

# The Periodic Ambiguity Function – Its Validity and Value

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**Abstract**—The periodic ambiguity function (PAF) relates to Woodward's ambiguity function (AF) like periodic autocorrelation relates to autocorrelation. AF is defined for a finite signal, processed by its matched filter. PAF can handle periodic signals with infinite (or large) number of periods, and a processor that is matched to fewer periods. PAF suits practical scenarios found in radar employing coherent pulse train or CW waveforms. This paper revisits the PAF and its properties, demonstrating its value and viability.

## I. INTRODUCTION

Woodward's AF [1] extends the one dimensional autocorrelation function of delay to a two-dimensional function of delay and Doppler. The PAF, suggested in [2] and studied in [3–6], performs a similar extension for the periodic autocorrelation. It turns out that for some practical radar scenes the PAF is more relevant than the AF. Prominent examples are continuous wave (CW) radar and pulse radar employing coherent pulse train waveforms. In both examples the transmitter emits (and the target returns) more periods, of duration  $T_r$ , of the waveform than the receiver processes coherently (Fig.1).

The periodicity of the identical transmitted and reference waveforms implies that their complex envelope obeys

$$u(t) = u(t + nT_r), \quad n = 0, \pm 1, \pm 2, \dots \quad (1)$$

As long as the delay  $\tau$  is shorter than the difference between the lengths of the transmitted (or received) waveform and the reference waveform,  $|\tau| \leq (P - N)T_r$ , the output response of a correlator receiver, in the delay-Doppler plane, is given by

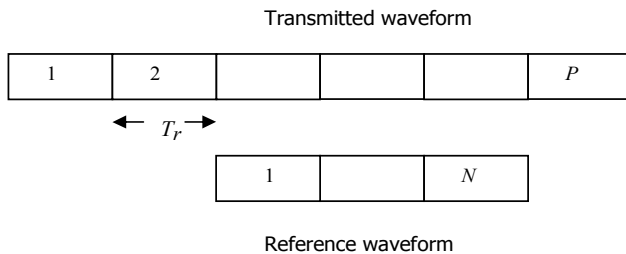


Fig. 1 Relative lengths of signal and reference

$$|\chi_{NT_r}(\tau, \nu)| = \frac{1}{NT_r} \left| \int_0^{NT_r} u(t - \tau) u^*(t) \exp(j2\pi\nu t) dt \right| \quad (2)$$

where  $\tau$  is assumed to be a constant and the delay rate of change is represented by the Doppler shift  $\nu$ . An important property of the PAF, proved in [3], says that a PAF with  $N$  reference periods is related to a PAF with a single reference period by a universal relationship, which is a function of Doppler only,

$$|\chi_{NT_r}(\tau, \nu)| = |\chi_{T_r}(\tau, \nu)| \left| \frac{\sin(N\pi\nu T_r)}{N \sin(\pi\nu T_r)} \right| \quad (3)$$

where the PAF with a single reference period is

$$|\chi_{T_r}(\tau, \nu)| = \frac{1}{T_r} \left| \int_0^{T_r} u(t - \tau) u^*(t) \exp(j2\pi\nu t) dt \right| \quad (4)$$

$$= \frac{1}{T_r} \left| \int_0^{\tau} u(t + T_r - \tau) u^*(t) \exp(j2\pi\nu t) dt + \int_{\tau}^{T_r} u(t - \tau) u^*(t) \exp(j2\pi\nu t) dt \right|$$

Another intuitively expected property is that for  $|\tau| \leq (P - N)T_r$  the PAF is periodic with period  $T_r$ .

## II. PAF AND AF OF A PHASE CODED SIGNAL

An example of the difference between the AF, single-period PAF and  $N$ -period PAF is shown in Figs. 2 to 5, which utilize a 19 element P4 waveform. Fig. 2 displays the phase evolution of the waveform. Fig. 3 displays its AF. Note the relatively high delay sidelobes at zero-Doppler, and the ridge-like AF, indicating lack of Doppler resolution. The zero-Doppler cut of the PAF of a single period (Fig. 4) exhibits no delay sidelobes due to the ideal periodic autocorrelation of P4 signals. The ridge remains and repeats itself every  $T_r$ . Doppler resolution is seen in Fig. 5, which is the PAF with a reference waveform containing 16 periods. Note the Doppler recurrent lobes at  $\nu = 1/T_r$  (and its multiples). The first Doppler null is seen at  $\nu = 1/(16T_r)$ . Still found in Fig. 4 are Doppler sidelobes. Those can be mitigated by adding weight on receive.

The PAF describes the delay-Doppler response when the reference waveform is identical to the transmitted waveform, except for the number of periods. Further deviations from the AF and the PAF are justified if the reference signal differs in other ways. The first modification is adding amplitude weighting to the reference.

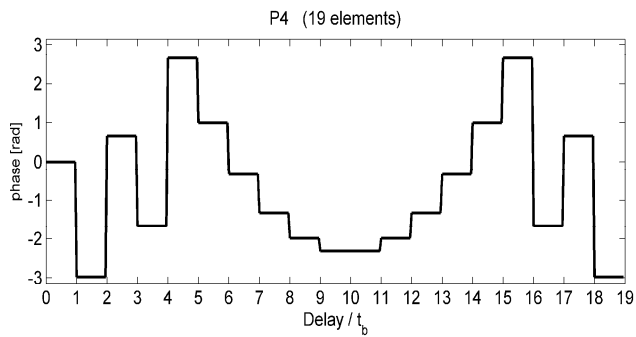


Fig. 2 Phase evolution of a 19 element P4 waveform

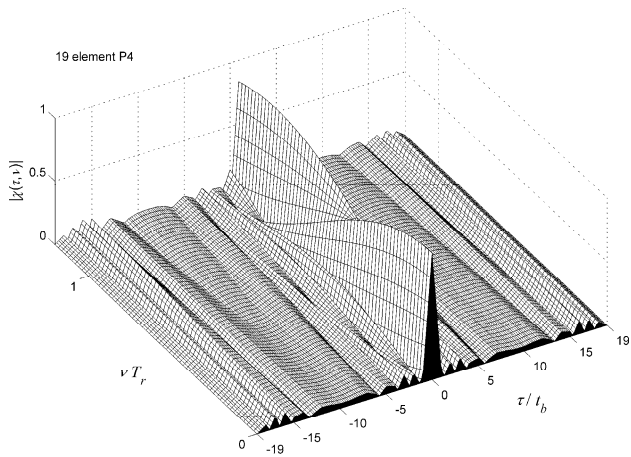


Fig. 3 AF of a 19 element P4 waveform

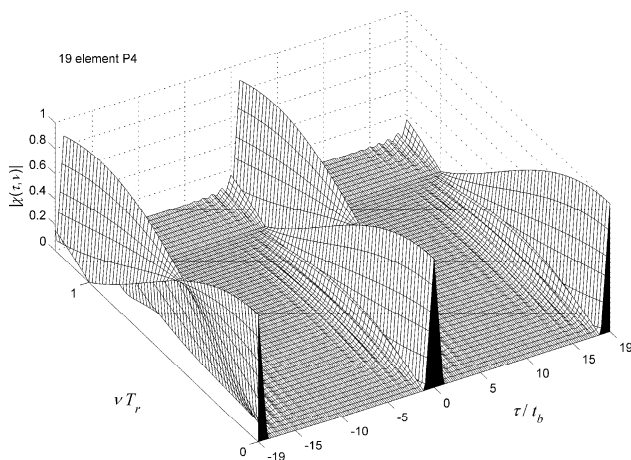


Fig. 4 PAF of a 19 element P4 waveform

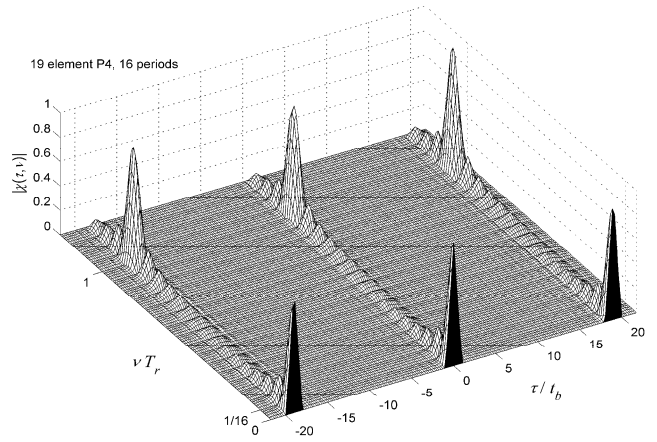


Fig. 5 PAF of 16 periods of a 19 element P4 waveform

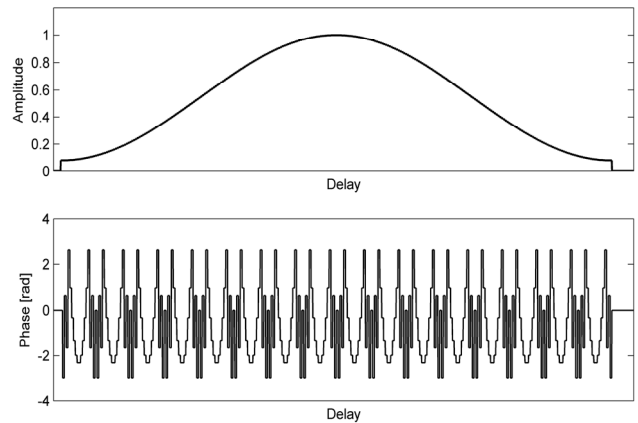


Fig. 6 Amplitude and phase of 16 periods of a 19 element P4 waveform (Hamming amplitude weight.)

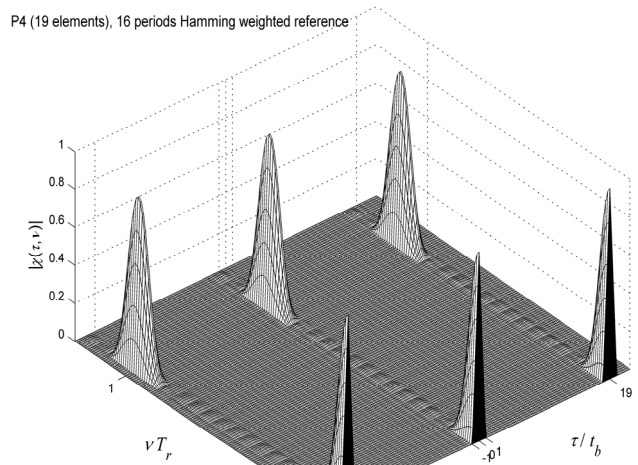


Fig. 7 Delay-Doppler response with a reference as in Fig. 6

Recall that correlation with the reference is usually performed in a digital stage of the receiver, where adding amplitude variation is relatively simple. Fig. 6 displays the amplitude and phase of 16 periods of a 19 element P4 waveform, with Hamming amplitude weight. The resulted delay-Doppler response (not PAF anymore) is shown in Fig. 7. Seen are a significant reduction in Doppler sidelobes and a widening of the Doppler main and recurrent lobes. In [6] it was proved that for the popular weight windows (Hamming, Hann, etc.), if the reference and the weight window extend over an integer number of periods, the zero-Doppler cut of the response remains sidelobe-free.

Adding amplitude weighting to the receiver's reference becomes even more important when the received signal's periodicity is not perfect. This can happen, for example, when the target's return is modulated by a rotating antenna pattern. For waveforms with ideal periodic autocorrelation (like P4) this will re-introduce sidelobes in the correlator response (-51 dB in Fig. 8). Adding weight window to the reference will greatly reduce those sidelobes (-67 dB in Fig. 9).

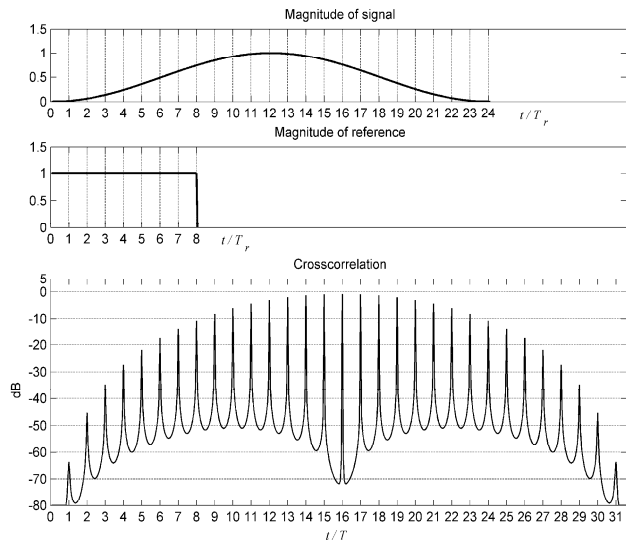


Fig. 8 Stronger crosscorrelation sidelobes, unweighted reference, reflected signal is amplitude modulated

For most CW periodic waveforms, analytical expressions of the PAF are difficult to derive. The resulted expressions are usually tedious and do not provide qualitative insight. A simple alternative are numerical calculations. MATLAB codes for calculating and plotting PAFs are listed in [7]. The software given there can also plot AF and delay-Doppler response (when the reference differs from the signal). The PAFs drawn in this paper were produced using those programs; so were the many PAF plots found in [8,9].

### III. USING PAF TO COMPARE TWO LFM-CW WAVEFORMS

Linear-FM is the most common type of CW waveform. It is used in low cost automotive radar as well as in advanced coastal radar.

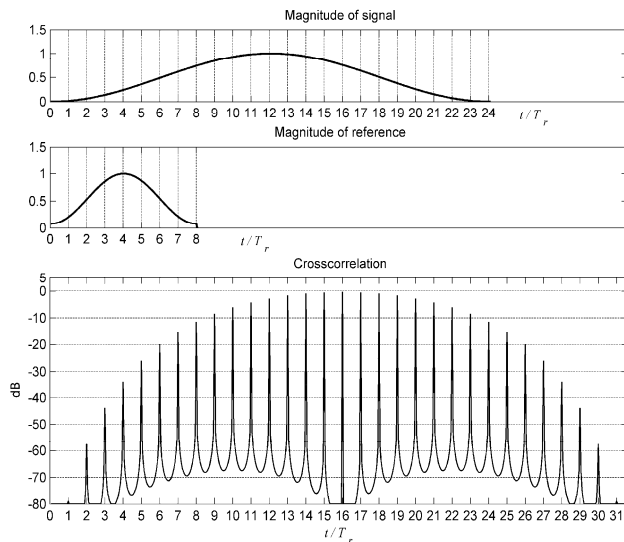


Fig. 9 Weaker crosscorrelation sidelobes, weighted reference, reflected signal is amplitude modulated

Using the PAF we will demonstrate attainable delay – Doppler responses when the coherent processing interval (CPI) contains only one long period (Fig. 10), or when the same CPI contains eight shorter periods (Fig. 11). The frequency deviation will be the same in both cases.

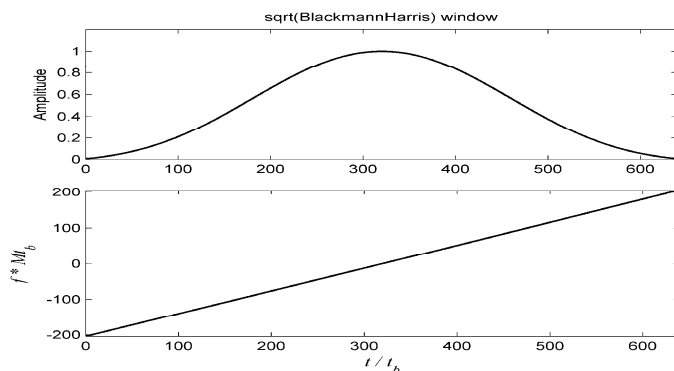


Fig. 10 Weighted single period LFM waveform

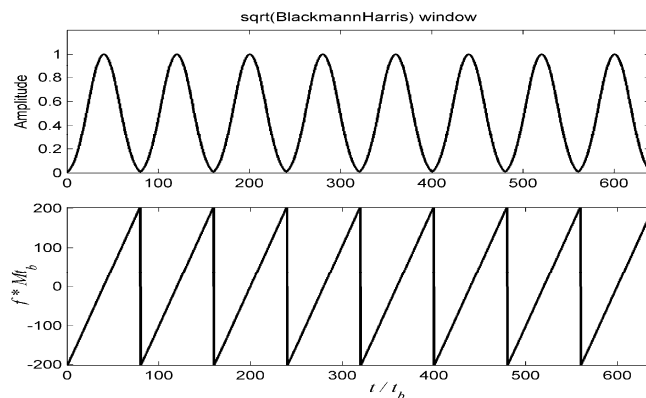


Fig. 11 Weighted 8 periods LFM waveform

Note that the weight window, which applies to both the transmitted signal and the reference signal, is a square root of the Blackmann-Harris window. This implies using a matched filter; hence the delay-Doppler response is indeed a PAF. In practical radar a full (rather than square root) weight window will be implemented only in the receiver. The mismatch will entail small SNR loss, but the delay-Doppler response will be similar to the PAF. Comparing Figs. 10 and 11, note that in both drawings the total duration of the CPI is 640 bits. The single frequency sweep in Fig. 10 is of that duration, while each sweep duration in Fig. 11 is 80 bits (frequency steps). The total frequency deviation, for both signals, is therefore

$$\Delta f = \frac{400}{640t_b} \quad (5)$$

implying a delay first null, without weighting, at

$$\tau_{\text{null}} = \frac{1}{\Delta f} = 1.6t_b \quad (6)$$

With amplitude weighting the mainlobe width would approximately double, yielding  $\tau_{\text{null}} \approx 3.2t_b$ .

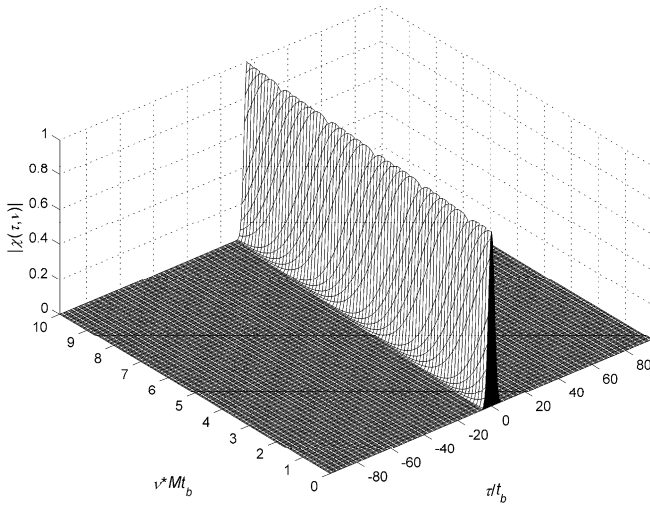


Fig. 12 PAF of a weighted single period LFM waveform

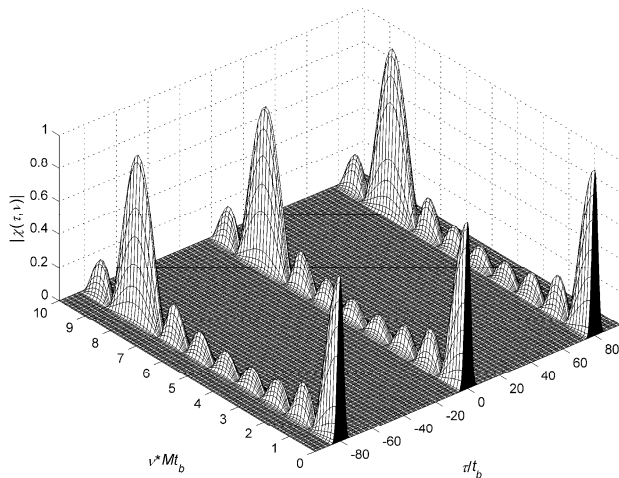


Fig. 13 PAF of weighted 8 periods LFM waveform

The PAFs in Figs. 12 and 13 indeed exhibit that same delay resolution. The main differences between the two PAFs are in Doppler resolution and unambiguous delay. The PAF of the single frequency sweep (Fig. 12) shows no Doppler resolution and (not seen) an unambiguous delay equal to the sweep duration (640 bit). The PAF of the 8 frequency sweeps (Fig. 13) shows Doppler resolution equal to the inverse of the CPI ( $=1/640t_b$ ), Doppler ambiguity equal to the inverse of the repetition period ( $=1/80t_b$ ) and delay ambiguity equal to the shorter sweep duration (80 bit).

The PAFs in the LFM example demonstrate what performances can be obtained from a periodic LFM-CW waveform, if processed by a matched filter. Other, less optimal processors, like stretch processing, often used with a single sweep LFM, may be simpler to implement but will achieve poorer response.

When the receiver is matched to a Doppler shifted version of its nominal signal, its delay-Doppler response is not a PAF anymore. Such a response, for the 8 periods LFM waveform, is given in Fig. 14. While not obvious from the drawing, the response is not symmetrical with respect to the origin, which is an important property of a PAF.

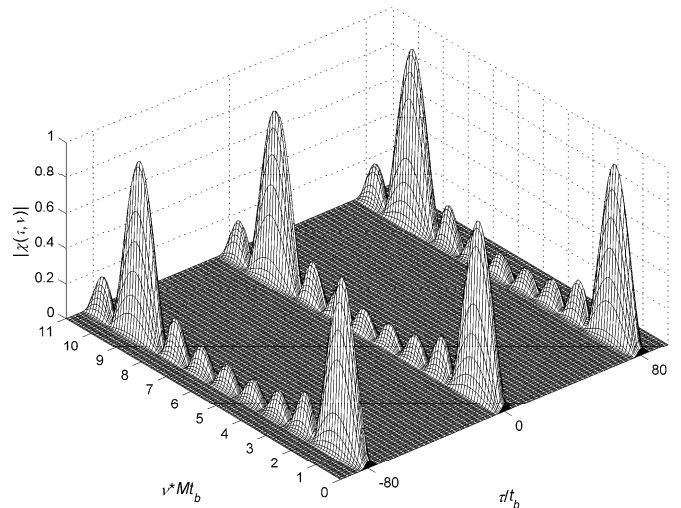


Fig. 14 Delay-Doppler response of a weighted filter matched to a Doppler shifted version of the 8 periods LFM waveform

#### IV. CODED PULSE TRAIN

The last example of analyzing periodic signals involves a periodic pulse train, in which both periodicity and a mismatch reference are necessary in order to extract the special features of the waveform. The example follows an Ipatov-coded pulse train described in [10]. The envelope of the binary-coded waveform, the reference and the resulted periodic cross-correlation are shown in Fig. 15. The combination of inter-pulse Ipatov coding and a mismatched reference, results in mitigating the cross-correlation peaks at the pulse repetition intervals. A section of the periodic delay-Doppler response is shown in Fig. 16. It shows how well the mitigation holds with Doppler. Doppler sidelobes would have been reduced if the reference was amplitude weighted.

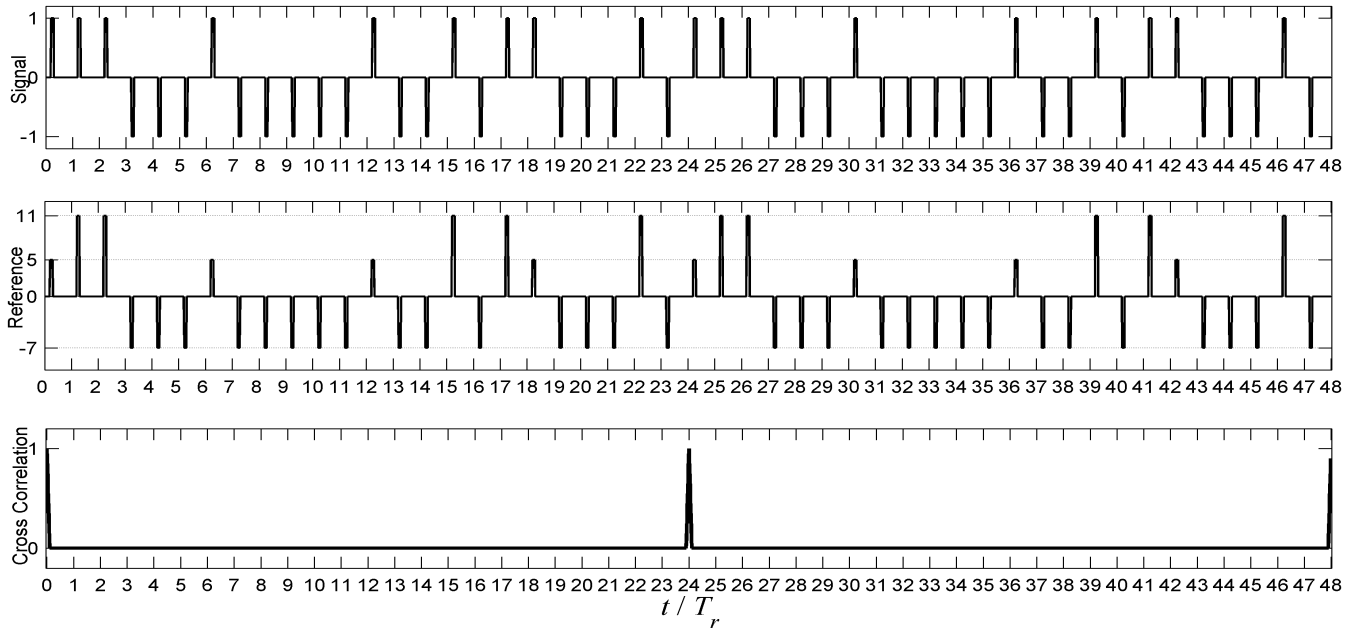


Fig. 15 Two periods out of a high PRF pulse train with inter-pulse coding based on Ipatov 24 sequence: Signal's envelope (top), reference (middle), periodic cross-correlation (bottom).

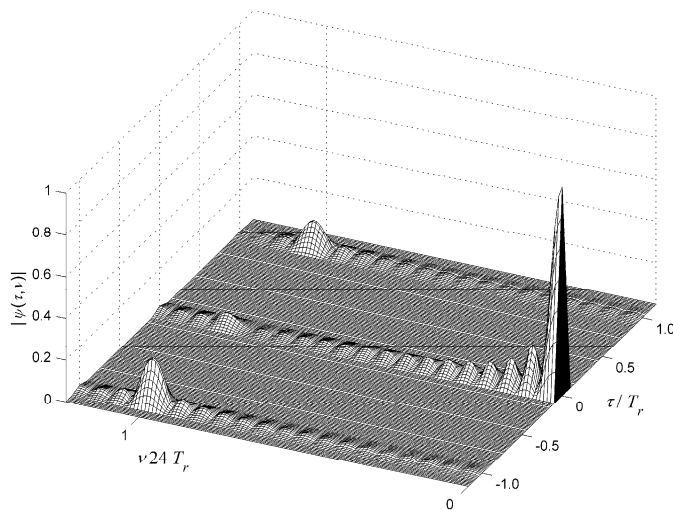


Fig. 16 Delay-Doppler periodic response of  $M=16$  periods of a signal based on an Ipatov 24 sequence. Zoom on  $|\tau| < 1.1T_r$ , and  $0 \leq \nu \leq 1.2/MT_r$ .

## V. CONCLUSIONS

The periodic version of Woodward's ambiguity function was revisited in order to show how important and useful it is for the analysis of CW waveforms and other periodic signals that are not necessarily CW. PAF analysis also applies to *finite* periodic coherent signals whenever the receiving filter is matched to fewer periods than the finite number of periods coherently transmitted. In cases when a mismatched reference is used (e.g., to reduce Doppler sidelobes) the delay-Doppler response is an extension of the periodic cross-correlation rather than the periodic autocorrelation. In those cases the term PAF cannot be used, despite the similarity.

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