Interferometry against differential Doppler: performance comparison of two emitter location airborne systems

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Abstract: A comparative random error analysis of two airborne systems that provide the location of a stationary emitter is presented. Both systems utilise only two airborne receivers. In the interferometric system, the separation between the receivers (baseline) is of the order of one wavelength, and the phase measurements yield bearing measurements. In the differential Doppler system, the baseline can extend to the full length of the aircraft, and phase measurements yield accumulated range difference. Both systems employ the Gauss–Newton iterative algorithm to obtain an unbiased, least-squares estimate of the emitter location. Simple analytical expressions are obtained for the random error of both systems when the observation section is centred near the point of closest approach. Results of Monte-Carlo computer simulations are also presented, which confirm the theoretical error expressions.

1 Introduction

Interferometric direction finding, the prevailing emitter location airborne system, suffers from a conflict between accuracy and ambiguity. A small separation ($<\lambda/2$) between the two receivers which comprise the interferometer yield poor bearing accuracy; a longer separation causes ambiguity. The ambiguity can be resolved by adding receivers. At least two bearing measurements, taken along the track, are required to obtain the location of a stationary emitter, in a plane defined by the emitter and the aircraft track. The farther the measurements are from each other, the higher the location accuracy.

Differential Doppler is another airborne positioning system that can be implemented with only two receivers. It can yield unambiguous location even though the two receivers are separated by many wavelengths. It requires, however, a relatively dense set of measurements, taken along the track.

Random error analysis of an emitter location system based on two or more bearing measurements was given by Torrieri [1]. The location accuracy in Reference 1 is given as a function of the individual bearing measurement accuracy and the position, relative to the emitter, of each bearing measurement. Differential Doppler accuracy is discussed in very general terms in Reference 2. The generality, however, prevents one from getting a clear idea of the expected location accuracy.

In this paper we present a random error analysis of an emitter location system based on many evenly spaced differential Doppler measurements (also called Doppler count). Simple results are obtained for the case in which the measurements are evenly spaced along a straight line, with the centre of the observation section near the point of closest approach to the emitter (PCA).

To obtain a perspective on the performance of the differential Doppler system, we also analyse an interferometric system operating under similar conditions, namely, with many evenly spaced measurements along a straight line and centred near the PCA. We then compare the accuracy of the two methods.

The two measurement systems can be described with the help of Fig. 1, which depicts the plane defined by the emitter and the aircraft track. The instrumented aircraft flies along the y axis. The emitter is located on the x axis at a distance x_0. Selecting the origin at the PCA does not effect the generality of the discussion. N measurements are taken along an observation section of length L, centred around y_c. Each measurement is actually an electrical phase difference measurement between the signals received by the two receivers. The phase difference $\Delta\theta$ is related to the range difference $\Delta R\,(=R_1 - R_2)$ as

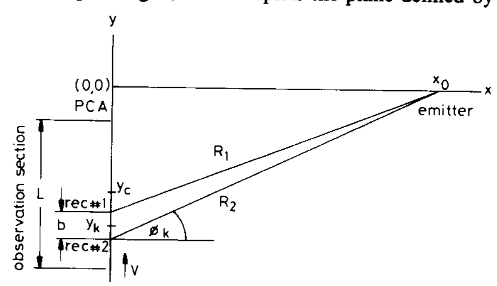

Fig. 1 *Geometry of emitter location from an aircraft*

$$\Delta\theta = 2\pi \frac{\Delta R}{\lambda} \text{ (modulo } 2\pi) \tag{1}$$

If the separation b between the two receivers is one half wavelength, $b = \lambda/2$, then the range difference and phase difference will be limited to

$$-\lambda/2 \leqslant \Delta R \leqslant \lambda/2, \quad -\pi \leqslant \Delta\theta \leqslant \pi \tag{2}$$

which ensures unambiguous results of bearing measurements over the angular range $-\pi/2 \leqslant \phi \leqslant \pi/2$.

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(Additional receivers and a judicious choice of separations can increase the effective baseline to several wavelengths without causing ambiguity.)

When the baseline b is small relative to the range to the emitter, and when the bearing angle ϕ is also small (which is the case when the observation section is located near the PCA) then the geometry is as depicted in Fig. 2.

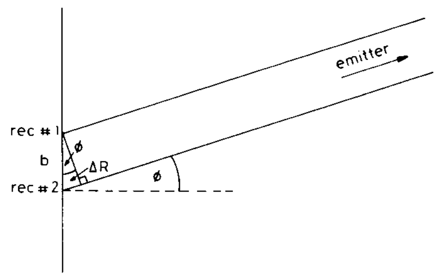

Fig. 2 *Relation between range difference and bearing for distant emitter*

This geometry yields a simple relation between the error in range difference measurement $\sigma_{\Delta R}$ and the bearing error (in radians) σ_ϕ;

$$\sigma_\phi \approx \frac{\sigma_{\Delta R}}{b}, \quad \phi \ll 1 \text{ rad} \tag{3}$$

The difference between the two methods begins with the length of the baseline b. In the interferometric direction finding utilizing two receivers, $b \leqslant \lambda/2$ and the set of phase difference measurements is converted to a set of independent and unambiguous bearing measurements $\phi_k, k = 0, \pm 1, \ldots, \pm M$.

In the differential Doppler (DD) system, the baseline between the two receivers can extend to the full length of the aircraft. The first phase difference measurement, taken at the start of the observation section, will yield a modulo λ range difference measurement. What can be obtained from consecutive phase difference measurements, *if they are taken at small enough intervals*, is the accumulated change in the range difference. A 2π jump in the phase difference implies an accumulation of one more λ in the range difference ΔR. A typical range difference history, and its modulo λ history are given in Fig. 3 for

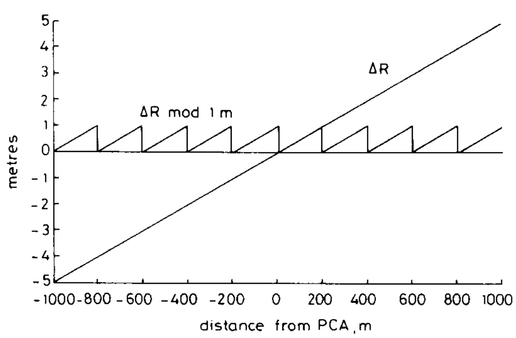

Fig. 3 *Range difference and its modulo λ history*

the case $x_0 = 10$ km, $b = 50$ m and $\lambda = 1$ m. The measurement density must be high enough to ensure that all the 2π phase jumps are identified and counted. For the example in Fig. 3 this can be ensured if measurements are taken at intervals smaller than approximately 60 metres along the flight track. If this is the case then the set of available measurements is

$$(\Delta R)_k - (\Delta R)_0, \quad k = \pm 1, \pm 2, \ldots, \pm M$$

By choosing the range of k to be between $-M$ and $+M$ we have placed the reference difference at the centre of the observation section.

In this paper we will compare the random positioning error of both systems under identical geometrical conditions, namely a set of many measurements taken along a straight line track. Simple closed-form expressions can be obtained if the observation section is centred around the PCA. Our comparison will therefore emphasise that geometry.

The noise induced error, in either bearing or range difference measurements, will be assumed to have a zero mean and a Gaussian distribution. A least-squares estimator (which in this case will also be a maximum likelihood estimator) can be implemented using a Gauss–Newton iterative method [1, 3]. A general summary of the estimation method, which applies to both systems, will be discussed in the next Section. Following sections will adapt it to the two measurement concepts.

2 Estimation method

In the co-ordinate system defined in Fig. 1, the two unknowns describing the emitter location relative to the centre of the observation section are x_0 and y_c. The vector of unknowns is therefore

$$\boldsymbol{x} = [x_0, y_c]^T \tag{4}$$

The relation between the (noiseless) measured parameter at the kth measurement and the two unknowns will be termed $h_k(x_0, y_c)$. It can easily be seen from Fig. 1 that for the interferometric measurement

$$\phi_k = h_k(x_0, y_c) = \tan^{-1}\left(\frac{-y_k}{x_0}\right) \tag{5}$$

where

$$y_k = y_c + k\Delta L, \quad k = 0, \pm 1, \ldots, \pm M \tag{6}$$

For the differential Doppler the noiseless measured parameter in the kth measurement is

$$\Delta\Delta R_k = h_k(x_0, y_c) = R_{1k} - R_{2k} - (R_{10} - R_{20}) \tag{7}$$

where

$$R_{1k} = [(y_c + b/2 + k\Delta L)^2 + x_0^2]^{1/2} \tag{8}$$

$$R_{2k} = [(y_c - b/2 + k\Delta L)^2 + x_0^2]^{1/2} \tag{9}$$

The $2M + 1$ noisy measurements f_k provide us with the same number of equations:

$$f_k(x_0, y_c) = h_k(x_0, y_c) + v_k, \quad k = 0, \pm 1, \ldots, \pm M \tag{10}$$

Using the terminology

$$h_{kx} = \frac{\partial h_k(x_0, y_c)}{\partial x_0}; \quad h_{ky} = \frac{\partial h_k(x_0, y_c)}{\partial y_c} \tag{11}$$

we define a matrix of partial derivatives

$$G = \begin{bmatrix} h_{-Mx} & h_{-My} \\ \vdots & \vdots \\ h_{kx} & h_{ky} \\ \vdots & \vdots \\ h_{Mx} & h_{My} \end{bmatrix} \tag{12}$$

and the four vectors

$$f = [f_{-M}, f_{-M+1}, \ldots, f_k, \ldots, f_M]^T \tag{13}$$

$$h = [h_{-M}, h_{-M+1}, \ldots, h_k, \ldots, h_M]^T \tag{14}$$

$$v = [v_{-M}, v_{-M+1}, \ldots, v_k, \ldots, v_M]^T \tag{15}$$

$$w = [\Delta x_0, \Delta y_c]^T \tag{16}$$

The last vector is the correction in the parameters estimation for the next iteration.

In the Gauss–Newton iterative method, the recent estimate of the unknowns is used in each iteration to calculate h and G.

If the noise covariance matrix is given by

$$\text{Cov } v = \sigma_v^2 I \tag{17}$$

where I is the identity matrix, then the correction vector is given by

$$w = (G^T G)^{-1} G^T (f - h) \tag{18}$$

and the next estimate is obtained from the last estimate using

$$\hat{x}_{i+1} = \hat{x}_i + w \tag{19}$$

The estimation process is terminated when w becomes negligible.

Our main interest in this work is to obtain a simple expression for the random location error. For that we will use one more result from the least-squares estimator which says that the error covariance matrix of the unknowns is given by

$$\text{Cov } x = \sigma_v^2 (G^T G)^{-1} \tag{20}$$

In the next two Sections we will develop eqn. (20) for the two measurement systems: interferometry and differential Doppler. We will detail the simple case where the observation section is centred at the PCA, namely when $y_c = 0$. Only the final results will be given from the more complicated analysis when $y_c \neq 0$.

3 Estimation accuracy of the interferometric system

In order to calculate the covariance matrix in eqn. (20) we need the partial derivative terms in G. Using eqns. (5) and (11) we get

$$h_{kx} = \frac{y_c + k\Delta L}{x_0^2 + (y_c + k\Delta L)^2} \tag{21}$$

$$h_{ky} = \frac{-x_0}{x_0^2 + (y_c + k\Delta L)^2} \tag{22}$$

Having the expressions for the partial derivatives, we can calculate the matrix $G^T G$, since

$$G^T G = \begin{bmatrix} \sum_{k=-M}^{M} (h_{kx})^2 & \sum_{k=-M}^{M} h_{kx} h_{ky} \\ \sum_{k=-M}^{M} h_{kx} h_{ky} & \sum_{k=-M}^{M} (h_{ky})^2 \end{bmatrix} \tag{23}$$

We will simplify the analysis by limiting it to a case in which the observation section is centered at the PCA. Setting $y_c = 0$ in eqns. (21) and (22) yields

$$h_{kx} = \frac{k\Delta L}{x_0^2 + (k\Delta L)^2}, \quad y_c = 0 \tag{24}$$

$$h_{ky} = \frac{-x_0}{x_0^2 + (k\Delta L)^2}, \quad y_c = 0 \tag{25}$$

Choosing $y_c = 0$ sets the off-diagonal elements of $G^T G$ equal to zero and makes the inversion very simple. We also note that the measurement error σ_v is a bearing error σ_ϕ. Thus, we obtain

$$(\text{Var } x_0)^{-1} = \sigma_\phi^{-2} \sum_{k=-M}^{M} \frac{(k\Delta L)^2}{[x_0^2 + (k\Delta L)^2]^2} \tag{26}$$

$$(\text{Var } y_c)^{-1} = \sigma_\phi^{-2} \sum_{k=-M}^{M} \frac{x_0^2}{[x_0^2 + (k\Delta L)^2]^2} \tag{27}$$

We will now make use of the second assumption, namely that the measurements are dense. We first multiply (within the sum) and divide (outside the sum) eqns. (26) and (27) by ΔL and then replace the sum with an integral, using

$$k\Delta L = y, \quad M\Delta L = L/2, \quad \Delta L = dy \tag{28}$$

The two expressions become

$$(\text{Var } x_0)^{-1} \approx \frac{1}{\sigma_\phi^2 \Delta L} \int_{-L/2}^{L/2} \frac{y^2}{(x_0^2 + y^2)^2} \, dy \tag{29}$$

$$(\text{Var } y_c)^{-1} \approx \frac{1}{\sigma_\phi^2 \Delta L} \int_{-L/2}^{L/2} \frac{x_0^2}{(x_0^2 + y^2)^2} \, dy \tag{30}$$

Using the assumption that

$$x_0 \gg L \tag{31}$$

and that the number of measurements N is

$$N = 2M + 1 \approx 2M = \frac{L}{\Delta L} \tag{32}$$

eqns. (29) and (30) reduce to

$$(\text{Var } x_0)^{-1} \approx \frac{NL^2}{12\sigma_\phi^2 x_0^4} \tag{33}$$

$$(\text{Var } y_c)^{-1} \approx \frac{N}{\sigma_\phi^2 x_0^2} \tag{34}$$

We finally get the two components of the positioning random error, assuming $y_c = 0$ and $x_0 \gg L$:

$$\sigma_x \approx \sqrt{\frac{12}{N}} \, \sigma_\phi \frac{x_0^2}{L} \tag{35}$$

$$\sigma_y \approx \frac{1}{\sqrt{N}} \sigma_\phi x_0 \tag{36}$$

It is customary to describe σ_x as the 'across-track' error, and σ_y as the 'along-track' error.

4 Estimation accuracy of the differential Doppler system

The measurement in the differential Doppler system is of $\Delta\Delta R$ as defined in eqn. (7). Such a measurement, however, contains a common error term ε, which is the error in the zeroth range difference measurement $R_{10} - R_{20}$. To cope with this common error term, we must incorporate it in the 'noiseless' measurement and redefine eqn. (7) as

$$h_k(x_0, y_c, \varepsilon) = R_{1k} - R_{2k} - (R_{10} - R_{20}) + \varepsilon \tag{37}$$

We must know estimate three unknowns and thus

$$x = [x_0, y_c, \varepsilon]^T \tag{38}$$

and G will become a $(2M + 1) \times 3$ matrix. The remaining measurement error v_k is now the error in measuring R_{1k}

$- R_{2k}$, and it meets the requirement in (17)

$$\text{Cov}\ \boldsymbol{v} = \sigma_{\Delta R}^2 \boldsymbol{I} \tag{39}$$

If we use the two simplifying assumptions, which we made in the previous Section, namely

$$y_c = 0, \quad M\Delta L \ll x_0 \tag{40}$$

we get

$$h_{kx} \approx -\frac{bk\Delta L}{x_0^2} \tag{41}$$

$$h_{ky} \approx -\frac{3b(k\Delta L)^2}{2x_0^3} \tag{42}$$

$$h_{k\varepsilon} = 1 \tag{43}$$

The matrix of partial derivatives now becomes

$$\boldsymbol{G} = \begin{bmatrix} h_{-Mx} & h_{-My} & 1 \\ \vdots & \vdots & \vdots \\ h_{kx} & h_{ky} & 1 \\ \vdots & \vdots & \vdots \\ h_{Mx} & h_{My} & 1 \end{bmatrix} \tag{44}$$

which yields

$$\boldsymbol{G}^T\boldsymbol{G} = \begin{bmatrix} \sum_{k=-M}^{M} (h_{kx})^2 & \sum_{k=-M}^{M} h_{kx}h_{ky} & \sum_{k=-M}^{M} h_{kx} \\ \sum_{k=-M}^{M} h_{kx}h_{ky} & \sum_{k=-M}^{M} (h_{ky})^2 & \sum_{k=-M}^{M} h_{ky} \\ \sum_{k=-M}^{M} h_{kx} & \sum_{k=-M}^{M} h_{ky} & \sum_{k=-M}^{M} 1 \end{bmatrix} \tag{45}$$

Assuming again that the measurements are dense, namely that ΔL is small, we can convert the sums into integrals, after dividing (outside the sum) and multiplying (inside the sum) by ΔL. This yields

$$\boldsymbol{G}^T\boldsymbol{G} \approx \begin{bmatrix} \frac{Nb^2L^2}{12x_0^4} & 0 & 0 \\ 0 & \frac{9Nb^2L^4}{320x_0^6} & \frac{NbL^2}{8x_0^3} \\ 0 & \frac{NbL^2}{8x_0^3} & N \end{bmatrix} \tag{46}$$

Inverting eqn. (46) yields

$$(\boldsymbol{G}^T\boldsymbol{G})^{-1} \approx \begin{bmatrix} \frac{12x_0^4}{Nb^2L^2} & 0 & 0 \\ 0 & \frac{80x_0^6}{Nb^2L^4} & -\frac{10x_0^3}{NbL^2} \\ 0 & -\frac{10x_0^3}{NbL^2} & \frac{9}{4N} \end{bmatrix} \tag{47}$$

Using eqns. (47) and (39) in eqn. (20), we find from the main diagonal terms that, for $y_c = 0$ and $x_0 \gg L$, the across-track and along-track errors are, respectively,

$$\sigma_x \approx \sqrt{\frac{12}{N}} \frac{\sigma_{\Delta R}}{b} \frac{x_0^2}{L} \tag{48}$$

$$\sigma_y \approx \sqrt{\frac{80}{N}} \frac{\sigma_{\Delta R}}{b} \frac{x_0^3}{L^2} \tag{49}$$

With the help of eqn. (3) we can compare the positioning errors of the differential Doppler (DD) system, as given in eqns. (48) and (49), with the corresponding errors in the interferometric system, as given in eqns. (35) and (36). We first note that the across-track error expressions σ_x are identical. However, we should recall that the separation b between the two receivers is much larger in the differential Doppler system. We can conclude that differential Doppler can yield a smaller across-track error. The opposite conclusion is reached with regard to the along-track error σ_y. There the DD system yields an error which is x_0^2/L^2 times larger than the error in the interferometric system. This ratio is usually so much larger than unity that even the larger b cannot compensate for it.

5 Errors when $y_c \neq 0$

When the observation section is not centred at the PCA, we cannot set $y_c = 0$ in eqns. (21) and (22). This will result in more elaborate elements in the $\boldsymbol{G}^T\boldsymbol{G}$ matrix, including the off-diagonal elements of eqn. (23) which will no longer be equal to zero. The same thing will happen in the differential Doppler matrix (45). The matrix inversion becomes more complicated. We will therefore present only the final results, which are still an approximation because of the assumptions $L \ll x_0$ and $y_c \ll x_0$.

For the interferometric system we get

$$\sigma_x \approx \sqrt{\frac{12}{N}} \sigma_\phi \frac{x_0^2}{L} \tag{50}$$

$$\sigma_y \approx \frac{1}{\sqrt{N}} \sigma_\phi x_0 \left(1 + \frac{12y_c^2}{L^2}\right)^{1/2} \tag{51}$$

For the differential Doppler system we get

$$\sigma_x \approx \sqrt{\frac{12}{N}} \frac{\sigma_{\Delta R}}{b} \frac{x_0^2}{L} \left(1 + 60\frac{y_c^2}{L^2}\right)^{1/2} \tag{52}$$

$$\sigma_y \approx \sqrt{\frac{80}{N}} \frac{\sigma_{\Delta R}}{b} \frac{x_0^3}{L^2} \tag{53}$$

The theoretical error analysis summarized in eqns. (50)–(53) was checked by a Monte-Carlo computer simulation using the Gauss–Newton iterative algorithm outline above. The 'measured' data was generated by adding range difference (or bearing) noise to a noiseless measurement history, calculated using the correct emitter co-ordinates. The first guess was set 50 m off the correct location. Convergence was assumed when $(\Delta x_0)^2 + (\Delta y_c)^2$ was less than 10^{-4} m^2; this usually happened after three iterations. The results of 1000 simulated flights, each containing 100 position evaluations evenly distributed along

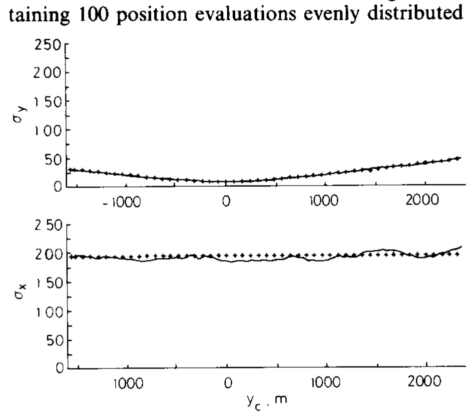

Fig. 4 *Random error of the emitter co-ordinates in the interferometric system with b = 2.5 m*

+ + + theoretical
—— Monte-Carlo results

the path, are presented as the solid lines in Figs. 4 and 5. The curves marked by crosses represent the respective theoretical errors (in metres) as given in eqns. (50) to (53).

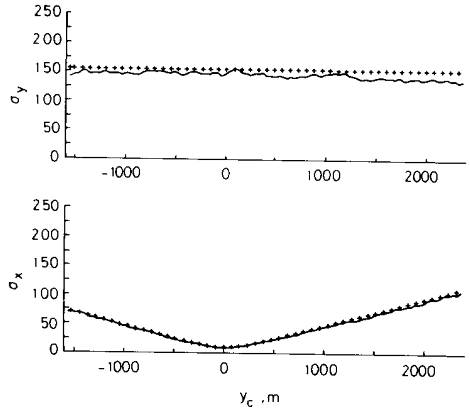

Fig. 5 *Random error of the emitter co-ordinates in the differential Doppler system with b = 50 m*
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The parameters for both systems were identical: $x_0 = 10$ km, $L = 1.6$ km, $N = 40$, and $\sigma_{\Delta R} = 0.0141$ m. The only different parameter was obviously the baseline b. In the interferometer we used $b = 2.5$ m, whereas in the differential Doppler system $b = 50$ m was chosen. σ_ϕ was obtained from $\sigma_{\Delta R}$ using eqn. (3).

The good agreement between the theoretical errors and those obtained from the computer simulation attests to the accuracy of the analytically obtained error expressions. The 'crossed' and solid lines deviate only at large y_c, where the assumptions yielding the theoretical results are violated.

Figs. 4 and 5 indicate a performance deterioration in both systems as the observation centre shifts away from the PCA. This is most pronounced in the across-track error of the differential Doppler system. We did not take into account the obvious fact that away from the PCA the signal-to-noise ratio drops also, which should further reduce performance there.

6 Conclusions

If we accept a baseline ratio of 20 between the two systems, which is the ratio used in Figs. 4 and 5, then we can conclude that the differential Doppler location system has an advantage with regard to the random error in the across-track co-ordinate, whereas the interferometric system has an advantage in the along-track co-ordinate. This suggests a new system that will utilise three receivers and that will combine the two techniques.

When making the comparison it should be remembered that the analysis assumed a dense set of measurements along the observation section. This was a requirement imposed for the benefit of the differential Doppler system. The DD system breaks down if the spacing between measurements is too large and does not allow proper sampling of the differential Doppler cycles; the interferometric system does not suffer from such a requirement. The along-track length of the differential Doppler cycle will increase as the separation between receivers decreases. This will alleviate the requirement for dense measurements at the cost of reduced accuracy. At the limit, as b is reduced to the order of a wavelength, the DD system becomes an interferometric system.

Another drawback of the comparison is its restriction to random error only. Bias errors may be the dominant ones. However, random error analysis points out sensitivities, which apply also to other error sources.

7 References

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