Multi-Objective Zero-Sum Games with Postponed Objective Preferences

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Abstract—The Multi-Objective Game (MOG) considered here involves a lack of knowledge on the part of both players regarding the objective preferences of the opponent. In addition, each player is assumed to postpone its own objective preferences. In contrast to the commonly used utility approach, in this case we propose to apply multi-objective optimization methods that are suited for the current MOG problem. In particular, this paper applies a worst-case domination relation among the payoff vectors. Moreover, the rationality of the players under postponed objective preferences is discussed and defined. The proposed solution approach results in the rational strategies under postponed objective preferences. This allows the players to further examine their alternative strategies before objective preferences are eventually made. Hence, this paper proposes a base for future work on decision support systems for players of such MOGs.

I. INTRODUCTION

In conventional game theory, the usual assumption is that decision-makers (players) make their decisions based on a scalar payoff. But in many practical problems in the fields of economics and engineering, decision-makers must cope with multiple objectives or payoffs. Furthermore, these objectives may be contradicting.

When a non-scalar payoff is considered, the game is commonly referred to as a Multi-Objective Game (MOG). If players in a MOG are able to specify their objective preferences, a utility function (usually a weighted sum) can be used to transform the MOG into a game with a single objective.

The case of a utility-based two persons, zero-sum MOG has been studied by several researchers, e.g., [1]–[4]. In contrast to MOG studies such as in [1]–[12], which involve complete information, our work considers a situation in which each player knows the payoff vectors of both sides, but postpones its objective preferences. This constitutes a case of incomplete information because the utilities are unknown. It should be noted that the lack of information in the current MOG study is about utilities and not about payoff functions as in [13].

The MOG defined here aims to support decision-making i.e., support the decision maker to eventually pick a solution. It is further noted that the assumption of a lack of information regarding the opponent’s objective preferences also holds for the decision-making stage. This means that it is assumed that the decision maker will have no knowledge on the opponent’s objective preferences at any stage of the decision-making process.

In studies on MOGs, selecting a strategy is based on one out of three different representation types of the strategy’s set of performance vectors. Namely, either by a scalar, a unique vector in the objective space, or by a sub-set in the same objective space. In most studies on MOGs, such as in [1]–[13], solving the problem involves the aggregation of payoffs by a weighted sum approach. In such studies, strategy representation belongs to the first type. Studies, using the second and third types of strategy representation, are rare. In works [5], [14], [15] strategy representation belongs to the second type. In these studies the unique representation vector is the nadir point. To the best of our knowledge, only [16] and the current work employ the third type of strategy representation. These two works employ a set-based worst-case optimization approach. The main difference between the current paper and [16] is that in [16] the game is solved by finding the optimal strategies from the perspective of one player only. In contrast, here the MOG with postponed objective preferences is considered from the perspectives of both players. In addition, the current paper provides an answer to the non-trivial question of how a rational player, who has no knowledge of the opponent’s preferences toward objectives and has made a decision to postpone preferences of objectives, should be defined.

II. DEFINITIONS AND NOTATIONS

A. Multi-Objective Optimization

In multi-objective optimization problems for which the objectives are conflicting, there is no universally accepted definition of an ‘optimum’ as in a single-objective optimization [17]. Pareto optimality, which is based on a domination relation (e.g., in [18]) simultaneously optimizes all the K objective
functions $F(z) = [f^{(1)}(z), \ldots, f^{(K)}(z)]^T \in \mathbb{R}^K$ subject to $z = [z_1, \ldots, z_q, \ldots, z_Q]^T \in \mathcal{Y} \subseteq \mathbb{R}^Q$, where $\mathcal{Y}$ is the feasible design space. Without loss of generality, the concept of domination for a minimization problem can be defined as follows: A vector $z \in \mathcal{Y}$ dominates a vector $z' \in \mathcal{Y}$ ($z \succ z'$), if $f^{(k)}(z) \leq f^{(k)}(z')$ for all $k \in [1, K]$ and there exists $k \in [1, K]$ for which $f^{(k)}(z) < f^{(k)}(z')$. A vector $z^* \in \mathcal{Y}$ is called Pareto optimal if no solution exists in $\mathcal{Y}$ that dominates $z^*$. The set of Pareto optimal solutions is the Pareto set $Z^* := \{z^* \in \mathcal{Y} : \neg \exists z' \in \mathcal{Y} : z' \succ z^*\}$. Its image, $F^* = F(Z^*)$, is called the Pareto front.

B. Domination Relations Among Sets

In the current paper domination relations among sets are used to compare strategies. Such a set comparison is not new. The following section provides the necessary background and notations that will be used here. For domination in a minimization problem, if a vector $y \in \mathcal{Y}$ dominates a vector $x \in \mathcal{Y}$, the relation among the vectors is denoted as: $x \succ y$ and for a maximization problem, if a vector $x \in \mathcal{Y}$ dominates a vector $y \in \mathcal{Y}$, the relation among the vectors is denoted as: $x \succ y$.

These relations are extended to define domination relations among sets as follows. Let $X = \{x_1, x_2, \ldots, x_m\} \in \mathcal{Y}$ and $Y = \{y_1, y_2, \ldots, y_n\} \in \mathcal{Y}$ be two sets. Comparing between two solutions represented by the two sets is based on the definition of domination among sets given in [19]. A set $X$ dominates a set $Y$ in a minimization problem: $X \succ Y := \{\forall y \in Y \exists x \in X \mid x \succ y\}$. In the same way, if the set $X$ dominates the set $Y$ in a maximization problem, the $\succ$ notation should be replaced by $\succ$. According to [20] a set $X^* \in \mathcal{Y}$ is labeled Pareto optimal if no other set exists in $\mathcal{Y}$ that dominates $X^*$. The set of all Pareto optimal sets is the Pareto set of sets, see [20]. In the case of minimization the set is defined as: $X^* := \{X \in \mathcal{Y} : \neg \exists Y \in \mathcal{Y} : Y \succ X\}$. In the case of maximization, the same definition will be used but $\succ$ will be replaced by $\succ$. The image of the set $X^*$ is $F(X^*)$, which is termed, in [20], Pareto layer.

C. Domination Relations in Worst-Case Optimization

Worst-case domination is a particular type of a relation among sets used in this study. In [21] and [22] worst-case optimization is utilized for finding a robust solution to a MOP that involves uncertainties. These two studies proposed a definition of a worst-case domination relation based on worst-case considerations.

In a minimization problem, $X \succ_{wc} Y$ means $X$ worst-case dominates set $Y$, indicating that $Y \succ X$. In a maximization problem, $X \succ_{wc} Y$ (set $X$ worst-case dominates set $Y$), means $Y \succ_{wc} X$. Without loss of generality, the concept of worst-case domination for a minimization problem can be defined as: A set $X \in \mathcal{Y}$ worst-case dominates a set $Y \in \mathcal{Y}$ ($X \succ_{wc} Y$), iff $X \succ_{wc} Y := \{\forall y \in Y \exists x \in X \mid x \succ y\}$. 

D. Rationality in Single Objective Games

According to [23], "A player is rational if his strategy choice maximizes his expected payoff." Such a definition implicitly assumes that the player will consider the opponent’s strategies when choosing its action. It is important to note that the above definition refers to a scalar utility or payoff. A more elaborate description is provided in [24], which suggests that a "decision-maker is rational" in the sense that it is aware of its alternatives, forms expectations about any unknowns, has clear preferences, and chooses his action deliberately after some process of optimization." A similar definition is given in [25] as follows: "Rationality simply means playing a strategy which is optimal with respect to the player’s belief." Such definitions indicate that the players determine the optimal outcome of a game from their viewpoint and from their belief about the opponent’s strategy. However, none of the above refers to MOGs. An intuitive approach to extend the definition of a rational player, as given in [23], is to simply change it from "expected payoff" to "expected payoffs." Yet one should realize that there is no one way to compare vectors of payoffs; hence, such a definition of rationality and optimality is ill-defined for MOGs. This issue is further discussed in Section VII, where an alternative definition is provided.

E. A Game of Incomplete Information

A game is said to be of incomplete information if the players, or at least some of them, lack full information about the basic mathematical structure of the game, [13], [26]. Usually this lack of information can be reduced to the case where the players have less than full information about each other’s payoff functions, [26], [27]. Here we consider lack of information on the utilities rather than on the payoff functions.

III. THE BASIC ASSUMPTIONS

The assumptions used in this study are summarized as follows:

1) There are two players, $P_1$ and $P_2$.
2) The game involves $K$ contradicting objectives (per player).
3) The game is zero-sum with respect to each of the objectives.
4) The game is pure strategy, meaning that no probability is assigned to the selection of strategies.
5) The game is single-act. That is, once a strategy is chosen, it is not altered during the game.
6) The players postpone making the decision regarding their preferences of objectives.
7) The game is non-cooperative, meaning that no agreement is made between the players.
8) The game is of imperfect information. This means that each player is unaware of the opponent’s chosen strategy.
9) The players do not and will not know their opponents’ preferences.
10) The game is of incomplete information, meaning that although each player knows both the available strategies and the corresponding payoff vectors of the opponent,
the players don’t know what the opponent’s objective preferences are or will be.

IV. MULTI-OBJECTIVE GAMES

In this study, a game between two players, $P_1$ and $P_2$, is considered. Each player may choose a strategy ($s_1^i$ for $P_1$ or $s_2^j$ for $P_2$) out of a set of strategies. Each strategy is a decision vector.

Let $s_1^i = [s_1^{i(1)}, \ldots, s_1^{i(n_1)}, \ldots, s_1^{i(N_1)}]^T \in \mathbb{R}^{N_1}$ and $s_2^j = [s_2^{j(1)}, \ldots, s_2^{j(n_2)}, \ldots, s_2^{j(N_2)}]^T \in \mathbb{R}^{N_2}$, where $N_1$, and $N_2$ are the number of strategy parameters for players $P_1$ and $P_2$ respectively.

Let $S_1$ and $S_2$ be the sets of all possible $I$ and $J$ strategies for these players respectively, such that: $S_1 = \{s_1^1, s_1^2, \ldots, s_1^i, \ldots, s_1^I\}$, $S_2 = \{s_2^1, s_2^2, \ldots, s_2^j, \ldots, s_2^J\}$, where $S_1 \subseteq \Omega \subseteq \mathbb{R}^{I \times N_1}$ and $S_2 \subseteq \Gamma \subseteq \mathbb{R}^{J \times N_2}$.

The interaction between the $i$-th strategy played by $P_1$ and the $j$-th strategy played by $P_2$ results in a game:

$$g_{i,j} := \{s_1^i, s_2^j\} \in \Phi \subseteq \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \quad (1)$$

All of the alternative interactions, among all strategies of both players, form the set of all possible games $G$:

$$G := \{g_{i,j} \in \Phi \mid s_1^i \in S_1 \land s_2^j \in S_2\} \quad (2)$$

Here, players choose their strategies based on the assessment of all possible games’ results. Each game is evaluated using an objective vector of performances (payoff vector), $\bar{f} = [f^{(1)}, f^{(2)}, \ldots, f^{(K)}]^T \in \Psi \subseteq \mathbb{R}^K$. More specifically, the result of a game between player $P_1$ using the $i$-th strategy and player $P_2$ using the $j$-th strategy, is assessed by:

$$F(g_{i,j}) = \bar{f}_{i,j} = [f^{(1)}_{i,j}, f^{(2)}_{i,j}, \ldots, f^{(K)}_{i,j}]^T \in \Psi \quad (3)$$

V. DOMINATION RELATIONS IN MOGS

Based on the definition of a multi-objective optimization problem in [16], a general MOG from the minimizer or the maximizer viewpoint can be evaluated as a minimization or a maximization problem, respectively. Formally:

$$\min \text{ or } \max F(g_{i,j}) \quad \text{subject to } g_{i,j} \in G \quad (4)$$

Without loss of generality, let us assume a maximization problem and consider two games $g_{i,j}, g_{i',j'} \in G$. Game $g_{i,j}$ is said to dominate $g_{i',j'}$ (abbreviated as $g_{i,j} \succ g_{i',j'}$) iff:

$$\forall p \in [1,K] : f^{(p)}_{i,j} \geq f^{(p)}_{i',j'} \land \exists q \in [1,K] : f^{(q)}_{i,j} > f^{(q)}_{i',j'} \quad (5)$$

VI. PROBLEM FORMULATION

A game with a non-dominated payoff vector should be considered by both the maximizer and the minimizer. However, each player must take into account the alternatives of the opponent. This leads to the need to consider two different optimization problems. The first problem is based on the minimizer ($P_1$) perspective, which aims at minimizing the objectives while considering the best opponent (the maximizer) strategies (that are aimed at maximizing the objectives). This problem can be defined as a MinMax problem. The second problem is based on the maximizer ($P_2$) perspective, which aims at maximizing the objectives while considering the best opponent (the minimizer) strategies. This leads to a MaxMin problem. The problems are defined in Equations 6 and 7.

For $P_1$ : $\min \max \bar{f}_{i,j}$ \quad (6)

For $P_2$ : $\max \min \bar{f}_{i,j}$ \quad (7)

In both cases, the problem can be defined as a worst-case optimization problem because each player considers the best performances of the opponent, which in the case of a zero-sum game are its own worst-cases.

VII. RATIONALITY IN MOGS WITH POSTPONED OBJECTIVE PREFERENCES

In a worst-case approach, the goal of each player is to minimize its loss instead of maximizing its gain in a game against an unpredictable opponent. This means that by adopting the worst-case optimization approach the player aims at securing its gain against any behavior of the opponent. The worst-case presumption is the result of basic assumptions 8-10 in Section III.

In such a game the player does not know what strategy the opponent is going to choose (assumption 8), and although the player knows the opponent’s payoff it does not know the opponent’s preferences (assumption 9), i.e., the player does not know how the opponent evaluates the payoff vector (assumption 10).

Consequently, in a MOG with postponed objective preferences, if the decision makers are “conservative” and do not wish to take any risks, they must use the worst-case set based approach as detailed in [21]. A particular definition for such a rational player in a MOG is given as follows:

Definition 1: A player in a MOG with postponed objective preferences is said to be rational in a worst-case sense if it seeks to play in a manner which does not result in a worst-case dominated payoff vector from its own viewpoint.

The solutions of the problems defined in Equation 6 and Equation 7 are the worst-case rational strategies of the minimizer and the maximizer, respectively. These are hereby termed in short, rational strategies.

VIII. SOLUTION PROCEDURE

The solution procedure is outlined in this section using an introductory example of a bi-objective MOG between two players, $P_1$ (minimizer) and $P_2$ (maximizer). The game is taking place in an arena as depicted in Figure 1.

The arena includes three rivers (gray stripes), two roads (white stripes) that cross the rivers on six bridges (in black) and three guarding posts (stars), each located at a dam (not shown). The first player $P_1$ is an agent that has to cross the
three rivers from the starting point and reach the target. The two strategies of the first player are choosing either the first or the second road. This player is interested in reaching the target as fast as possible, as well as to stay as far as possible from the guard (second player) in order to reduce the risk of being detected.

The second player $P_2$ is the guard. This player may choose one of the three guarding posts. The first goal of the guard is to cause the maximal delay to the first player. At each post there is a switch that allows flooding the adjacent river by opening the associated dam. The flooding of the river is expected to cause a different level of damage to the associated bridges. This means that different delays are expected to be inflicted on the first player, according to the selected road. The second goal of the guard is to increase the likelihood of detecting the first player. This likelihood depends on the distance of the post from the road at which the first player will be moving on.

In summary, the two payoffs of this MOG are the distance $f^{(1)} = -\text{Distance}$ and the time $f^{(2)} = \text{Time}$.

As the first and second player have two and three different strategies respectively ($I = 2$ and $J = 3$). There are six possible interactions (all possible combinations $I \times J = 6$). The results of these interaction are depicted in Table I.

Figure 2 describes the results (the vectors in Table I) in the objective space. The different strategies of $P_1$ are represented by different shapes: triangle for $s_1^1$ (road 1) and square for $s_1^2$ (road 2). The different strategies of $P_2$ are represented by different fillings: white for $s_2^1$ gray for $s_2^2$ and black for $s_2^3$ (guarding post 1, 2 and 3 respectively). A combination between a shape and a filling represents a game; e.g., in the introductory example, the result of the game between strategies $s_1^1$ (road 2) and $s_2^3$ (guarding post 3) is denoted as a black square labelled by (2,3).

In considering the $i$-th strategy of $P_1$, there are $J$ finite possible strategies of $P_2$ to consider. All possible games that involve the $i$-th $P_1$ strategy form a set of $J$ possible games associated with the $P_1$ $i$-th strategy: $G_{s_1} := \{g_{i,1}, g_{i,2}, \ldots, g_{i,J}\} \subset G$. For example, in Figure 2, $G_{s_1}^2$ is the set of the three squares (labelled by (2,1), (2,2) and (2,3)). These are the three possible results of selecting road 2.

$P_2$ aims at maximizing the objectives ($f^{(1)}$ and $f^{(2)}$) while playing with the $P_1$ $i$-th strategy. This means that $P_2$ aims at:

$$\max_{s_2 \in S_2} f_{i,j} \quad \forall j \in [1, J]$$

(8)

For the $P_1$ $i$-th strategy, a set of $P_2$ strategies exist that may serve as solutions to equation 8. As the maximizer, $P_2$ aims at the worst-case for the minimizer, $P_1$. These strategies form the $i$-th anti-optimal set of strategies $C_{s_1}^{-*}$. Here, the sign $-*$ is used to emphasize the fact that this set is the optimal set of the maximization problem and therefore is anti-optimal from the minimizer’s viewpoint.

$$C_{s_1}^{-*} := \{s_2^j \in S_2 \mid -\exists s_2^j, j' \in [1, J] : g_{i,j'}^{\text{max}} > g_{i,j}\} \quad \forall j \in [1, J]$$

(9)

Note that according to the definition in [21] and [22], the anti-optimal fronts of $P_1$ are the worst cases of $P_1$.

Mapping this anti-optimal set to the objective space forms the $i$-th anti-optimal front of $P_1$, $F_{s_1}^{-*}$ where:

$$F_{s_1}^{-*} := \{f_{i,j} \in \Psi \mid s_2^j \in C_{s_1}^{-*}\}$$

(10)

All the anti-optimal fronts $F_{s_1}^{-*}$ in the introductory example of Figure 2 are depicted in Figure 3, encircled by dashed
The set of rational strategies of $P_1$ includes all $s_1$'s strategy that are associated with a dominating anti-optimal front in the maximization problem.

The set of rational strategies of $P_1$ is the relative complement of $SC_1$ and $C_{1^{w.c}}$:

$$C_1^R = SC_1 - C_{1^{w.c}}$$

The set $C_1^R$ includes all $P_1$'s strategy that are associated with a non-dominating anti-optimal front in the maximization problem.

Each of the rational strategies is represented in the objective space by its related anti-optimal front, which is termed as the front of the rational strategies:

$$F_{s_1^1}^R := \{ \mathbf{f} | s_1 \in C_{s_1^1}^- \wedge C_{s_1^1}^- \in C_1^R \}$$

The given example includes two rational strategies for the minimizer. This means that any of the roads is a rational choice for $P_1$. None of the two anti-optimal fronts dominates the other anti-optimal front in the maximization problem. Therefore, as depicted in Figure 4 by the gray shaded areas, the set of rational strategies includes two fronts. Namely $F_{s_1^1}^s$ (triangle $(1,1)$) and $F_{s_1^1}^c$ (squares $(2,1)$ and $(2,3)$).

The union of all the fronts of the rational strategies $F_{s_1^1}^R$ form the rational layer of $P_1$:

$$F_1^R := \bigcup_{i=1}^l F_{s_1^i}^R$$

After the problem is solved for the minimizer, the problem can be solved considering the game from the $P_2$ perspective, as detailed in the next section.

B. Problem Solution for the Maximizer - $P_2$

In considering the j-th strategy of $P_2$ (maximizer), I finite possible strategies of $P_1$ (minimizer) must be considered. All possible games that involve the j-th $P_2$ strategy form a set of $I$ possible games associated with the $P_2$ j-th strategy: $G_{s_2^j} := \{ g_{1,j}, g_{2,j}, \ldots, g_{i,j}, \ldots, g_{l,j} \} \subset G$. For example, in Figure 2, $G_{s_2^j}$ is the set of the two white shapes (labelled by $(1,1)$ and $(2,1)$). These are the two possible results of the decision to choose guarding post 1 while the first player is choosing one of the roads. The minimizer ($P_1$) aims at minimizing the objectives ($f^{(1)}$ and $f^{(2)}$) while playing with the maximizer ($P_2$) j-th strategy. This means that $P_1$ aims at:
\[
\min_{s^i_1 \in S_1} \bar{f}_{i,j} \quad \forall i \in [1, I]
\] (16)

For the maximizer’s j-th strategy, a set exists of minimizer strategies that may serve as solutions to equation 16. As the minimizer (P_1) aims at worst-case for the maximizer (P_2), these strategies will form the j-th anti-optimal set of strategies \(C_{s^j_2}^{-*}\).

\[
C_{s^j_2}^{-*} := \{s^i_1 \in S_1 | ~\exists s^{i'}_1, i' \in [1, I] : g_{i',j} \geq g_{i,j}, ~\forall i \in [1, I]\}
\] (17)

Mapping these anti-optimal strategies of P_2 to the objective space forms the j-th anti-optimal front: \(F_{s^j_2}^{-*}\) where:

\[
F_{s^j_2}^{-*} := \{\bar{f}_{i,j} \in \Psi | s^i_1 \in C_{s^j_2}^{-*}\}
\] (18)

All the anti-optimal fronts \(F_{s^j_2}^{-*}\) of the introductory example are depicted in Figure 5, encircled by dashed curves. \(F_{s^j_2}^{-*}\), includes the white triangle (1,1) and square (2,1). This front represents the best responses (road 1 or 2) of P_1 assuming P_2 chooses guarding post 1. The anti-optimal front \(F_{s^j_2}^{-*}\) includes the gray triangle (1,2). This front represents the best response (road 1) of P_1 assuming P_2 chooses guarding post 2. The anti-optimal front \(F_{s^j_2}^{-*}\) includes the black triangle and square and ((1,3) and (2,3) respectively. As in the case of strategy \(s^j_2\) this front represents the best responses (road 1 or 2) of P_1 assuming P_2 chooses guarding post 3.

The next phase is to search for the rational strategies of P_2. The first step for finding these strategies is to construct the union of all of the anti-optimal sets of P_2.

\[
SC_2 = \{C_{s^j_2}^{-*} \ | \ j \in [1, I]\}
\] (19)

Next, sorting \(SC_2\) and selecting the dominating sets in the minimization problem will result in the maximizer’s (P_2) irrational strategies, which are the worst-case dominated strategies:

\[
C_{\text{w.c.}} := \{s^j_2 \in S_2 | \sim \exists C_{s^j_2}^{-*} \in SC_2 : C_{s^j_2}^{-*} \cap C_{s^j_2}^{-*} \max \text{ } C_{s^j_2}^{-*}\}
\] (20)

The set \(C_{\text{w.c.}}\) includes all P_2’s strategy that are associated with a dominating anti-optimal front in the minimization problem.

The set of rational strategies of P_2 is the relative complement of \(SC_2\) and \(C_{\text{w.c.}}\):

\[
C^R_2 = SC_2 - C_{\text{w.c.}}
\] (21)

The set \(C^R_2\) includes all P_2’s strategies associated with a non-dominating anti-optimal front in the minimization problem.

Each of the maximizer rational strategies is represented in the objective space by its related anti-optimal front, which is termed as the maximizer rational front:

\[
F^R_{s^j_2} := \{\bar{f}_{i,j} \in \Psi | s^i_1 \in C_{s^j_2}^{-*}\}
\] (22)

Referring to the example, the only strategy excluded from the rational set is \(s^j_2\) (gray) because its related anti-optimal front \(F_{s^j_2}^{-*}\) (the gray triangle) dominates (worst-case domination) the other two anti-optimal fronts \(F_{s^j_2}^{-*}\) and \(F_{s^j_2}^{-*}\) of \(s^1_2\) (white) and \(s^3_2\) (black) respectively. This means that no matter which road the first player is choosing, selecting the guarding post 2 yields a payoff vector that is strictly inferior when comparing with the results of choosing post 1 or 3. Therefore, as depicted in Figure 6 by the gray shaded areas, the set of rational strategies includes two fronts. Namely \(F_{s^j_2}^{-*}\) (white triangle and square (1,1), (2,1) respectively) and \(F_{s^j_2}^{-*}\) (black triangle and square (1,2), (2,2) respectively). Note that in this case \(F^R_{s^j_2} = \emptyset\).

The union of all of the rational fronts \(F^R_{s^j_2}\) of the rational strategies forms the rational layer of P_2:

\[
F^R_2 := \bigcup_{j=1}^{J} F^R_{s^j_2}
\] (23)

The rational layer \(F^R_2\) is represented by the gray shaded areas in Figure 6.

**C. Rational Games**

When the minimizer uses one of its rational strategies and the maximizer uses one of its rational strategies, the game they are playing is considered a rational game. The game is rational in the sense of “worst-case” optimization, meaning that this game earns each player at least the “worst-case” payoff vector that it took into consideration when it chose its rational strategy.

The set of all rational games is built from all the possible combinations of the rational strategies of both players:

\[
G^R := \{g_{i,j} \in G | s^i_1 \in C^{R}_{s^j_1}, \ s^j_2 \in C^{R}_{s^j_2}\}
\] (24)

In the case of the introductory example the set of rational games \(G^R\) will include the following games: \(g_{1,1}, g_{1,3}, g_{2,1}\) and \(g_{2,3}\), which are depicted in Figure 7.
D. Implications

When choosing a rational strategy, the minimizer guarantees that whichever strategy the maximizer chooses (even a strategy that is not included in its set of rational strategies), the performances of the related games will be at least as good as the performances of the games associated with the front. In Figure 8 the two rational fronts of $P_1$ are encircled. All games associated with the rational strategy, $s_1$ (triangle) either belong to the related front ($g_{1,1}$) or dominate at least one of its members in the minimization problem (e.g., games $g_{1,2}$ and $g_{1,3}$ dominate game $g_{1,1}$ in the minimization problem). For clarity, the area that contains the games that dominate at least one of the games in the rational front $F_{s_1}^R$ is marked by light grey.

Note that the game $g_{1,2}$ is not a member in the set of the rational games $G^R$ because strategy $s_2$ is not a rational strategy for the maximizer. Therefore, strategy $s_2$ is not a rational choice, but even if the maximizer is irrational, the minimizer can ensure that its performances will be at least as good as the performances of the games associated with strategy $s_1$ (triangle). Similarly, for strategy $s_1$ (square) the area is marked by dark grey.

By choosing a rational strategy, the maximizer guarantees that whichever strategy the minimizer chooses (even a strategy that is not included in the set of the minimizer’s rational strategies), the performances of the related games will be at least as good as the performances of the games associated with the rational front. In Figure 9 the two rational fronts of $P_2$ are encircled. All games associated with the rational strategy, $s_2$ (black fill) either belong to the related rational fronts ($g_{1,3}$ and $g_{2,3}$) or dominate at least one of its members in the maximization problem. The area that contains the games that dominate the games in the rational front $F_{s_2}^R$ is marked by light grey. Similarly, for the white strategy $s_2$ the area is marked by dark grey.

Knowledge of all the rational games and their tradeoffs may be utilized by the maximizer while deciding which of the rational strategies to choose. For example, if $P_2$ (the guard) has a higher preference for $f^{(2)}$ over $f^{(1)}$ (meaning that it is more important for $P_2$ to delay the first player), it would choose to use the strategy ($s_1$) associated with the white filling and to locate it’s guard on guarding post 1. Further decision support considerations are beyond the scope of the current paper and are expected to be part of future work.

IX. SUMMARY AND FUTURE WORK

In contrast to most studies on MOGs that use a weighted sum approach to resolve the multiplicity of objectives, here non-domination among sets of payoff vectors is used. As neither player knows the opponent’s preferences, each player has to adopt the worst-case approach as in playing against nature. Hence each strategy is evaluated by assessing its representative anti-optimal front with respect to the anti-optimal fronts of the other strategies. This is accomplished by amalgamating two paradigms together, game theory and set-based multi-objective worst-case optimization.

Future work may include:

- Equilibrium between players that have postponed objective preferences.
- Develop and solve the case of non-zero-sum MOGs with postponed objective preferences.
- Develop decision support systems for players of such MOGs.
- Develop algorithms in order to solve MOGs with postponed objective preferences.
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