can be used to fine-tune the model using the discrepancy between the measurements and the model's output (see, e.g., [8,3,9] and the references therein).

4. An example: territorial behavior

There are at least two reasons for modeling the behavior of animals (and humans) using fuzzy sets and fuzzy logic. The first is that for many actions of animals the all-or-none law does not hold—the behavior itself is "fuzzy". For example, the response to a (low-intensity) stimulus might be what Heinroth called *intention movements*, that is, a slight indication of what the animal is tending to do.

Tinbergen [26, Chapter IV] states:

> As a rule, no sharp distinction is possible between intention movements and more complete responses; they form a continuum. [3]

The second reason is that researches studying animal behavior many times provide a *linguistic* description of both their observations and interpretations. Thus, we demonstrate the fuzzy modeling approach using an example from ethology.

Territory has a major role in social animal behavior [29] and results in a rich set of phenomena, but how is the territory created? Nobel Laureate Konrad Lorenz describes a specific example:

> … a real stickleback fight can be seen only when two males are kept together in a large tank where they are both building their nests. The fighting inclinations of a stickleback, at any given moment, are in direct proportion to his proximity to his nest. … The vanquished fish invariably flees homeward and the victor … chases the other furiously, far into its domain. The farther the victor goes from home, the more his courage ebbs, while that of the vanquished rises in proportion. Arrived in the precincts of his nest, the fugitive gains new strength, turns right about and dashes with gathering fury at his pursuer …. [14, p. 47]

Note that Lorenz provided us with a complete linguistic description including the players (the fish), the relevant factors in their environment (their nests), and their behavior and interaction. Furthermore, he also described the resulting patterns as observed in nature:

> The pursuit is repeated a few times in alternating directions, swinging back and forth like a pendulum which at last reaches a state of equilibrium at a certain point.

We demonstrate the fuzzy modeling approach using it to derive a mathematical model of this system. In the first step, we identify the state-variables: fish i, $i=1,2$, is located at $\mathbf{x}^i(t)\in\mathbb{R}^n$, and has a *fighting inclination* $w_i(t)\in\mathbb{R}$, and a nest located at $\mathbf{c}^i\in\mathbb{R}^n$.

Next, we transform Lorenz's description of the change in fighting inclination into the following fuzzy rules:

- if $near_i(\mathbf{x}^i,\mathbf{c}^i)$ then $\dot{w}_i=+1$,
- if $far_i(\mathbf{x}^i,\mathbf{c}^i)$ then $\dot{w}_i=-1$,

that is, the *fighting inclination* increases (decreases) when the fish is near (far) its nest.

Similarly, the description of the movement of fish i is transformed into:

- if $near_i(\mathbf{x}^i,\mathbf{x}^j)$ and $high_i(w_i)$ then $\dot{\mathbf{x}}^i=\mathbf{x}^j-\mathbf{x}^i$,
- if $near_i(\mathbf{x}^i,\mathbf{x}^j)$ and $low_i(w_i)$ then $\dot{\mathbf{x}}^i=\mathbf{c}^i-\mathbf{x}^i$,

where $\mathbf{x}^j$ is the location of the other fish. That is, when the other fish is near me, and my fighting inclination is high (low), I move in the direction of the other fish (my nest).

The third step is to determine the membership functions for the fuzzy terms. We define $near_i(\mathbf{x},\mathbf{y})$ using the membership function $n_i(\mathbf{x},\mathbf{y}) = \exp(-||\mathbf{x}-\mathbf{y}||^2/k_i^2)$ (the parameter $k_i>0$ determines the spread of the Gaussian function). The term $high_i$ is defined using $h_i(w) = \frac{1}{2}(1+\tanh(w/a_i))$ (the

[3] It is interesting to recall that Zadeh defined a fuzzy set as "a class of objects with a continuum of grades of membership" [31].

parameter $a_i > 0$ determines the slope of h_i). Note that $\lim_{w\to-\infty} h_i(w) = 0$ and $\lim_{w\to+\infty} h_i(w) = 1$. The opposite terms far_i and low_i are defined using $f_i(x, y) = 1 - n_i(x, y)$, and $l_i(w) = 1 - h_i(w)$, respectively.

Finally, using multiplication for the "and" operator, and center of gravity defuzzification, we obtain the mathematical model:

$$\dot{w}_i = 2\exp\left(-\frac{||\mathbf{x}^i - \mathbf{c}^i||^2}{k_i^2}\right) - 1,$$

$$\dot{\mathbf{x}}^i = \mathbf{c}^i - \mathbf{x}^i + h_i(w_i)(\mathbf{x}^j - \mathbf{c}^i) \tag{1}$$

for $i = 1, 2$.

This set of differential equations constitutes a complete mathematical model of the stickleback behavior described above, and can be used to simulate and analyze the system.

4.1. Simulations

We first simulated the one-dimensional case ($n = 1$), using the following parameters:

$$c^1 = -1, \quad c^2 = 1, \quad a_1 = a_2 = k_1 = k_2 = 1 \tag{2}$$

and initial values $x^1(0) = -0.4$, $x^2(0) = 0.8$, $w_1(0) = w_2(0) = 1$.

Fig. 1 depicts $x^1(t)$ and $x^2(t)$ as a function of t. It may be seen that the fish follow an oscillatory movement with one fish advancing, the other retreating until a point is reached where they switch roles. Finally, they converge to a steady-state point at $\bar{x}^1 = -0.1674$ and $\bar{x}^2 = 0.1674$.

We also simulated the three-dimensional case ($n = 3$), this being the one actually found in nature. Fig. 2 depicts the behavior of the model for the parameter values:

$$\mathbf{c}^1 = (-1, -1, -1)^{\mathrm{T}}, \quad \mathbf{c}^2 = (1, 1, 1)^{\mathrm{T}}, \quad a_1 = a_2 = k_1 = k_2 = 1$$

and initial values $\mathbf{x}^1(0) = (-0.5, 0.4, -1)^{\mathrm{T}}$, $\mathbf{x}^2(0) = (0.5, 1, 1)^{\mathrm{T}}$, and $w_1(0) = w_2(0) = 1$. It may be seen that again, we have an oscillatory movement converging to a steady-state point $\mathbf{x}^i(t) \to \bar{\mathbf{x}}^i$, $i = 1, 2$, with $\bar{\mathbf{x}}^i$ on the line connecting the two nests.

Note that the oscillatory movement of the fish in our fuzzy model and the behavior observed in nature, as described by Lorenz, are congruent.

4.2. Analysis

We analyze the case $n = 1$, that is—the fish live in a one-dimensional world, and the system is described by four state variables: x^1, x^2, w_1, and w_2. We assume, without loss of generality, that $c^2 = c$ and $c^1 = -c$, for some $c > 0$. In this case, it is easy to verify that (1) admits an equilibrium point:

$$\bar{x}^1 = -c + k_1\sqrt{\log 2},$$

$$\bar{x}^2 = c - k_2\sqrt{\log 2},$$

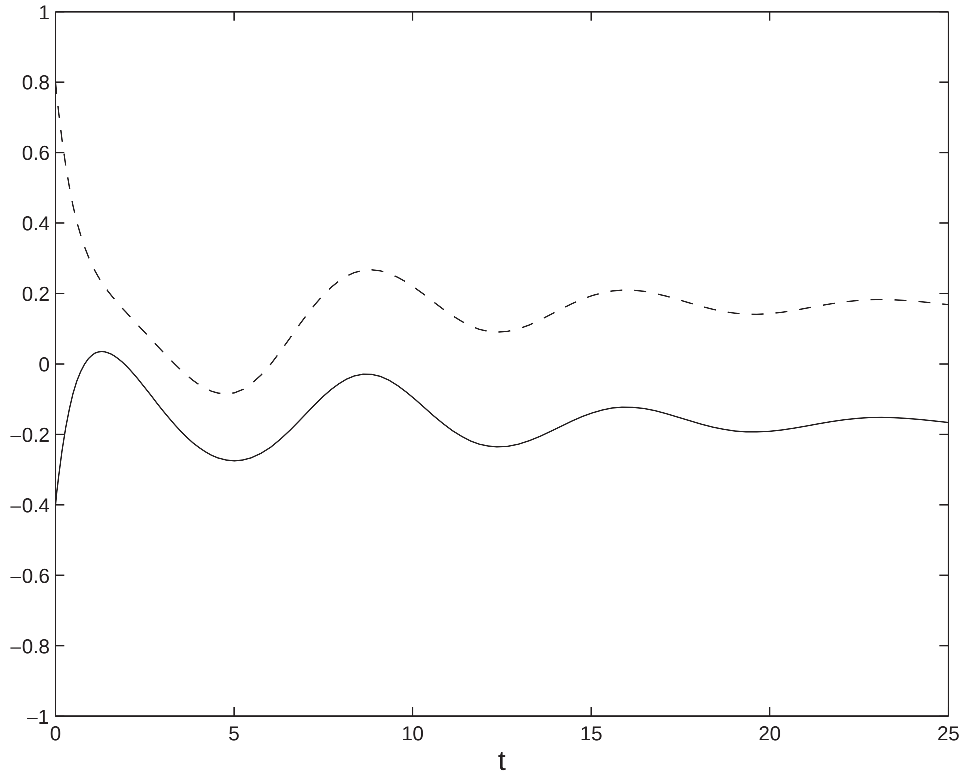

Fig. 1. The trajectories $x^1(t)$ (solid line) and $x^2(t)$ (dashed line) for initial positions $x^1(0) = -0.4$ and $x^2(0) = 0.8$. The two nests are at $c^1 = -1$ and $c^2 = 1$.

$$\bar{w}_1 = a_1 \tanh^{-1}\left(2\,\frac{\bar{x}^1 + c}{\bar{x}^2 + c} - 1\right),$$

$$\bar{w}_2 = a_2 \tanh^{-1}\left(2\,\frac{\bar{x}^2 - c}{\bar{x}^1 - c} - 1\right), \tag{3}$$

where $\tanh^{-1}$ denotes the inverse hyperbolic tangent function. We assume that the values of the parameters are chosen so that $-c < \bar{x}^1 < \bar{x}^2 < c$ (that is, the equilibrium point is between the two nests, with each fish closer to its nest than the other fish), which implies that the $\bar{w}_i$'s are well defined.

The next result shows that the arrangement (nest1, fish1, fish2, nest2) is an invariant of the mathematical model.

Proposition 1. *If* $-c < x^1(0) < x^2(0) < c$, *then*

$$-c \leqslant x^1(t) \leqslant x^2(t) \leqslant c \quad \text{for all } t \geqslant 0. \tag{4}$$

Proof. See the appendix. □

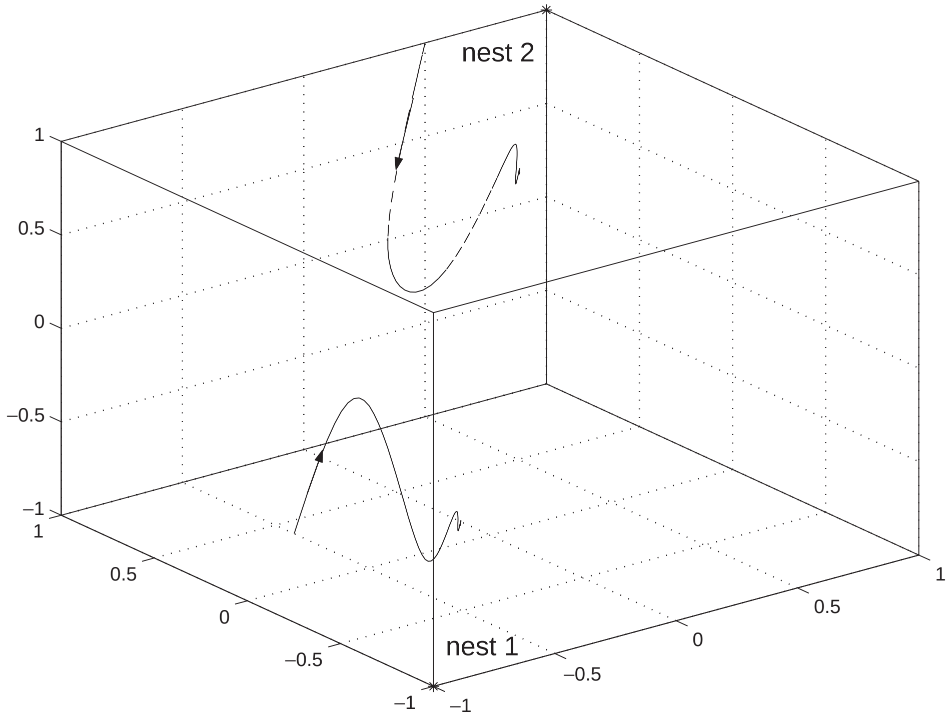

Fig. 2. The trajectories $\mathbf{x}^1(t)$ (solid line) and $\mathbf{x}^2(t)$ (dashed line) for initial positions $\mathbf{x}^1(0) = (-0.5, 0.4, -1)^T$, $\mathbf{x}^2(0) = (0.5, 1, 1)^T$. The nests are at $\mathbf{c}^1 = (-1, -1, -1)^T$ and $\mathbf{c}^2 = (1, 1, 1)^T$.

Proposition 2. *The equilibrium point* (3) *is locally asymptotically stable.*

Proof. See the appendix. □

It is possible, of course, to further analyze (1) using tools from the qualitative theory of ordinary differential equations. However, we will not pursue this further, as our goal is not to analyze this specific model, but to show how fuzzy modeling can be use to transform Lorenz's linguistically stated observations and insights into a form amenable to mathematical analysis.

4.3. Parameter influence

Once a mathematical model such as (1) is derived, we can easily study the effect of different parameters on the system's behavior.

Consider for example the case $n = 1$ and the parameter k_i which determines the spread of the Gaussian function defining the term $near_i$. As k_i decreases, the Gaussian function becomes narrower so the rule:

- if $near_i(\mathbf{x}^i, \mathbf{c}^i)$ then $\dot{w}_i = +1$

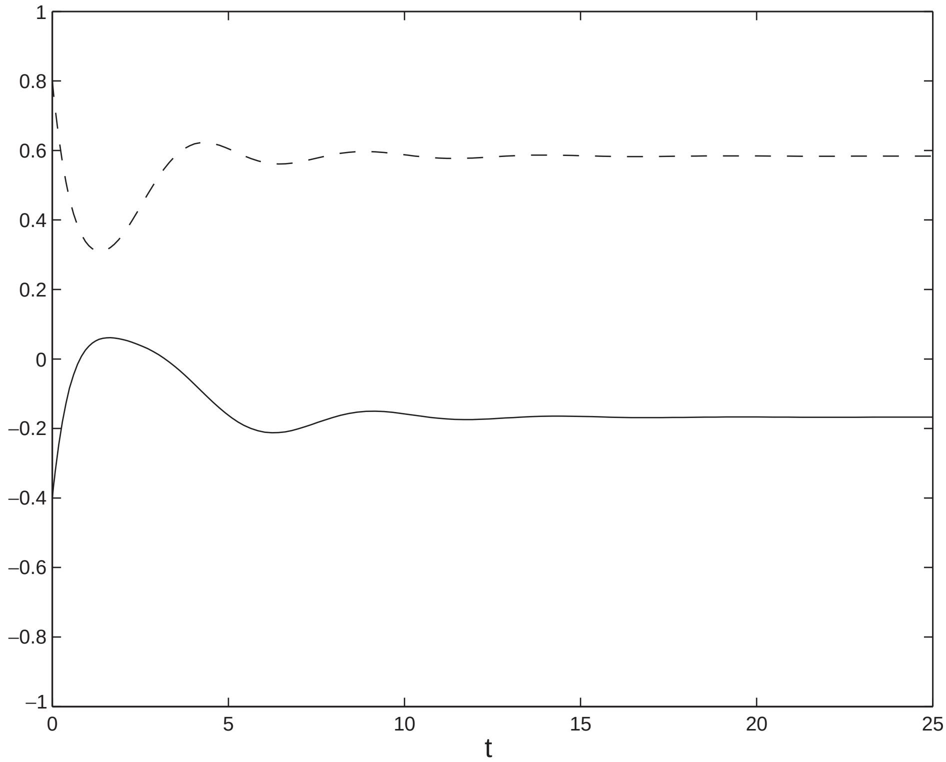

Fig. 3. The trajectories $x^1(t)$ (solid line) and $x^2(t)$ (dashed line) for $k_1 = 1$, $k_2 = 0.5$. The two nests are at $c^1 = -1$ and $c^2 = 1$.

will fire only when $\mathbf{x}^i$ is very close to $\mathbf{c}^i$. In other words, the fighting inclination will begin to increase only very near to the nest. Thus, we expect the fish to have a somewhat lower "fighting potential".

Fig. 3 depicts $x^1(t)$ and $x^2(t)$ for the same values as in (2) except that we decreased k_2 from 1.0 to 0.5. It may be seen that in this case the convergence to the steady state is faster and that $x_1(t) \to -0.1674$ (as before) but $x^2(t) \to 0.5387$, that is, the equilibrium point is no longer symmetrical. Indeed, it follows from (3) that when k_2 decreases, $\bar{x}^2$ increases. If we define the territory of fish 2 as $[\bar{x}^2, c^2]$, then decreasing k_2 yields a decrease in the size of the territory.

In fact, this too is congruent with the behavior observed in nature:

> ... the relative fighting potential of the individual is shown by the size of the territory which he keeps clear of rivals. [14]

4.4. Prediction

We can also use the mathematical model, such as (1), to analyze and simulate new scenarios that were not described by the observer. The results can be regarded as predictions of the behavior of the real system.

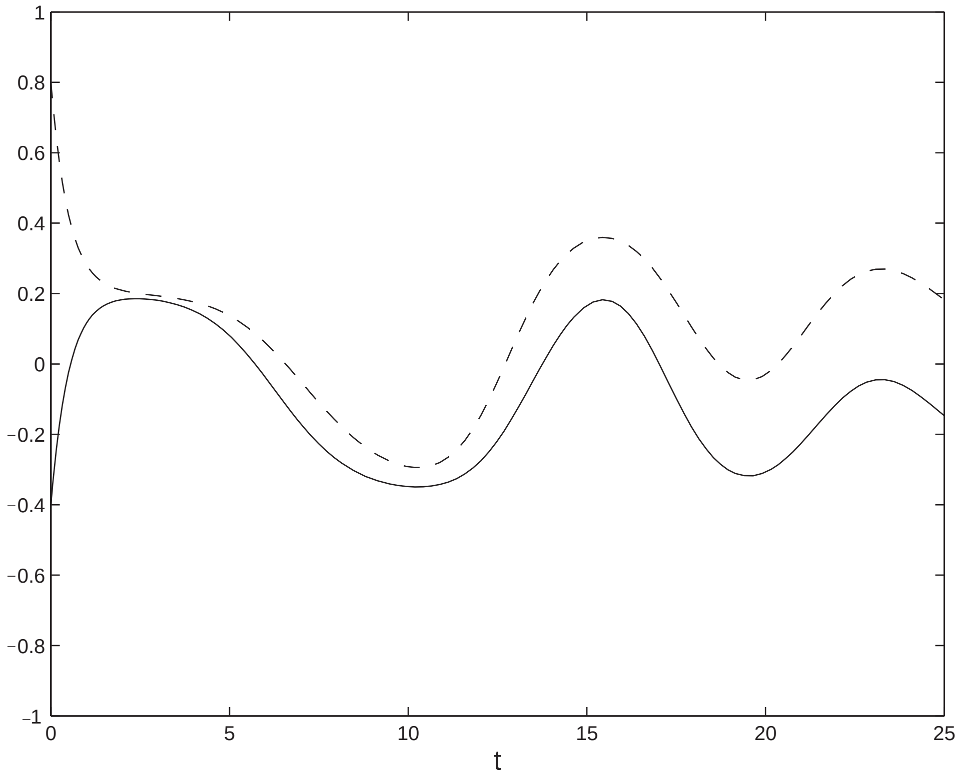

Fig. 4. The trajectories $x^1(t)$ (solid line) and $x^2(t)$ (dashed line) for $w_1(0) = w_2(0) = 3$.

For example, suppose that before the fish are placed in the aquarium, they are "irritated" in a form that increases their fighting inclinations. How will this affect their mutual behavior? We can simulate this scenario in our model by increasing the values $w_i(0)$, $i = 1, 2$.

Fig. 4 depicts $x^1(t)$ and $x^2(t)$ for the same parameters as in Section 4.1 but with $w_1(0) = w_2(0) = 3$ instead of $w_1(0) = w_2(0) = 1$. Comparing this with Fig. 1, we can see that now the fish come closer together before they eventually retreat, and that the amplitude of their oscillations is increased.

Thus, the model provides a prediction of the outcome of new experiments, as well as stimulus for further study.

5. Concluding remarks

Numerous real-world phenomena were described and explained linguistically by human observers. The derivation of suitable mathematical models is a necessary step in order to study these phenomena in a systematic manner. The resulting model can be used to analyze, simulate, test the influence of parameters, and predict the behavior of the system.

In this paper we pointed out that fuzzy modeling is the most suitable approach for constructing the mathematical model. Furthermore, fuzzy modeling offers a unique advantage—the close relationship between the linguistic description and the mathematical model can be used to verify the validity of the verbal explanation suggested by the observer.

To demonstrate these ideas, we used Lorenz's linguistic description of the mechanisms behind territorial behavior in fish. Using fuzzy modeling, we transformed this linguistic description into a mathematical model. The simulations and the mathematical analysis of the model are congruent with the behavior observed in nature, and suggest that the insights in the verbal description are indeed correct.

We believe that this approach can be utilized in many fields of science, including biology, economics, psychology, sociology and more. In such fields, many linguistic models exist in the research literature, and they can be directly transformed into mathematical models using the method described herein. Among other examples currently under study are the flocking patterns of birds, and the behavior of human drivers in congested highways.

An interesting research topic is the implementation of fuzzy modeling tools in existing simulation environments (e.g., *StarLogo* [5]) allowing the observer to intuitively program, simulate, and refine her *verbal* model. This can make fuzzy modeling more accessible to researchers in many fields of science.

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Appendix A.

Proof of Proposition 1. Seeking a contradiction, let $\tau > 0$ be the first time for which (4) is false. Thus, either (1) $x^2(\tau) > c$; or (2) $x^1(\tau) < -c$; or (3) $x^2(\tau) < x^1(\tau)$. If case (1) holds, then there exists a time $\tau' < \tau$ such that $x^1(\tau') \leqslant x^2(\tau') = c$, but then $\dot{x}^2(\tau') = h_2(w_2(\tau'))(x^1(\tau') - c) \leqslant 0$ so we cannot get $x^2(\tau) > c$. Similarly, the other two cases also yield a contradiction. □

Proof of Proposition 2. Defining new variables $y^1 = (x^1 - \bar{x}^1)/k_1$, $y^2 = (\bar{x}^2 - x^2)/k_2$, and $v_i = w_i - \bar{w}_i$, $i = 1, 2$, we get

$$\dot{v}_1 = 2\exp(-(y^1 + \sqrt{\log 2})^2) - 1,$$

$$\dot{v}_2 = 2\exp(-(y^2 + \sqrt{\log 2})^2) - 1,$$

$$\dot{y}^1 = -y^1 - g_1(v_1 + \bar{w}_1)y^2 + (g_1(v_1 + \bar{w}_1) - g_1(\bar{w}_1))(\alpha - \sqrt{\log 2}),$$

$$\dot{y}^2 = -y^2 - g_2(v_2 + \bar{w}_2)y^1 + (g_2(v_2 + \bar{w}_2) - g_2(\bar{w}_2))(\beta - \sqrt{\log 2}), \tag{A.1}$$

where $\alpha = 2c/k_2$, $\beta = 2c/k_1$, $k = k_2/k_1$, $g_1(x) = kh_1(x)$, and $g_2(x) = h_2(x)/k$.

Denoting $\mathbf{m} = (v_1, v_2, y^1, y^2)^{\mathrm{T}}$, and linearizing (A.1) around the equilibrium point $\mathbf{0}$, we get $\dot{\mathbf{m}} = A\mathbf{m}$, with

$$A = \begin{pmatrix} 0 & 0 & -2\sqrt{\log 2} & 0 \\ 0 & 0 & 0 & -2\sqrt{\log 2} \\ \sqrt{\log 2}t_1/a_1 & 0 & -1 & k(t_1-2)/2 \\ 0 & \sqrt{\log 2}t_2/a_2 & (t_2-2)/(2k) & -1 \end{pmatrix},$$

where $t_i = 1 - \tanh(\overline{w_i}/a_i)$. Note that $t_i \in (0,2)$.

Calculating the characteristic polynomial $P(s) = \det(sI - A)$, we get

$$P(s) = s^4 + 2s^3 + \left(\frac{t_1}{2} + \frac{t_2}{2} - \frac{t_1 t_2}{4} + \frac{2t_1 \log 2}{a_1} + \frac{2t_2 \log 2}{a_2}\right) s^2 + \left(\frac{2t_1 \log 2}{a_1} + \frac{2t_2 \log 2}{a_2}\right) s + \frac{4t_1 t_2 \log^2 2}{a_1 a_2}$$

and the first column of the Routh–Hurwitz array (see [18] and the references therein) of $P(s)$ is

$$1, 2, \frac{t_1}{2} + \frac{t_2}{2} - \frac{t_1 t_2}{4} + \frac{t_1 \log 16}{4a_1} + \frac{t_2 \log 16}{4a_2}, \frac{num}{den}, 4\frac{t_1 t_2 \log^2 2}{a_1 a_2},$$

where $den = t_1/2 + t_2/2 - t_1 t_2/4 + t_1 \log 2/a_1 + t_2 \log 2/a_2$ and $num = 2\log 2((t_1/a_1 - t_2/a_2)^2 \log 2 + (t_1/a_1 + t_2/a_2)(t_1/2 + t_2/2 - t_1 t_2/4))$. It is easy to see that all these numbers, as well as all the coefficients of $P(s)$, are strictly positive, so we conclude that $\dot{\mathbf{m}} = A\mathbf{m}$ is asymptotically stable. Thus, the origin is a locally asymptotically stable equilibrium point of (A.1). □

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