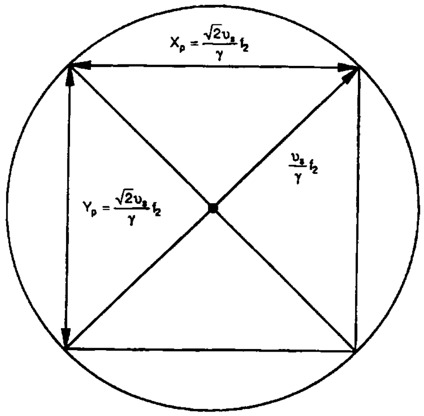

Fig. 4. Maximum rectangular aperture for $\Delta\theta_p = 360°$, $f_1 = 0$.

(Fig. 4). Then

$$X_p = Y_p = \sqrt{2}v_s f_2/\gamma$$

are the dimensions of the maximum area inscribed rectangle, which is therefore a square. Then, from (2) and (3),

$$\rho_x = \rho_y = \frac{c}{2\sqrt{2}f_2}. \tag{17}$$

If we define $f_0 = f_2/2$, then

$$\rho_x = \rho_y = \frac{\lambda_0}{4\sqrt{2}}. \tag{18}$$

This is the limiting resolution obtainable with a rectangular aperture.

VI. QUALIFICATIONS

The resolution limits derived in this correspondence are theoretical limitations based on aperture geometry and FFT processing. They are useful for systems analysis and preliminary design. Since the derivation starts with Walker's equation for phase, the same limitations and approximations used in his paper apply here. The most important is that the size of the image is small compared with the range between radar and object, as is most often the case for airborne or spaceborne SAR. Other practical limitations may prevent obtaining these resolutions. For example, the scatterer may be directional, or it may be masked during a portion of the aperture time. And, of course, amplitude and phase errors introduced by the environment and by the signal processing algorithms ultimately limit resolution.

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Properties of the Periodic Ambiguity Function

The periodic ambiguity function (PAF) is a tool for analyzing the performance of a radar system utilizing a periodically modulated CW signal and a receiver matched to one or several periods thereof. Alternative definitions for the PAF are given and the properties of the PAF are elaborated. The properties shown include periodicity, symmetry with respect to the origin, and the volume limitation property.

I. INTRODUCTION AND DEFINITIONS

The well-known ambiguity function, first presented in [1], is a useful tool for analyzing the response of a receiver, matched to a finite duration transmission signal, to a delayed and Doppler shifted replica of that signal. In [2], the periodic ambiguity function (PAF) has been introduced for the purpose of analyzing the response of a filter matched to a finite interval T_f of a CW signal modulated by a periodic waveform $u(t)$.

Let $u(t)$ be a periodic signal with a period T:

$$u(t - nT) = u(t) \quad \text{for any integer } n. \tag{1}$$

The PAF of a signal, of which the complex envelope is $u(t)$, was defined as

$$\chi_{T_f}(\tau,\nu) = \frac{1}{T_f}\int_0^{T_f} u\left(t+\frac{\tau}{2}\right)u^*\left(t-\frac{\tau}{2}\right)e^{j2\pi\nu t}\,dt \tag{2}$$

where T_f is the filter response duration, τ is the delay, and ν is the Doppler shift. This definition is not unique. Alternative definitions can be adopted. Such definitions are

$$^{1}\chi_{T_f}(\tau,\nu) = \frac{1}{T_f}\int_0^{T_f} u(t)u^*(t-\tau)e^{j2\pi\nu t}\,dt \tag{3}$$

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or

$$^2\chi_{T_f}(\tau,\nu) = \frac{1}{T_f}\int_{-T_f/2}^{T_f/2} u(t)u^*(t-\tau)e^{j2\pi\nu t}\,dt. \qquad (4)$$

The definition of $^1\chi_{T_f}$ in (3) represents the straightforward implementation of a filter matched to a signal $u(t)$ delayed by τ and Doppler shifted by ν. The other definitions represent an off-line, noncausal implementation. These definitions are not equivalent, as demonstrated hereunder.

II. PROPERTIES

In this section we discuss properties of the PAF. The first two properties, A and B, were mentioned in [2], and are repeated briefly for the sake of completeness. The last three properties C–E are new and are discussed more elaborately.

A. Relation Between Single and Multiple-Period PAF

The multiperiod PAF is defined as

$$\chi_{NT}(\tau,\nu) \triangleq \frac{1}{NT}\int_0^{NT} u\left(t+\frac{\tau}{2}\right) u^*\left(t-\frac{\tau}{2}\right) e^{j2\pi\nu t}\,dt$$

with N an integer greater than 1. It can be shown that

$$\chi_{NT}(\tau,\nu) = \chi_T(\tau,\nu)\frac{\sin(\pi\nu NT)}{N\sin(\pi\nu T)}e^{j\pi\nu(N-1)T}$$

where $\chi_T(\tau,\nu)$ is the PAF for a single period. Hence

$$|\chi_{NT}(\tau,\nu)| = |\chi_T(\tau,\nu)|\left|\frac{\sin(\pi\nu NT)}{N\sin(\pi\nu T)}\right|. \qquad (5)$$

The single-period PAF $-\chi_T(\tau,\nu)$ is multiplied by a function very similar to the sampling function presented by Rihaczek [3]. However, this sampling function is not applied only to the central part of the function but to the PAF as a whole. Equation (5) applies to all three definitions of the PAF.

B. Cuts Along Principal Axes

The cut along the delay axis is the periodic autocorrelation of the signal $u(t)$:

$$\chi_{T_f}(\tau,0) = \frac{1}{T_f}\int_0^{T_f} u\left(t+\frac{\tau}{2}\right) u^*\left(t-\frac{\tau}{2}\right) dt$$

which, for $T_f = NT$

$$\chi_{NT}(\tau,0) = \frac{1}{NT}\int_0^{NT} u\left(t+\frac{\tau}{2}\right) u^*\left(t-\frac{\tau}{2}\right) dt$$
$$= \frac{1}{T}\int_0^{T} u\left(t+\frac{\tau}{2}\right) u^*\left(t-\frac{\tau}{2}\right) dt. \qquad (6)$$

The cut along the Doppler axis is

$$\chi_{T_f}(0,\nu) = \frac{1}{T_f}\int_0^{T_f} |u(t)|^2 e^{j2\pi\nu t}\,dt$$

which for a constant amplitude signal, $|u(t)| = 1$

$$|\chi_{T_f}(0,\nu)| = \left|\frac{\sin(\pi\nu T_f)}{\pi\nu T_f}\right| \quad \text{and} \quad |\chi_{T_f}(0,0)| = 1. \qquad (7)$$

Equation (7) applies to all three definitions of the PAF.

C. Basic Periods

Since the signal $u(\cdot)$ is periodic with period T, the PAF will also be periodic, in the delay axis direction. The period is T for the definitions given in (3) and (4). Thus for n an integer, making use of (1), it can be shown

$$^i\chi_{T_f}(\tau+nT,\nu) = {}^i\chi_{T_f}(\tau,\nu) \qquad (8)$$

where $^i\chi_{T_f}$ is either $^1\chi_{T_f}$ or $^2\chi_{T_f}$.

The periodicity of χ_{T_f} defined in (2) is, on the other hand, $2T$ because of the $\tau/2$ term in (2):

$$\chi_{T_f}(\tau+2nT,\nu) = \chi_{T_f}(\tau,\nu). \qquad (9)$$

If the signal $u(t)$ is a constant magnitude signal ($|u(t)| = 1$), which is the class of signals most widely utilized, periodicity of a higher frequency can be found for special cuts of the PAF. Taking $\tau = nT$, with n an integer:

$$\chi_{T_f}(nT,\nu) = \frac{1}{T_f}\int_0^{T_f}\left|u\left(t+\frac{nT}{2}\right)\right|^2 e^{j2\pi\nu t}\,dt$$
$$= \chi_{T_f}(0,\nu). \qquad (10)$$

If $T_f = NT$, an integer multiple of the basic period T, it can be shown [5], that

$$\left|\chi_{NT}\left(\tau+nT,\frac{m}{T}\right)\right| = \left|\chi_{NT}\left(\tau,\frac{m}{T}\right)\right|. \qquad (11)$$

D. Symmetry with Respect to the Origin

Changing the signs of τ and ν in (2) yields

$$\chi_{T_f}(-\tau,-\nu) = \frac{1}{T_f}\int_0^{T_f} u\left(t-\frac{\tau}{2}\right) u^*\left(t+\frac{\tau}{2}\right) e^{-j2\pi\nu t}\,dt$$
$$= \chi^*_{T_f}(\tau,\nu). \qquad (12)$$

Hence, as far as the function magnitude is concerned

$$|\chi_{T_f}(-\tau,-\nu)| = |\chi_{T_f}(\tau,\nu)|. \qquad (13)$$

This property of symmetry with respect to the origin is identical to that of the non-PAF. However, it should be noted that this property would not hold had a nonsymmetric definition, like those given in (3) and (4), been chosen for the PAF.

Taking the definition of ${}^1\chi_{T_f}$ in [3] the PAF at $(-\tau,-\nu)$ is, for the case $T_f = NT$:

$$ {}^1\chi_{NT}(-\tau,-\nu) = e^{j2\pi\nu\tau}\left[{}^1\chi_{NT}(\tau,\nu) + \frac{e^{j2\pi\nu NT}-1}{NT} \times \int_0^{\tau} u^*(t-\tau)u(t)e^{+j2\pi\nu t}\,dt\right]^*. \quad (14) $$

And for ${}^2\chi_{T_f}$ in [4]

$$ {}^2\chi_{NT}(-\tau,-\nu) = e^{j2\pi\nu\tau}\left[{}^2\chi_{NT}(\tau,\nu) + \frac{e^{j2\pi\nu NT}-1}{NT} \times \int_{-T_f/2}^{-T_f/2+\tau} u^*(t-\tau)u(t)e^{j2\pi\nu t}\,dt\right]^*. \quad (15) $$

From (14) and (15) it can be deduced that in this case symmetry with respect to the origin namely:

$$ |{}^i\chi_{NT}(-\tau,-\nu)| = |{}^i\chi_{NT}(\tau,\nu)| \quad (16) $$

exists for the following values of ν and τ:

$\nu = \frac{k}{NT}$; with k an integer

$\tau = NT$, which is valid, being periodic, to any $\tau = mNT$.

Figs. 1 and 2 demonstrate the basic period and the symmetry properties of both symmetric and asymmetric definitions of the PAF. The examples are given for a signal based on Frank code of length 16. A single period of the signal is composed of 16 chips of duration t_c, in which the phase of the signal is constant. Thus

$$ u(t) = \sum_{k=-\infty}^{\infty} u_k(t-kt_c) $$

$$ u_k(t) = \begin{cases} e^{j\phi_n} & 0 \le t \le t_c \\ 0 & \text{elsewhere} \end{cases} \qquad n = k \bmod 16 \quad (17) $$

$$ \phi_{4q+l} = 2\pi\frac{ql}{4} \qquad q,l = 0,\ldots,3. $$

Fig. 1 shows a contour map (contours 0.1, 0.3, 0.5, 0.7 and 0.9) of the PAF for this signal with the PAF defined according to the symmetric definition of (2). The $2T = 32t_c$ periodicity and the symmetry with respect to the origin can clearly be observed. On the other hand, a periodicity of $T = 16t_c$ is seen in Fig. 2, which shows the PAF of the Frank signal computed according to (3). It is also obvious that in this case the function is not symmetric.

It should be noted however that, as can be seen from (5) above and demonstrated in [2, 4, 5], the bulk of the multiperiod PAF volume is concentrated in ridges located along the lines $\nu = k/T$ parallel to the delay axis. Above those lines symmetry with respect to the origin holds for all of the above-mentioned definitions. Hence the multiperiod PAF looks symmetric independent of the PAF definition. The same effect can be observed regarding the basic periods of the function. Above those lines the basic period is T and not $2T$. This fact is shown in Fig. 3, which is the 10-period PAF of the same signal. No difference can be observed between the PAFs computed according to either definition.

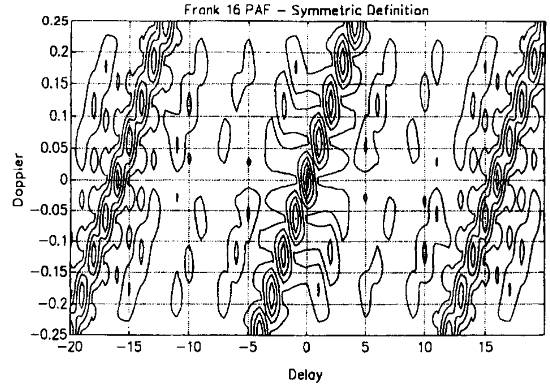

Fig. 1. Symmetrically defined PAF $|\chi_T(\tau,\nu)|$ of a length 16 Frank signal.

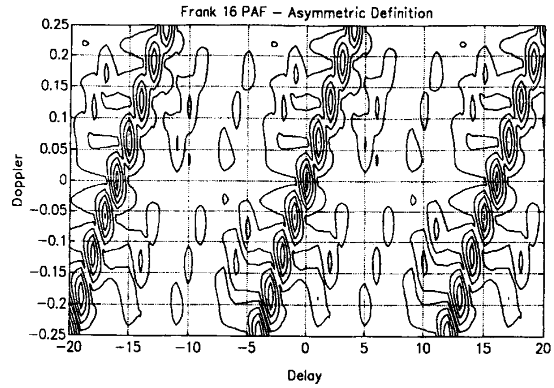

Fig. 2. Asymmetrically defined PAF $|{}^1\chi_T(\tau,\nu)|$ of a length 16 Frank signal.

E. Volume Limitation

As in the case of the non-PAF, there is a constraint on the volume under the PAF squared magnitude. However in the periodic case, this volume has to be calculated only for a single period of the PAF, that is on a strip extending parallel to the Doppler axis. We begin with a strip of width $2T$ in the delay direction. The symmetric definition of the PAF (2) has been

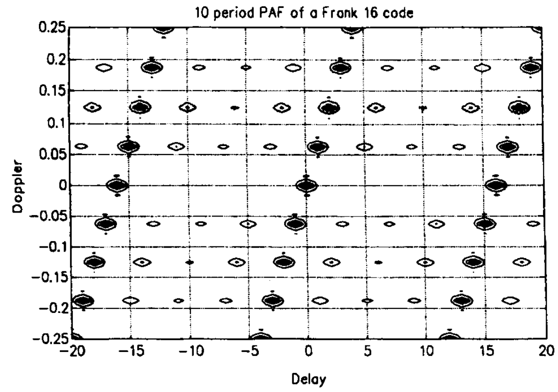

Fig. 3. Multiperiod PAF $|\chi_{NT}(\tau,\nu)|$ with $N = 10$, (defined either symmetrically or asymmetrically) of a length 16 Frank signal.

chosen here, without loss of generality

$$V_{2T} = \int_{\nu=-\infty}^{\infty}\int_{\tau=-T}^{T} |\chi_{T_f}(\tau,\nu)|^2 d\tau\, d\nu.$$

Replacing χ_{T_f} with the definition (2), and performing the integration over ν and one of the inner time variables yields:

$$V_{2T} = \frac{1}{T_f^2}\int_{-T}^{T}\int_{0}^{T_f} \left|u\left(t+\frac{\tau}{2}\right)u^*\left(t-\frac{\tau}{2}\right)\right|^2 dt\, d\tau.$$

For a uniform magnitude ($|u(t)| = 1$ for all t) waveform:

$$V_{2T} = \frac{1}{T_f^2}\int_{-T}^{T}\int_{0}^{T_f} 1\cdot dt\, d\tau = \frac{2T}{T_f}. \tag{18}$$

Since either a periodicity of T or a symmetry with respect to the origin holds for all the definitions of $\chi_{T_f}(\tau,\nu)$, it can be concluded that the volume within a strip of width T is one-half of the volume V_{2T}:

$$V_T = \frac{1}{2}V_{2T} = \frac{T}{T_f}. \tag{19}$$

For a single period, $T_f = T$, this volume is limited to $V_T = 1$, but for larger filter length the volume is reduced according to the period-to-filter length ratio. The unlimited reduction of V_T as $T_f \to \infty$ is the result of the unlimited improvement of the Doppler resolution, causing the Doppler mainlobe (at $\nu = 0$) and the Doppler sidelobes (at $\nu = n/T$) to be of negligible Doppler width.

III. CONCLUSIONS

In this paper some of the more important properties of the PAF are given. These properties include the multiple-single period PAF relationship, the basic periods, cuts along the principal axes, symmetry with respect to the origin and volume limitation.

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Clutter Covariance Smoothing by Subaperture Averaging

Space-time processing has been proposed as an approach by which an airborne radar can adaptively detect small targets in the presence of ground clutter and jammers, provided the covariance matrix of the interference is available. One of the difficulties in estimating the covariance matrix is the need to obtain a sufficient number of independent samples, especially for high pulse-repetition-frequency (PRF) systems. It is shown here that this difficulty can be overcome by combining subaperture averaging, or spatial smoothing, of the received data with conventional range bin averaging.

I. INTRODUCTION

The detection of small targets with an airborne radar is complicated by the presence of significant

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