

# Bounded Expected Delay in Arithmetic Coding

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**Abstract**—We address the problem of delay in an arithmetic coding system. Due to the nature of the arithmetic coding process, source sequences causing *arbitrarily large* encoding or decoding delays exist. This phenomena raises the question of just how large is the expected input to output delay in these systems, i.e., once a source sequence has been encoded, what is the expected number of source letters that should be further encoded to allow full decoding of that sequence. In this paper, we derive several new upper bounds on the expected delay for a memoryless source, which improve upon a known bound due to Gallager. The bounds provided are uniform in the sense of being independent of the sequence’s history. In addition, we give a sufficient condition for a source to admit a bounded expected delay, which holds for a stationary ergodic Markov source of any order.

## I. INTRODUCTION

Arithmetic coding has been introduced by Elias [1], as simple means to sequentially encode a source at its entropy rate, while significantly reducing the extensive memory usage characterizing non-sequential schemes. The basic idea underlying this technique is the successive mapping of growing source sequences into shrinking intervals of size equal to the probability of the corresponding sequence, and then representing those intervals by a binary expansion. Other coding schemes reminiscent of Elias’ arithmetic coding have been suggested since, aimed mostly to overcome the precision problem of the original scheme [2][3].

Delay in the classical setting of arithmetic coding stems from the discrepancy between source intervals and binary intervals, which may prohibit the encoder from producing bits (encoding delay) or the decoder from reproducing source letters (decoding delay). On top of its usual downside, delay also increases memory usage, and therefore a large delay may turn the main advantage of arithmetic coding on its head. As it turns out, for most sources there exists infinite number of source sequences for which the delay is infinite, where each sequence usually occurs with probability zero. A well known example demonstrating this phenomena is that of a uniform source over a ternary alphabet  $\{0, 1, 2\}$ . The source sequence  $111\dots$  is mapped into shrinking intervals that always contain the point  $\frac{1}{2}$ , and so not even a single bit can be encoded. This observation leads to the question of just how large is the expected delay (and consequently, the expected memory usage) of the arithmetic coding process for a given source, and if it is bounded at all.

The problem of delay can be practically dealt with by insertion of a fictitious source letter into the stream to “release” bits from the encoder or letters from the decoder,

whenever the delay exceed some predetermined threshold. Another possibility is coding of finite length sequences, so that a prefix condition is satisfied at the expense of a slightly higher redundancy, and blocks can be concatenated [4]. Nevertheless, it is still interesting to analyze the classical sequential setting in terms of expected delay.

In his lecture notes [5], Gallager has provided an upper bound for the expected delay in arithmetic coding for a memoryless source, which was later generalized to coding over cost channels [6]. Gallager’s bound is given by

$$\mathbb{E}(D) \leq \frac{\log(8e^2/\beta)}{\log(1/\alpha)} \triangleq \mathcal{D}_g(\alpha, \beta)$$

where  $\alpha$  and  $\beta$  are the maximal and minimal source letter probabilities respectively. Notice that this bound is independent of the sequence’s history, as shall be the case with all the bounds presented in this paper. In Theorem 1 (section III) we derive a new upper bound for the expected delay, given by

$$\mathbb{E}(D) \leq 1 + \frac{4\alpha(1 - \alpha + \log(1/\alpha))}{(1 - \alpha)^2} \triangleq \mathcal{D}_1(\alpha)$$

which depends only on the most favorable source letter. Following that, we show that the dependence on the least favorable letter in Gallager’s bound is unnecessary, and provide (section IV) a uniformly tighter version of the bound given by  $\mathcal{D}_{mg}(\alpha) \triangleq \mathcal{D}_g(\alpha, \alpha)$ . In Theorem 2 (section V) we derive another bound  $\mathcal{D}_2(\alpha)$  uniformly tighter than  $\mathcal{D}_1(\alpha)$ , which is also shown to be tighter than  $\mathcal{D}_{mg}(\alpha)$  for most sources, and looser only by a small multiplicative factor otherwise.

Our technique is extended to sources with memory, and in Theorem 3 (section VI) we provide a new sufficient condition for a source to have a bounded expected delay under arithmetic coding. Specifically, this condition is shown to hold for any stationary ergodic Markov source over a finite alphabet.

## II. ARITHMETIC CODING IN A NUTSHELL

Consider a discrete source over a finite alphabet  $\mathcal{X} = \{0, 1, \dots, K - 1\}$  with positive letter probabilities  $\{p_0, p_1, \dots, p_{K-1}\}$ . A finite source sequence is denoted by  $x_m^n = \{x_m, x_{m+1}, \dots, x_n\}$  with  $x^n = x_1^n$ , while an infinite one is denoted by  $x^\infty$ . An arithmetic coder maps the sequences  $x^n, x^{n+1}, \dots$  into a sequence of nested *source intervals*  $\mathcal{I}(x^n) \supset \mathcal{I}(x^{n+1}) \supset \dots$  in the unit interval that converge to a point  $y(x^\infty) = \bigcap_{n=1}^\infty \mathcal{I}(x^n)$ . The mapping is

defined as follows:

$$\begin{aligned} f_1(i) &= \sum_{j=0}^{i-1} p_j, \quad f(x^1) = f_1(x_1) \\ f(x^n) &= f(x^{n-1}) + f_1(x_n) \Pr(x^{n-1}) \\ \mathcal{I}(x^n) &= [f(x^n), f(x^n) + \Pr(x^n)) \end{aligned}$$

Notice that  $|\mathcal{I}(x^n)| = \Pr(x^n)$  and that source intervals corresponding to different sequences of the same length are disjoint. Following that, a random source sequence  $X^n$  is mapped into a random interval  $\mathcal{I}(X^n)$ , which as  $n$  grows converges to a random variable  $Y(X^\infty)$  that is uniformly distributed over the unit interval.

For any sequence of binary digits  $b^k = \{b_1, b_2, \dots, b_k\}$  we define a corresponding *binary interval*

$$\mathcal{J}(b^k) = [0.b_1b_2, \dots, b_k0, 0.b_1b_2, \dots, b_k1) \quad (1)$$

and the midpoint of  $\mathcal{J}(b^k)$  is denoted by  $m(b^k)$ .

The process of arithmetic coding is performed as follows. The encoder maps the input letters  $x^n$  into a source interval according to (1), and outputs the bits representing the smallest binary interval  $\mathcal{J}(b^k)$  containing the source interval  $\mathcal{I}(x^n)$ . This process is performed sequentially so the encoder produces further bits whenever it can. The decoder maps the received bits into a binary interval, and outputs source letters that correspond to the minimal source interval that contains that binary interval. Again, this process is performed sequentially so the decoder produces further source letters whenever it can.

### III. MEMORYLESS SOURCE

In this section, we provide a new bound on the expected delay of an arithmetic coding system for a memoryless source, as a function of the probability of the most likely source letter

$$\alpha \triangleq \max p_k.$$

All logarithms in this paper are taken to the base of 2.

*Theorem 1:* Assume a sequence of  $n$  source letters  $x^n$  has been encoded, and let  $D$  be the number of extra letters that need to be encoded to allow  $x^n$  to be fully decoded. Then

$$\Pr(D > d) \leq 4\alpha^d (1 + d \log(1/\alpha)) \quad (2)$$

independent of  $x^n$ . The expected delay is correspondingly bounded by

$$\mathbb{E}(D) \leq 1 + \frac{4\alpha(1 - \alpha + \log(1/\alpha))}{(1 - \alpha)^2} \triangleq \mathcal{D}_1(\alpha). \quad (3)$$

Let us first outline the idea behind the proof. The sequence  $x^n$  has been encoded into the binary sequence  $b^k$  which represents the minimal binary interval  $\mathcal{J}(b^k)$  satisfying  $\mathcal{I}(x^n) \subseteq \mathcal{J}(b^k)$ . The decoder has so far been able to decode only  $m < n$  letters, where  $m$  is maximal such that  $\mathcal{J}(b^m) \subseteq \mathcal{I}(x^m)$ . After  $d$  more source letters are fed to the encoder,  $x^{n+d}$  is encoded into  $b^{k'}$  where  $k' \geq k$  is maximal such that  $\mathcal{I}(x^{n+d}) \subseteq \mathcal{J}(b^{k'})$ . Thus, the entire sequence  $x^n$  is decoded if and only if

$$\mathcal{I}(x^{n+d}) \subseteq \mathcal{J}(b^{k'}) \subseteq \mathcal{I}(x^n). \quad (4)$$

Now, consider the middle point  $m(b^k)$ , which is always contained inside  $\mathcal{I}(x^n)$  as otherwise another bit could have been encoded. If  $m(b^k)$  is contained in  $\mathcal{I}(x^{n+d})$  (but not as an edge), then condition (4) cannot be satisfied, and the encoder cannot yield even one further bit. This observation can be generalized to a set of points which, if contained in  $\mathcal{I}(x^{n+d})$ ,  $x^n$  cannot be completely decoded. For each of these points the encoder outputs a number of bits which may enable the decoder to produce source letters, but not enough to fully decode  $x^n$ . The encoding and decoding delays are therefore treated here simultaneously, rather than separately as in [6].

We now introduce some notations and prove a Lemma, required for the proof of Theorem 1. Let  $[a, b) \subseteq [0, 1)$  be some interval, and  $p$  some point in that interval. In the definitions that now follow we sometime omit the dependence on  $a, b$  for brevity. We say that  $p$  is *strictly contained* in  $[a, b)$  if  $p \in [a, b)$  but  $p \neq a$ . We define the *left-adjacent* of  $p$  w.r.t.  $[a, b)$  to be

$$\ell(p) \triangleq \min \{x \in [a, p) : \exists k \in \mathbb{Z}^+, x = p - 2^{-k}\}$$

and the *t-left-adjacent* of  $p$  w.r.t.  $[a, b)$  as

$$\ell^{(t)}(p) \triangleq \overbrace{(\ell \circ \ell \circ \dots \circ \ell)}^t(p), \quad \ell^{(0)}(p) \triangleq p$$

Notice that  $\ell^{(t)}(p) \rightarrow a$  monotonically with  $t$ . We also define the *right-adjacent* of  $p$  w.r.t.  $[a, b)$  to be

$$r(p) \triangleq \max \{x \in (p, b) : \exists k \in \mathbb{Z}^+, x = p + 2^{-k}\}$$

and  $r^{(t)}(p)$  as the *t-right-adjacent* of  $p$  w.r.t.  $[a, b)$  similarly, where now  $r^{(t)}(p) \rightarrow b$  monotonically. For any  $\delta < b - a$ , the *adjacent  $\delta$ -set* of  $p$  w.r.t.  $[a, b)$  is defined as the set of all adjacents that are not "too close" to the edges of  $[a, b)$ :

$$S_\delta(p) \triangleq \left\{ x \in [a + \delta, b - \delta) : \exists t \in \mathbb{Z}^+ \cup \{0\}, \right. \\ \left. x = \ell^{(t)}(p) \vee x = r^{(t)}(p) \right\}$$

Notice that for  $\delta > p - a$  this set may contain only right-adjacents, for  $\delta > b - p$  only left-adjacents, for  $\delta > \frac{b-a}{2}$  it is empty, and for  $\delta = 0$  it is infinite.

*Lemma 1:* The size of  $S_\delta(p)$  is bounded by

$$|S_\delta(p)| \leq 1 + 2 \log \frac{|b-a|}{\delta} \quad (5)$$

*Proof:* It is easy to see that the number of t-left-adjacents of  $p$  that are larger than  $a + \delta$  is the number of ones in the binary expansion of  $(p - a)$  up to resolution  $\delta$ . Similarly, the number of t-right-adjacents of  $p$  that are smaller than  $b - \delta$  is the number of ones in the binary expansion of  $(b - p)$  up to resolution  $\delta$ . Defining  $\lceil x \rceil^+ \triangleq \max(\lceil x \rceil, 0)$ , we get:

$$\begin{aligned} |S_\delta(p)| &\leq \lceil \log \frac{p-a}{\delta} \rceil^+ + \lceil \log \frac{b-p}{\delta} \rceil^+ \\ &\leq \begin{cases} 2 + \log \frac{(p-a)(b-p)}{\delta^2} & , \delta < p-a, b-p \\ 1 + \log \frac{|b-a|}{\delta} & , o.w. \end{cases} \\ &\leq 1 + 2 \log \frac{|b-a|}{\delta} \end{aligned} \quad (6)$$

as desired.  $\blacksquare$

*Proof of Theorem 1:* Assume the source sequence  $x^n$  has been encoded into the binary sequence  $b^k$ , and let  $Y = Y(x^\infty)$ . Given  $x^n$ ,  $Y$  is uniformly distributed over  $\mathcal{I}(x^n)$ , and thus for any interval  $\mathcal{T}$

$$\Pr(Y \in \mathcal{T} | x^n) = \frac{|\mathcal{T} \cap \mathcal{I}(x^n)|}{|\mathcal{I}(x^n)|} \leq \frac{|\mathcal{T}|}{|\mathcal{I}(x^n)|} \quad (7)$$

The size of the interval  $\mathcal{I}(x^{n+d})$  for  $d \geq 0$  is bounded by

$$\begin{aligned} |\mathcal{I}(x^{n+d})| &= \Pr(x^{n+d}) = \Pr(x_{n+1}^{n+d} | x^n) \Pr(x^n) \\ &= \Pr(x_{n+1}^{n+d} | x^n) |\mathcal{I}(x^n)| \leq \alpha^d |\mathcal{I}(x^n)| \end{aligned} \quad (8)$$

Combining (7) and (8), we have that for any point  $p \in \mathcal{I}(x^n)$

$$\begin{aligned} \Pr(p \in \mathcal{I}(X^{n+d}) | x^n) &\leq \Pr(|Y - p| \leq |\mathcal{I}(x^n)| \alpha^d | x^n) \\ &\leq \frac{2\alpha^d |\mathcal{I}(x^n)|}{|\mathcal{I}(x^n)|} = 2\alpha^d \end{aligned} \quad (9)$$

where the probabilities are taken w.r.t. to ‘‘future’’ source letters. For any interval  $\mathcal{T} \subseteq \mathcal{I}(x^n)$  that shares an edge with  $\mathcal{I}(x^n)$  we have that

$$\Pr(\mathcal{T} \cap \mathcal{I}(X^{n+d}) \neq \phi) \leq \frac{\alpha^d |\mathcal{I}(x^n)| + |\mathcal{T}|}{|\mathcal{I}(x^n)|} = \alpha^d + \frac{|\mathcal{T}|}{|\mathcal{I}(x^n)|} \quad (10)$$

For any  $\delta \geq 0$ , let  $S_\delta$  denote the adjacent  $\delta$ -set of  $m(b^k)$  w.r.t. the interval  $\mathcal{I}(x^n)$ . Given  $x^n$ , the probability that the delay  $D$  is larger than  $d$  is the probability that (4) is not satisfied, which in turn is equal to the probability that the intersection  $S_0 \cap \mathcal{I}(X^{n+d})$  is not empty. This fact is explained as follows. As already shown, if  $m(b^k)$  is strictly contained in  $\mathcal{I}(X^{n+d})$  then the encoder emits no further bits, and the delay is larger than  $d$ . Otherwise, assume  $\mathcal{I}(X^{n+d})$  lies on the left side of  $m(b^k)$ . Obviously, if  $\mathcal{I}(X^{n+d}) \subseteq [\ell(m(b^k)), m(b^k))$ , then  $x^n$  is fully decoded since (4) is satisfied. However, if  $\ell(m(b^k))$  is strictly contained in  $\mathcal{I}(X^{n+d})$  then (4) is not satisfied,  $x^n$  cannot be decoded and the delay is larger than  $d$ . The same rationale also applies to  $r(m(b^k))$ . Continuing the argument recursively, it is easy to see that  $x^n$  can be decoded if and only if no point of  $S_0$  is strictly contained in  $\mathcal{I}(X^{n+d})$ .

Now, notice that  $S_\delta \subseteq S_0$ , and that  $S_0 \setminus S_\delta$  is contained in two intervals of length  $\delta$  both sharing an edge with  $\mathcal{I}(x^n)$  (the situation is illustrated in Figure 1). Letting  $\mathcal{J}_B$  denote a general binary interval, we bound the delay’s tail probability:

$$\begin{aligned} \Pr(D > d | x^n) &= \Pr(\mathcal{I}(X^{n+d}) \not\subseteq \mathcal{J}_B, \forall \mathcal{J}_B \subseteq \mathcal{I}(x^n) | x^n) \\ &= \Pr(S_0 \cap \mathcal{I}(X^{n+d}) \neq \phi | x^n) \leq \\ &\leq \Pr(S_\delta \cap \mathcal{I}(X^{n+d}) \neq \phi | x^n) + \\ &\quad + \Pr((S_0 \setminus S_\delta) \cap \mathcal{I}(X^{n+d}) \neq \phi | x^n) \\ &\leq 2\alpha^d |\mathcal{S}_\delta| + 2 \left( \alpha^d + \frac{\delta}{|\mathcal{I}(x^n)|} \right) \leq \\ &\leq 2\alpha^d \left( 1 + 2 \log \frac{|\mathcal{I}(x^n)|}{\delta} \right) + \\ &\quad + 2 \left( \alpha^d + \frac{\delta}{|\mathcal{I}(x^n)|} \right) \end{aligned} \quad (11)$$

Lemma 1 and equations (9),(10) were used in the transitions. Taking the derivative of the right-hand-side of (11) w.r.t.  $\delta$  we find that  $\delta = 2\alpha^d |\mathcal{I}(x^n)|$  minimizes the bound. We get:

$$\begin{aligned} \Pr(D > d | x^n) &\leq 2\alpha^d \left( 1 + 2 \log \frac{1}{2\alpha^d} \right) + 6\alpha^d \\ &= 4\alpha^d (1 + d \log(1/\alpha)) \end{aligned}$$

and (2) is proved. Now, the expectancy of  $D$  given  $x^n$  can be bounded accordingly

$$\begin{aligned} \mathbb{E}(D | x^n) &= \sum_{d=1}^{\infty} d \Pr(D = d | x^n) = \sum_{d=1}^{\infty} \Pr(D \geq d | x^n) \\ &\leq 1 + \sum_{d=1}^{\infty} \Pr(D > d | x^n) \\ &\leq 1 + 4 \sum_{d=1}^{\infty} \alpha^d (1 + d \log(1/\alpha)) \\ &= 1 + \frac{4\alpha(1 - \alpha + \log(1/\alpha))}{(1 - \alpha)^2} \end{aligned} \quad (12)$$

and (3) is proved. Notice that both of the bounds above are uniform so the dependence on  $x^n$  can be removed.  $\blacksquare$

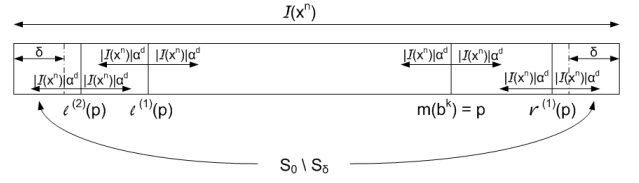


Fig. 1. Source interval illustration

#### IV. IMPROVING GALLAGER’S BOUND

Gallager [5] provided an upper bound for the expected delay in arithmetic coding of a memoryless source, given by

$$\mathbb{E}(D) \leq \frac{\log(8e^2/\beta)}{\log(1/\alpha)} \triangleq \mathcal{D}_g(\alpha, \beta)$$

where  $\alpha \triangleq \max p_k$  and  $\beta \triangleq \min p_k$ . Notice that our bound  $\mathcal{D}_1(\alpha)$  in (3) depends only on the most likely source letter, while Gallager’s bound  $\mathcal{D}_g(\alpha, \beta)$  depends also on the least likely source letter. Moreover, holding  $\alpha$  constant we find that  $\mathcal{D}_g(\alpha, \beta) \xrightarrow{\beta \rightarrow 0} \infty$ . This phenomena is demonstrated in the following example.

*Example:* Consider a ternary source with letter probabilities  $(p, \frac{1-p}{2}, \frac{1-p}{2})$ . Both bounds for that source are depicted in Figure 2 as a function of  $p$ , together with a modified bound derived in the sequel. As can be seen, Gallager’s bound is better for most values of  $p$ , but becomes worse for small  $p$ , due to its dependence on the least probable source letter. In fact, the bound diverges when  $p \rightarrow 0$ , which is counterintuitive since we expect the delay in this case will approach that of a uniform binary source (for which  $\mathcal{D}_g(\alpha, \beta)$  is finite). In contrast, the new bound which depends only on the most likely letter tends to a constant when  $p \rightarrow 0$ , which equals its value for the corresponding binary case.

Intuition suggests that least likely letters are those that tend to accelerate the coding/decoding process, and that the dominating factor influencing the delay should be the most likely source letters. Motivated by that, we turn to examine the origin of the term  $\beta$  in Gallager's derivations.

Gallager's bound for the expected delay is derived via a corresponding bound on the *information delay*, i.e., the difference in self-information between a source sequence and an extended source sequence needed to ensure that the original sequence is completely decoded. We remind the reader that the self information of a sequence  $x^n$  is just  $-\log(\Pr(x^n))$ . We now follow the derivations in [6], replacing notations with our own and modifying the proof to remove the dependence on  $\beta$ . Notice that [6] analyzes the more general setting of cost channels which reduces to that of [5] and to ours and by setting  $N = 2, C = c_i = c_{max} = 1$  (in the notation therein).

Consider a source sequence encoded by a binary sequence  $b^k$ . A bound on the expected self-information of that sequence with the last letter truncated is given by [6, equations 10,11]

$$\mathbb{E}(I(x^{n(k)-1})|b^k) \leq k + \log(2e) \quad (13)$$

where  $n(k)$  is the number of source letters emitted by the source, and  $I(x^{n(k)-1})$  is the self-information of the corresponding source sequence without the last letter. Using the relation  $I(x^n) \leq I(x^{n-1}) + \log(1/\beta)$ , we get a bound on the self-information of the sequence [6, equation 14]:

$$\mathbb{E}(I(x^{n(k)})|b^k) \leq k + \log(2e/\beta) \quad (14)$$

This is the only origin of the term  $\beta$ . In order to obtain a bound on the expected information delay, there seems to be no escape from the dependence on  $\beta$ . However, we are interested in the delay in source letters. We therefore continue to follow [6] but use (13) in lieu of (14) to bound the information delay up to one letter before the last needed for decoding. This approach eliminates the dependence on the least likely letter, which if appears last may increase the self-information considerably but meanwhile contribute only a single time unit to the delay.

Consider a specific source sequence  $x^n$ . A bound on the expected number of bits  $k(n)$  required to decode that sequence is given by [6, equation 15]:

$$\mathbb{E}(k(n)|x^n) \leq I(x^n) + \log(4e) \quad (15)$$

Now, let  $b^{k(n)}$  be the binary sequence required to decode  $x^n$ . Using (13) (instead of (14) used in [5] and [6]) we have that

$$\mathbb{E}(I(x^{n+D-1})|b^{k(n)}, x^n) \leq k(n) + \log(2e) \quad (16)$$

where  $D$  is the number of extra letters needed to ensure the encoder emits the necessary  $k(n)$  bits. Using (15) and taking the expectation w.r.t.  $k(n)$  we find that

$$\mathbb{E}(I(x^{n+D-1})|x^n) - I(x^n) \leq \log(8e^2) \quad (17)$$

and the modified bound for the delay in source letters follows through by dividing (17) by the minimal letter self-information  $\log(1/\alpha)$  and rearranging the terms:

$$\mathbb{E}(D|x^n) \leq 1 + \frac{\log(8e^2)}{\log(1/\alpha)} \triangleq \mathcal{D}_{mg}(\alpha) \quad (18)$$

Notice that the modified Gallager bound  $\mathcal{D}_{mg}(\alpha) = \mathcal{D}_g(\alpha, \alpha)$  is uniformly lower  $\mathcal{D}_g(\alpha, \beta)$ , and coincides with it only for uniformly distributed sources.

*Example (continued):* The modified Gallager bound for the ternary source converges for  $p \rightarrow 0$ , as illustrated in Figure 2. It is also easy to verify that it converges to the same value it takes for a uniform binary source.

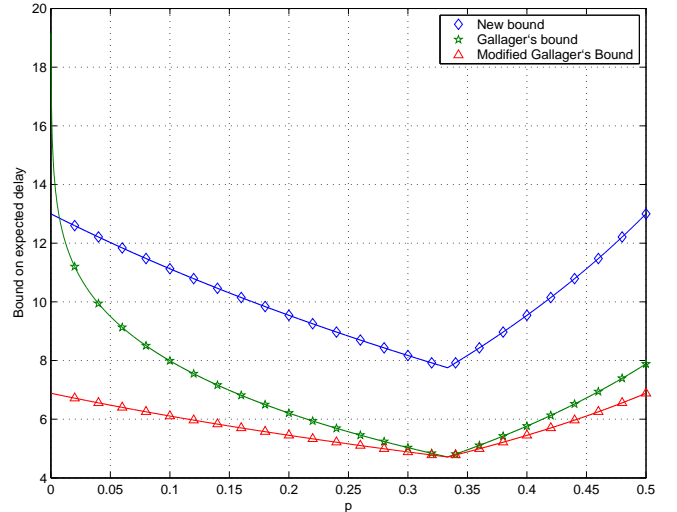


Fig. 2. Bounds for the ternary source

The ratio of our bound to the modified Gallager bound is depicted in Figure 3, together with two tighter bounds introduced in the following section. Comparing the bounds, we find that  $\mathcal{D}_1(\alpha)$  is at most  $\sim 2.4$  times worse than  $\mathcal{D}_{mg}(\alpha)$ , and is even better for small (below  $\sim 0.069$ ) values of  $\alpha$ . For  $\alpha \rightarrow 0$  the ratio tends to unity, since both  $\mathcal{D}_1(\alpha)$  and  $\mathcal{D}_{mg}(\alpha)$  approach 1, the minimal possible delay for a source that is not 2-adic. Indeed, for a very small  $\alpha$  it is intuitively clear that even when a single extra letter is encoded, the source interval decreases significantly which enables decoding of the preceding source interval with high probability.

## V. IMPROVING OUR BOUND

As we have seen,  $\mathcal{D}_1(\alpha)$  is good for small values of  $\alpha$  (the probability of the most likely letter) and becomes worse for larger values. The source of this behavior lies in a somewhat loose analysis of the size of  $S_\delta$  for large  $\delta$ , and also since for large  $\alpha$  and small  $d$  the bound (2) may exceed unity. A more subtle analysis enables us to improve our bound for large  $\alpha$ , and the result is now stated without proof.

*Theorem 2:* Let  $d_0 = \lfloor \frac{2}{\log(1/\alpha)} \rfloor$ , and define  $d_1 \geq d_0$  to be the largest such integer for which every integer  $d_0 < d \leq d_1$  (if there are any) satisfies

$$2\alpha^d (1 + 2d \log(1/\alpha)) > 1$$

The expected delay of an arithmetic coding system for a

memoryless source is bounded by

$$\mathbb{E}(D) \leq \mathcal{D}_2(\alpha) \triangleq \frac{1 + d_1 + \frac{2\alpha^{d_1+1}}{1-\alpha} + \frac{4\alpha^{d_1+1}(d_1(1-\alpha) + 1) \log(1/\alpha)}{(1-\alpha)^2}}{1} \quad (19)$$

An explicit bound  $\mathcal{D}_3(\alpha)$  (though looser for large  $\alpha$ ) can be obtained by substituting  $d_1 = d_0$ . The ratio of our original bound  $\mathcal{D}_1(\alpha)$ , the modified bound  $\mathcal{D}_2(\alpha)$  and its looser version  $\mathcal{D}_3(\alpha)$  to the modified Gallager bound  $\mathcal{D}_{mg}(\alpha)$  are depicted in Figure 3. As can be seen,  $\mathcal{D}_2(\alpha)$  is tighter than  $\mathcal{D}_{mg}(\alpha)$  for values of  $\alpha$  smaller than  $\sim 0.71$ , and for larger values is looser but only up to a multiplicative factor of  $\sim 1.04$ . Notice again that all of the bounds coincide for  $\alpha \rightarrow 0$ , as in this case they all tend to 1 which is the best possible general upper bound.

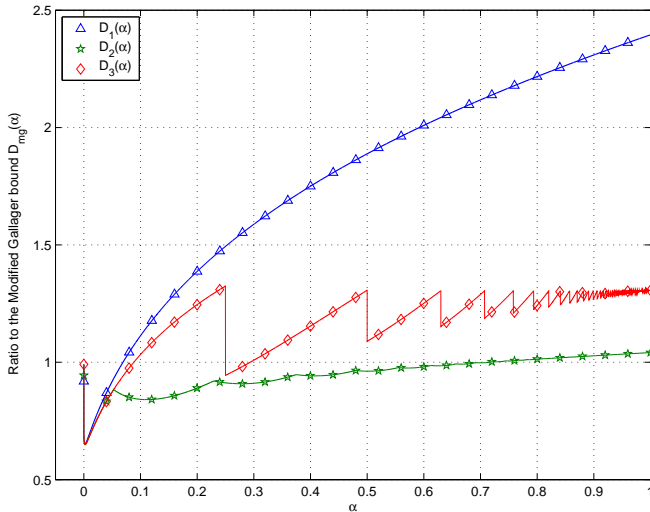


Fig. 3. The ratio of the different bounds to the modified Gallager bound

## VI. SOURCES WITH MEMORY

The discussion of section III is easily generalized to sources with memory. The only point in the proof that needs to be reestablished is the definition of  $\alpha$ , which was the probability of the most likely source letter in the memoryless case.

*Theorem 3:* Consider an arithmetic coding system for a source with a probability distribution  $p(x^n)$  over a finite alphabet  $\mathcal{X}$ . Let

$$\gamma(d) \triangleq \sup_n \max_{x^{n+d} \in \mathcal{X}^{n+d}} p(x^{n+d} | x^n)$$

If  $\gamma(d) = o(d^{-(1+\varepsilon)})$  for some  $\varepsilon > 0$ , then the expected delay of the system is bounded.

*Proof:* The derivations for the memoryless case can be repeated, with  $\alpha^d$  replaced by  $\gamma(d)$ . The bound (12) becomes

$$\mathbb{E}(D | x^n) \leq 1 + 4 \sum_{d=1}^{\infty} \gamma(d) \left( 1 + \log \frac{1}{\gamma(d)} \right)$$

If the sum above converges, then we have the bounded expected delay property. The condition given in Theorem 3 is sufficient to that end. ■

For a memoryless source,  $\gamma(d) = \alpha^d$  and the condition is satisfied. It is also fulfilled for any source with memory whose conditional letter probabilities are bounded away from 1, and thus such sources admit a bounded expected delay. This fact was already observed in [6] with the additional requirement for the conditional probabilities to be bounded away from 0 as well (a byproduct of the dependency on the least favorable letter). The condition in Theorem 3 is however more general. As an example, consider a stationary ergodic first order Markov source. Such a source satisfies

$$p(x^{n+|X|} | x^n) < 1, \quad \forall x^{n+|X|} \in \mathcal{X}^{n+|X|} \quad (20)$$

since otherwise the source would have a deterministic cycle which contradicts the ergodic assumption. Define:

$$\xi \triangleq \max_{x^{n+|X|} \in \mathcal{X}^{n+|X|}} p(x^{n+|X|} | x^n)$$

We have from (20) that  $\xi < 1$ , and since the source is stationary,  $\xi$  is also independent of  $n$ .  $\gamma(d)$  is monotonically non-increasing and therefore  $\gamma(d) \leq \xi^{\lfloor \frac{d}{|X|} \rfloor}$  and is exponentially decreasing with  $d$ , thus satisfying the condition in Theorem 3. This result can be generalized to any Markov order.

*Corollary 1:* The expected delay of arithmetic coding for a finite alphabet, stationary ergodic Markov source of any order is bounded.

## VII. SUMMARY

New upper bounds on the expected delay of an arithmetic coding system for a memoryless source were derived, as a function of the probability  $\alpha$  of the most likely source letter. In addition, a known bound due to Gallager that depends also on the probability of the least likely source letter was uniformly improved by disposing of the latter dependence. Our best bound was compared to the modified Gallager bound, and shown to be tighter for  $\alpha < 0.71$  and looser by a multiplicative factor no larger than  $\sim 1.04$  otherwise. The bounding technique was generalized to sources with memory, providing a sufficient condition for a bounded delay. Using that condition, it was shown that the bounded delay property holds for a stationary ergodic Markov source of any order.

Future research calls for a more precise characterization of the expected delay in terms of the entire probability distribution, which might be obtained by further refining the bounding technique presented in this paper. In addition, a generalization to coding over cost channels and finite-state noiseless channels in the spirit of [6] can be considered as well.

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