Achieving the Empirical Capacity for Any Noise Sequence Using Feedback

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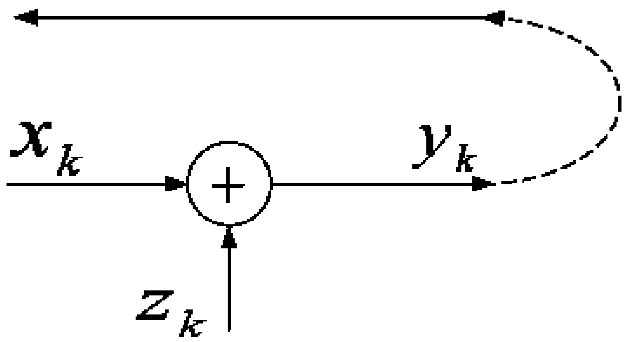
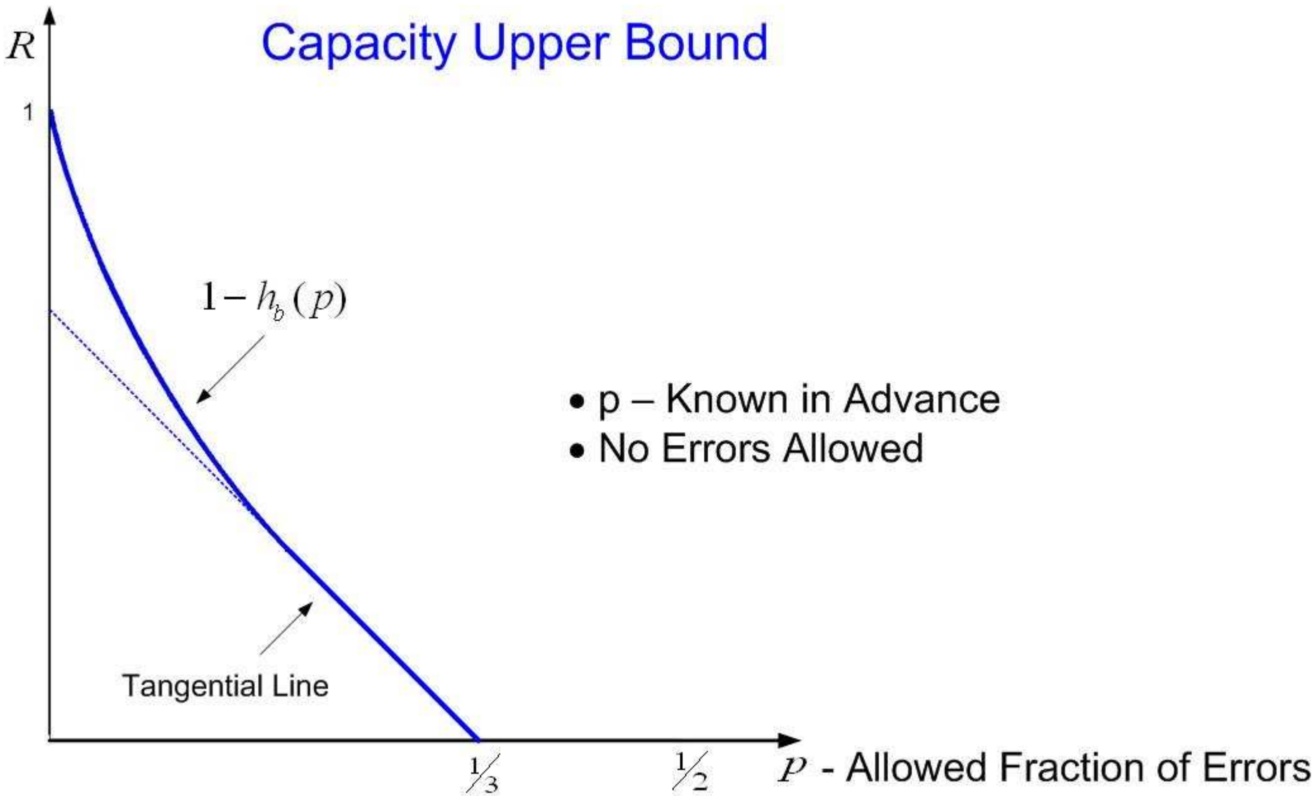
Problem Setting

A binary modulo-additive channel with noiseless feedback:



- Individual noise sequence $\underline{z}$ of length n
- May be generated by an adversary that knows the message
- Goal - Reliable communications with maximal rate R

Berlekamp's Classical Result



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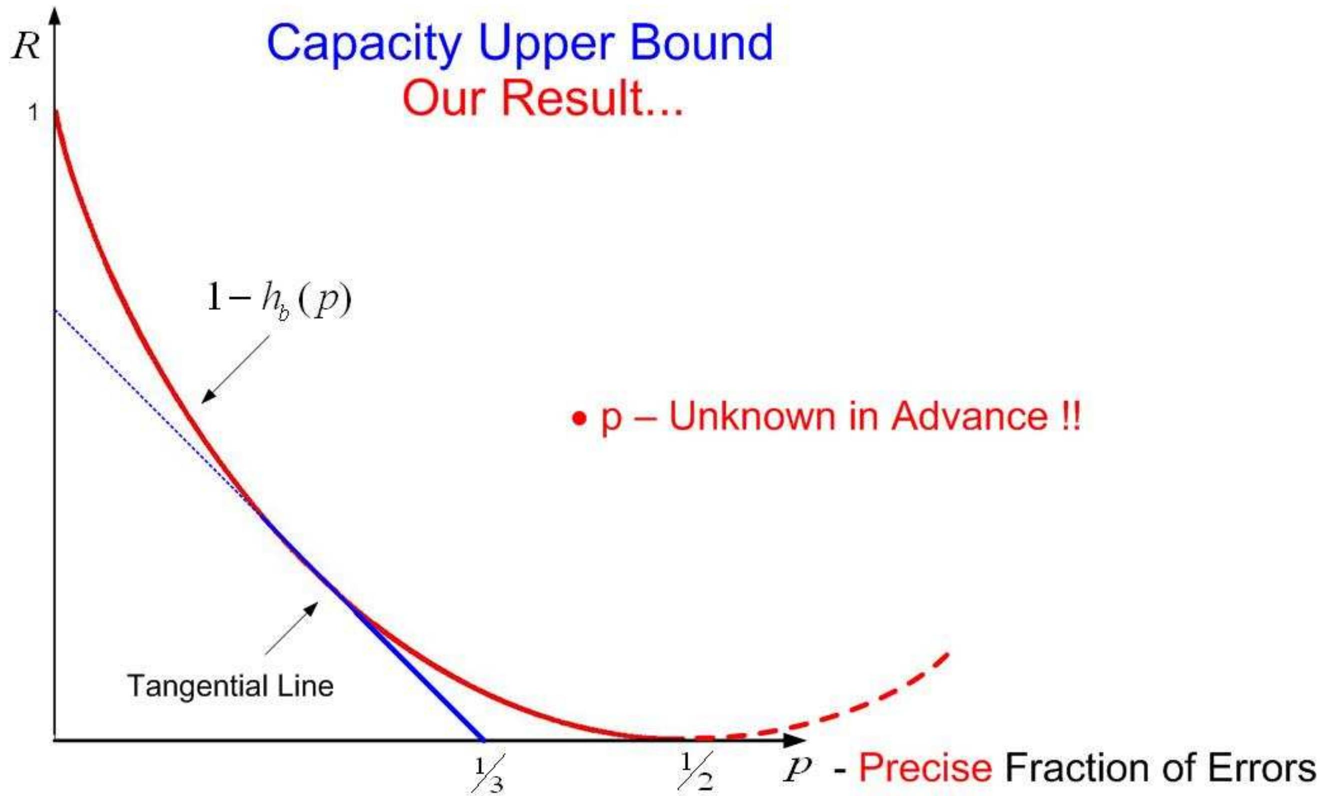

Previous Result for Unknown p

- Arbitrarily Varying Channel (AVC) setting with 2 states (clean/inverting) and feedback
- This AVC is *symmetrizable* [CsiszárNarayan‘88] $\Rightarrow$

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The Capacity is Zero !

Our Result

For an individual noise sequence $\underline{z}$ with any empirical probability p **unknown in advance**:

$$\boxed{R \cong 1 - h_b(p)}$$

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Specifically: *We introduce a* ***randomized*** *sequential transmission scheme, attaining a rate* $R = R(\underline{\boldsymbol{z}})$ *so that for some* $\delta(n) \to 0$

$$P_r\Big(R \geq 1 - h_b(p) - \delta(n)\Big) \xrightarrow[n\to\infty]{} 1$$

with a ***vanishing error probability*** *w.r.t. the randomization*

Background - The Horstein Scheme

[Horstein‘63]

- Capacity achieving scheme for the BSC_p with feedback
- The message is represented by a point $\omega \in [0, 1)$
- The transmitter steers the receiver in the direction of ω
- The receiver gradually zooms in on ω

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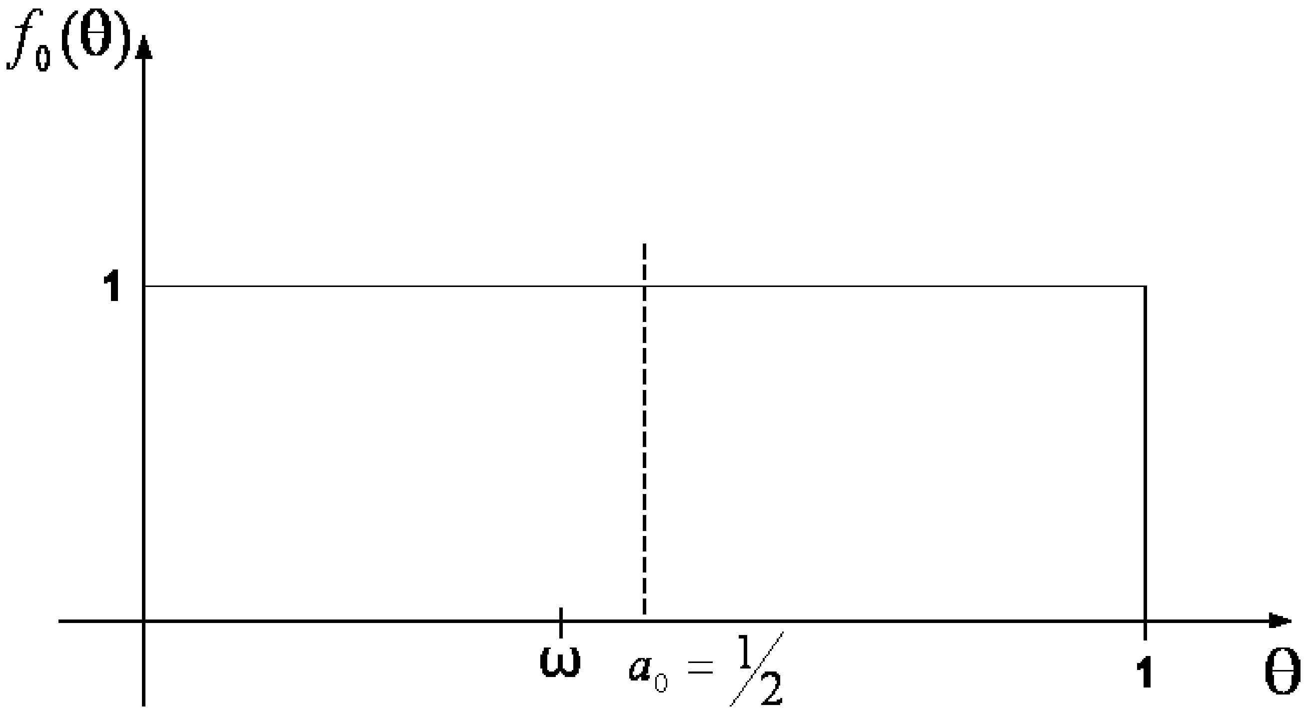
a_0 - Median point w.r.t. $f_0(\theta)$



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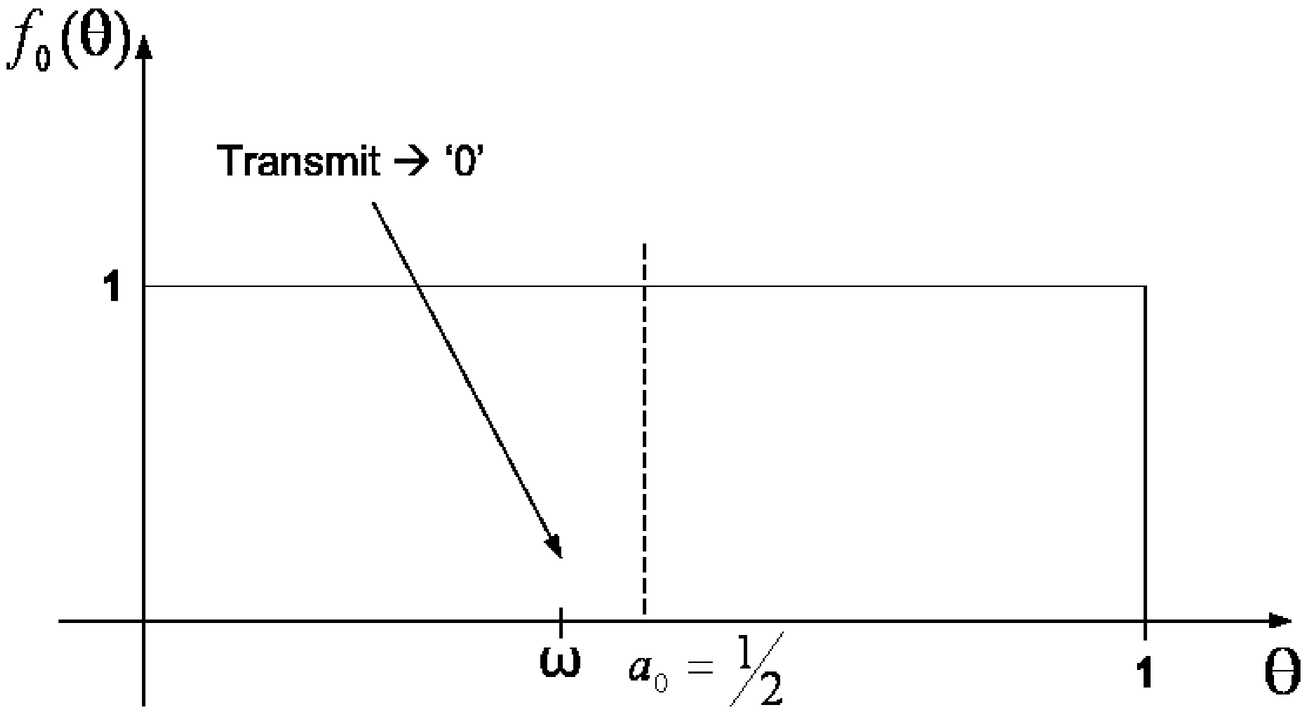

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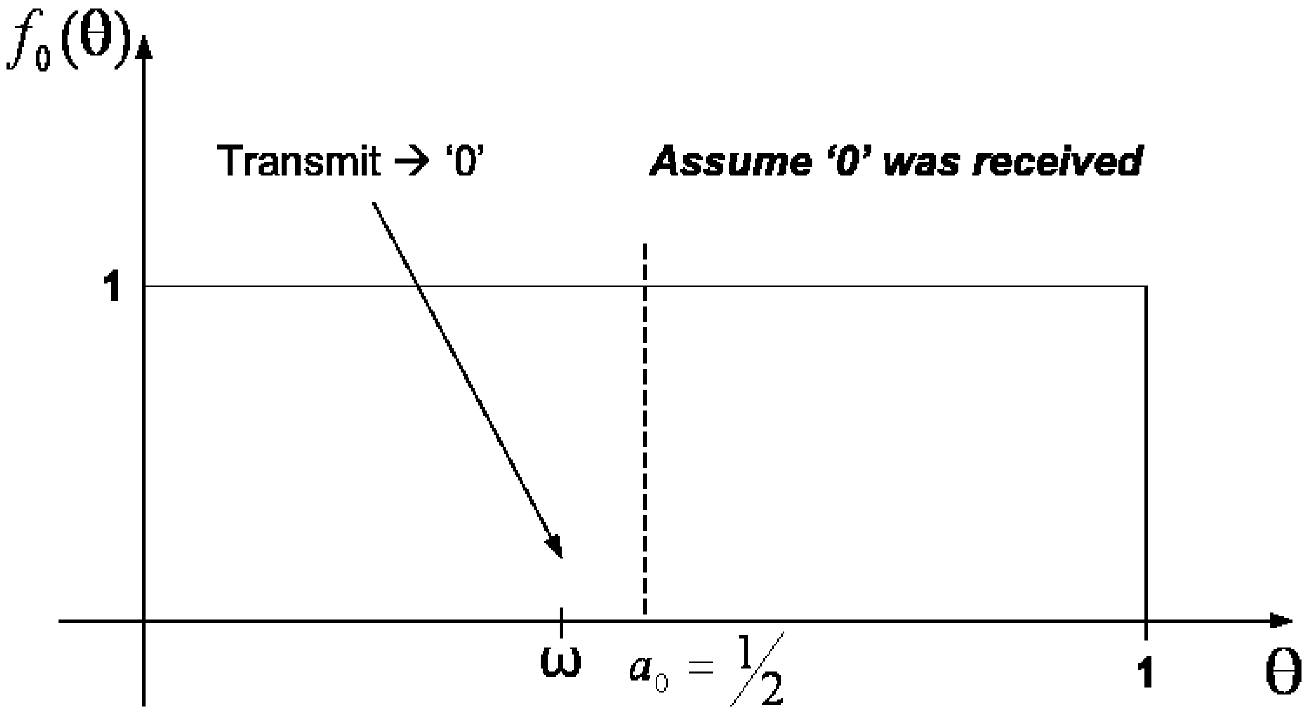

$f_1(\theta)$ - a-posteriori density given the received bit

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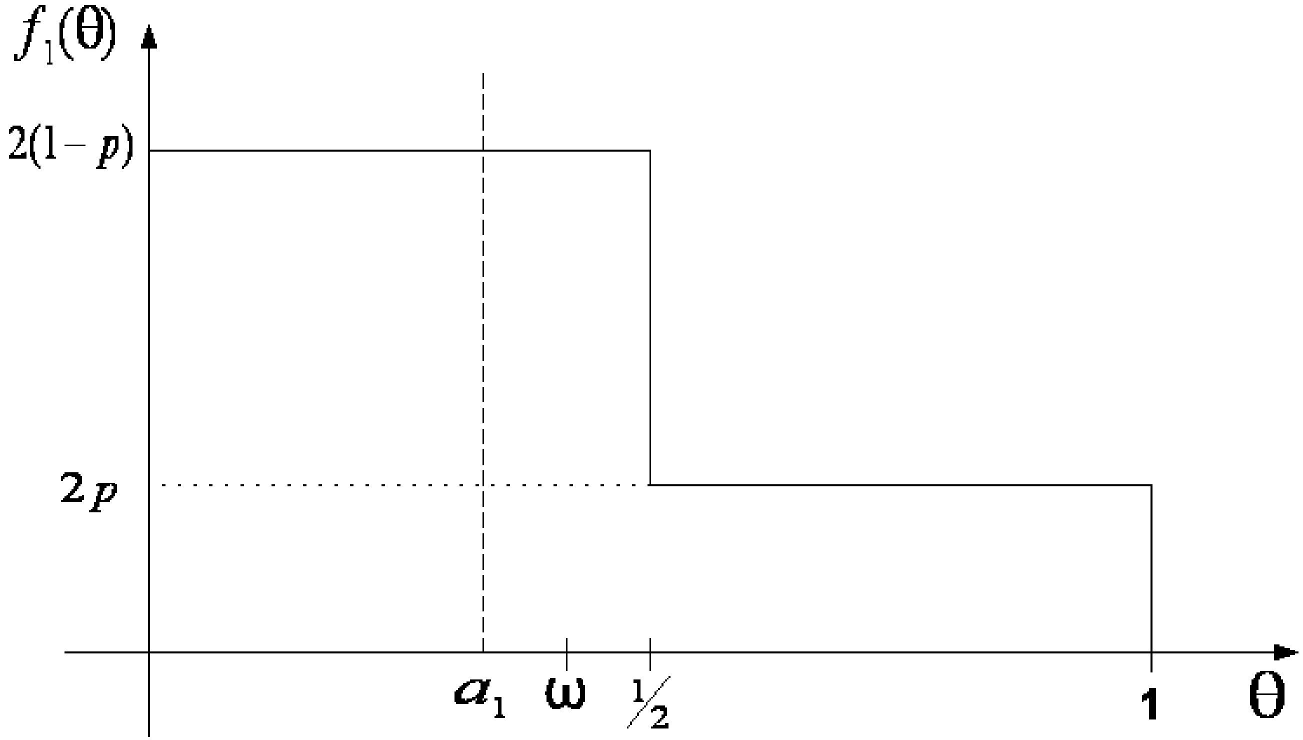

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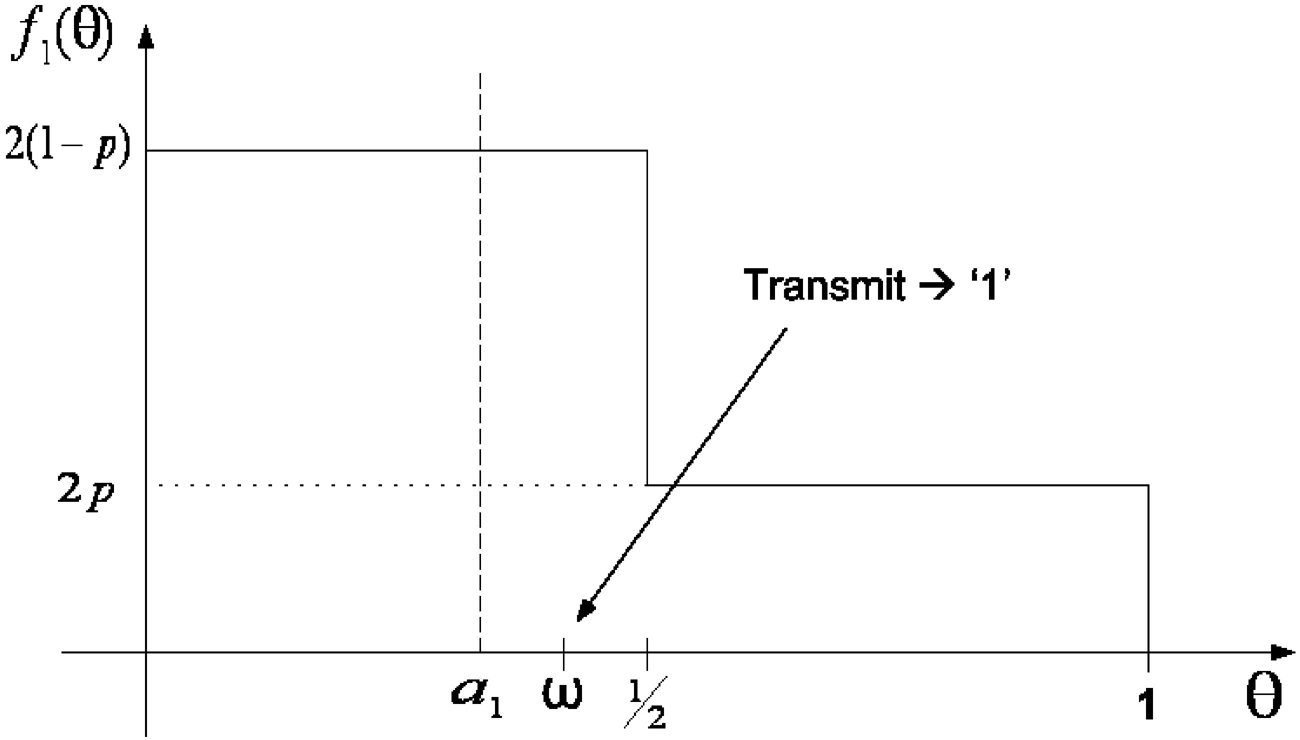

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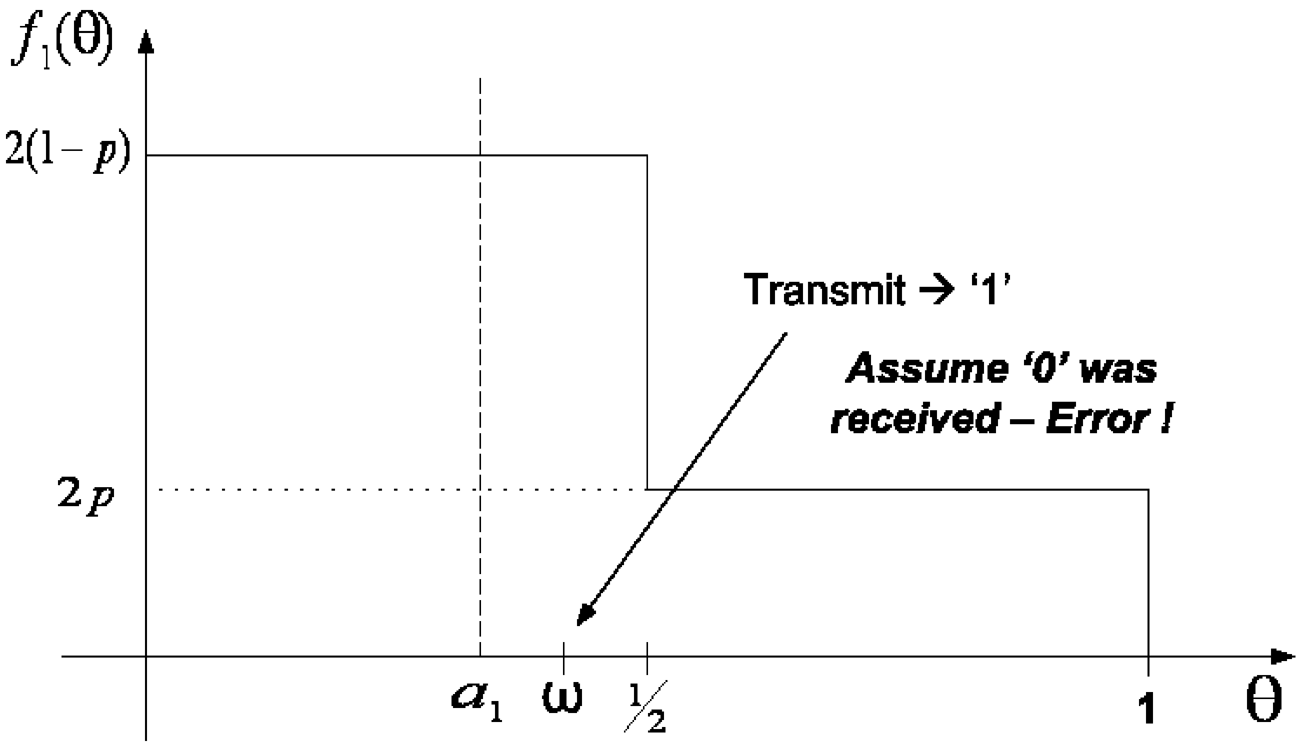

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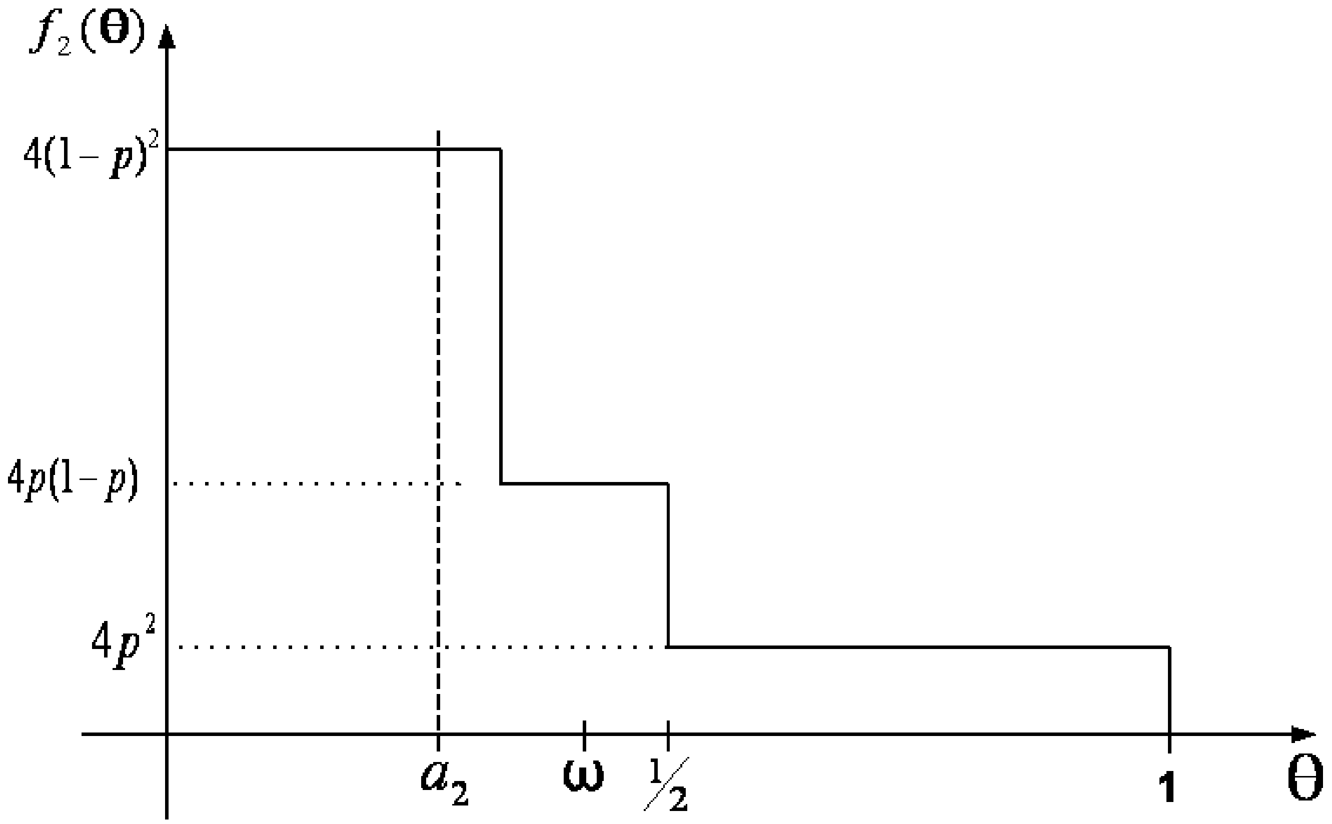
a_2 - Median point w.r.t. $f_2(\theta)$



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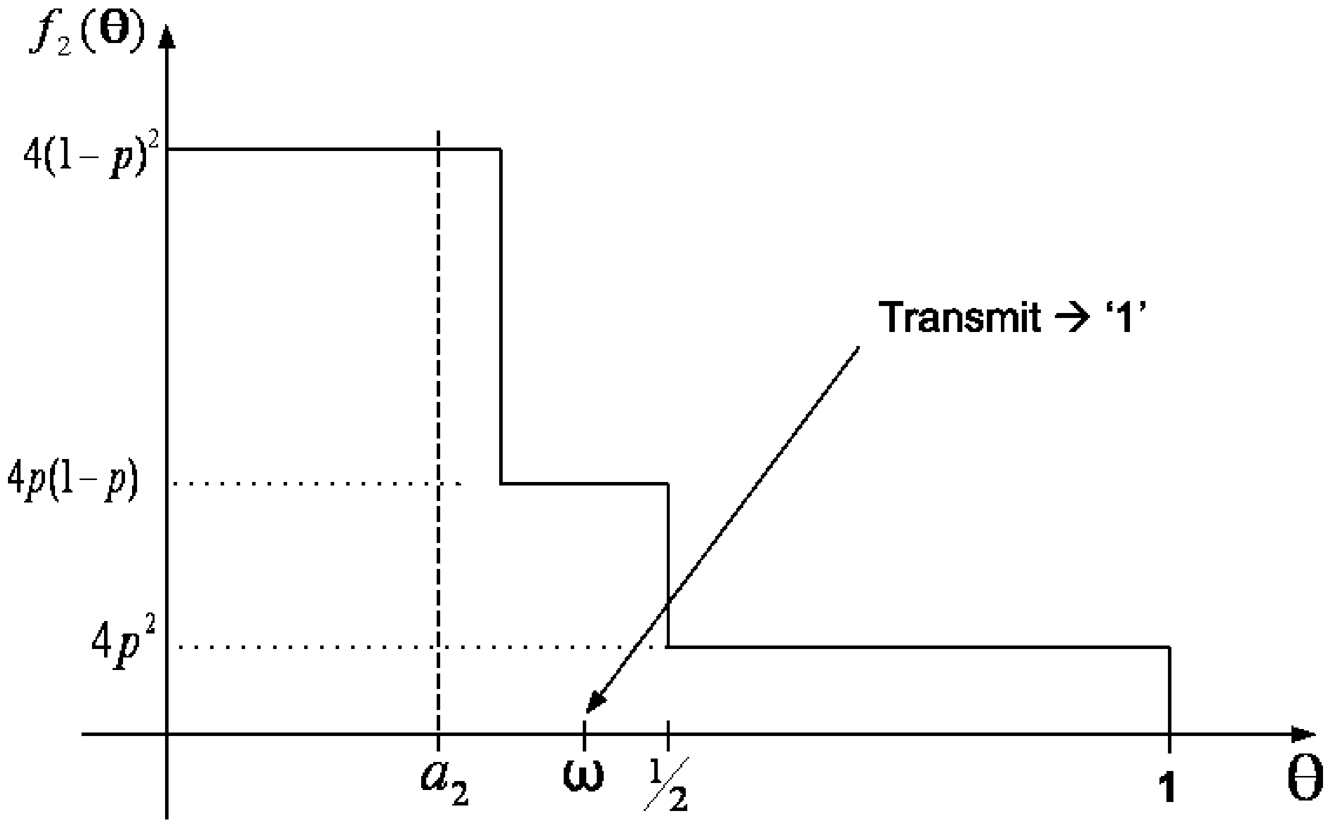

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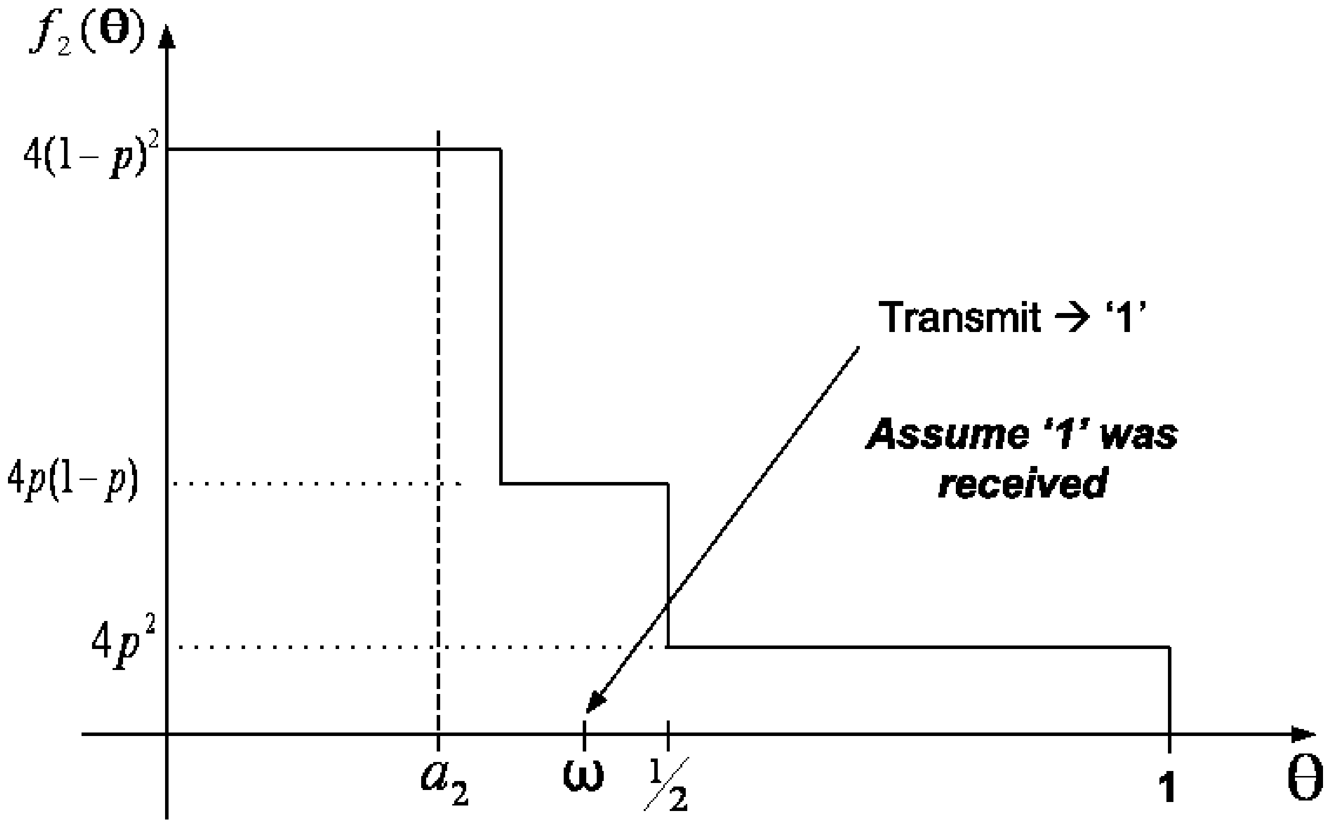
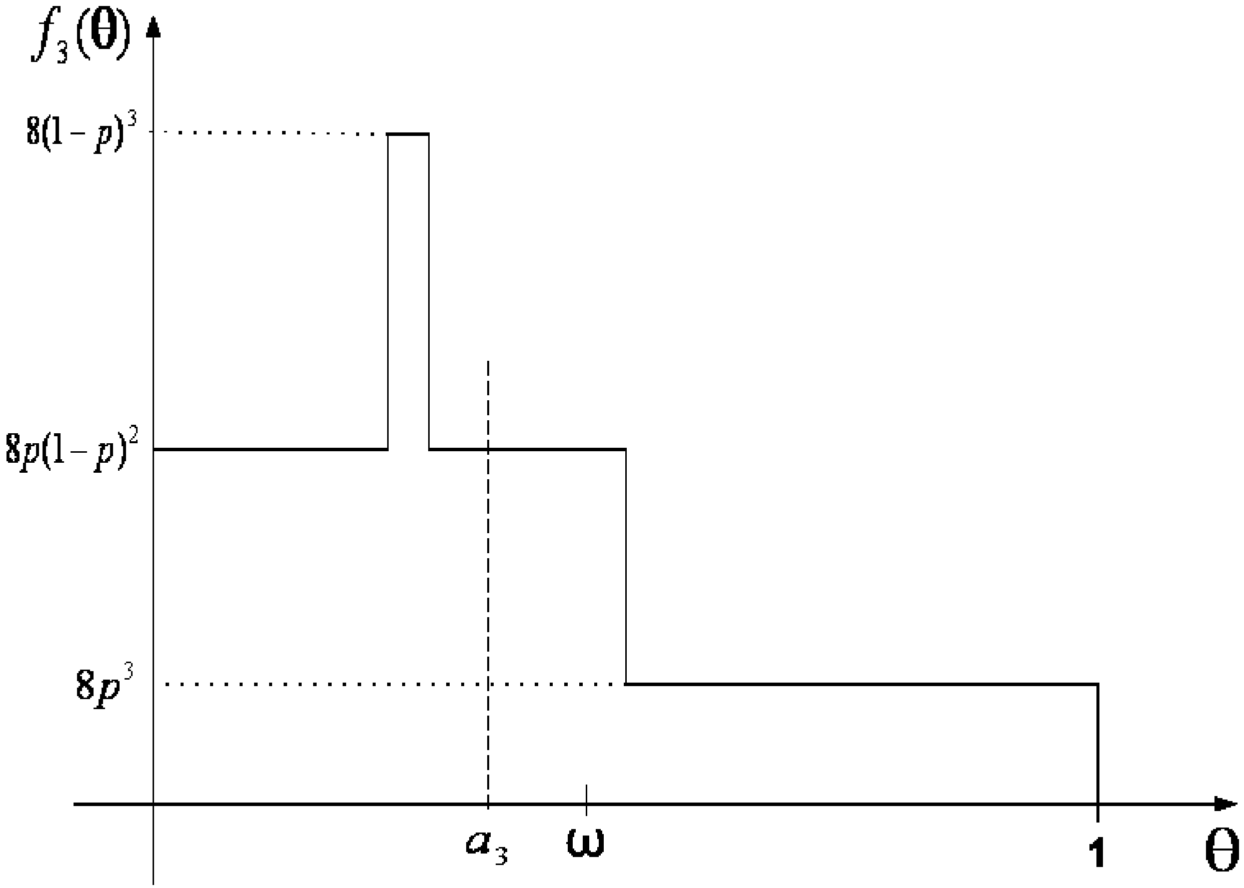
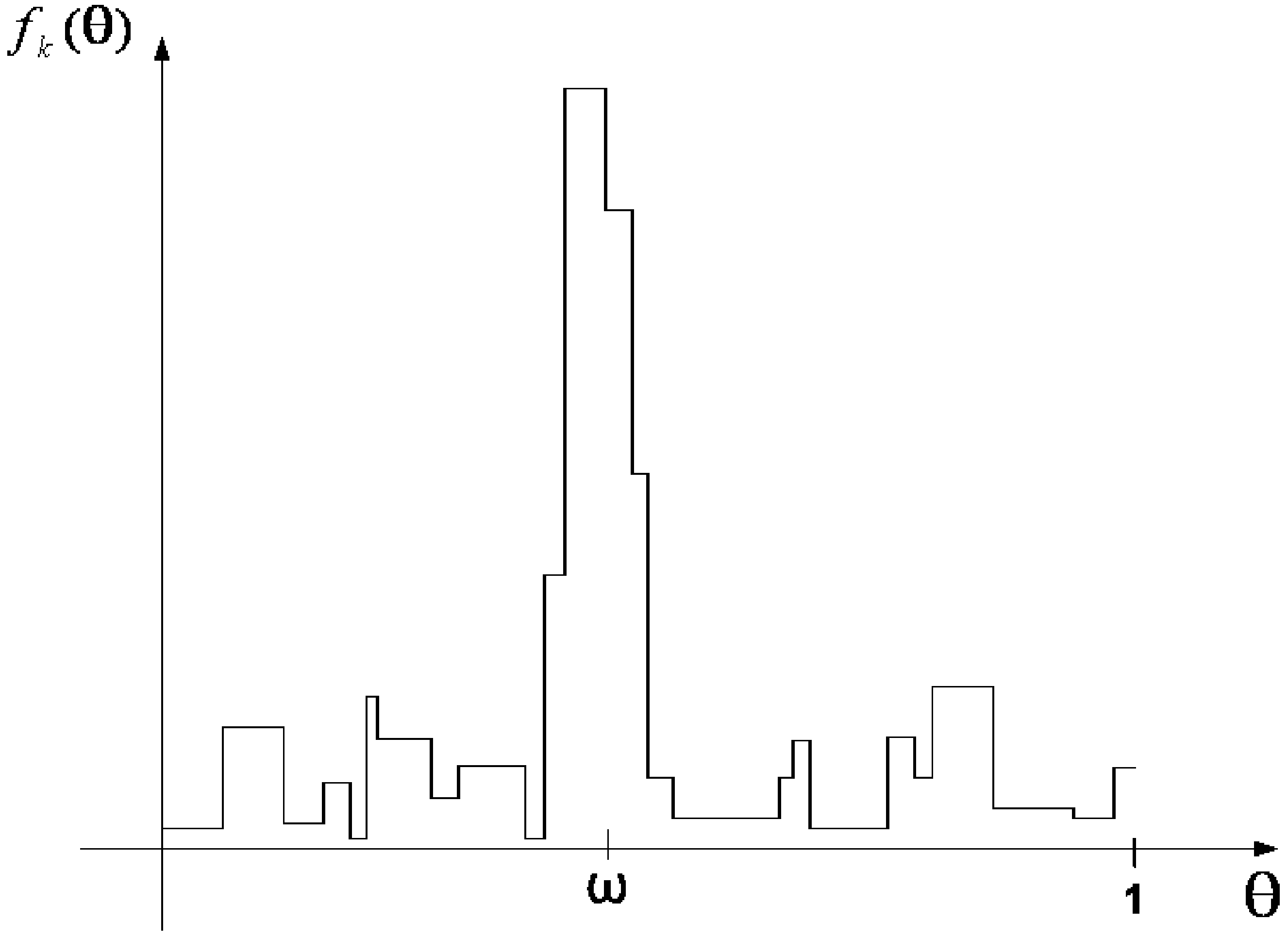


$f_3(\theta)$ - a-posteriori density given three received bits

a_3 - Median point w.r.t. $f_3(\theta)$



Hopefully after many channel uses...



Note...

- After n Horstein iterations there are $n+1$ intervals within each $f_n(\theta)$ is constant

- Defining $n_1 \triangleq \sum z_k$ we have

$$f_n(\theta = \omega) = 2^n p^{n_1}(1-p)^{n-n_1} = 2^n P_r(\underline{\boldsymbol{z}})$$

Background - The KT Probability Estimator

[KrichevskyTrofimov'81]

- Goal - Sequential probability assignment to a sequence $\underline{z}$

$$\hat{p}_{k+1} = \frac{\sum_1^k z_j + \frac{1}{2}}{k+1} \quad \Rightarrow \quad \widehat{P}(\underline{z}) = \prod_{k=1}^{n} \hat{p}_k^{\,z_k} (1 - \hat{p}_k)^{1-z_k}$$

- Known result - If $\underline{z}$ has np ones then

$$-\frac{1}{n} \log \widehat{P}(\underline{z}) \leq h_b(p) + \frac{\log n}{2n}$$

Our Transmission Scheme

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Solution - Slow estimate update, randomization (Later...)

Assume (for now) the KT estimate is available at the receiver

- Horstein iterations - Replace $(p, 1-p)$ with $(\hat{p}_k, 1-\hat{p}_k)$

- At the end of the block

$$f_n(\theta=\omega) = 2^n \underbrace{\prod_{k=1}^{n} \hat{p}_k^{\,z_k}(1-\hat{p}_k)^{1-z_k}}_{\substack{\text{The probability assigned to the noise} \\ \text{sequence by the KT estimator}}} = 2^n \widehat{P}(\underline{\boldsymbol{z}})$$

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Assume (for now) the receiver knows the interval containing ω

Let $2^{-\ell}$ be the size of the interval containing ω :

$$2^{-\ell} \cdot 2^n \widehat{P}(\underline{\boldsymbol{z}}) \leq 1 \quad \Rightarrow \quad \ell \geq n + \log \widehat{P}(\underline{\boldsymbol{z}})$$

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$$\boxed{R \stackrel{\triangle}{=} \frac{\ell}{n} \ \ge \ 1 + \frac{1}{n} \log \widehat{P}(\underline{\boldsymbol{z}}) \ \ge \ 1 - h_b(p) - \frac{\log n}{2n}}$$

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Solution - Communicate this together with the KT estimate...

Estimate+Interval index = Update Information (UI)

How Much UI is Needed?

- Divide into blocks of size $\sqrt{n}$, send UI once per block
- Estimator redundancy increased by $\sim \frac{\log n}{\sqrt{n}}$ (negligible)
- Estimate updates require $\sim \frac{1}{2} \log n$ bits per block
- Message interval index require $\sim \log n$ bits per block

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Total UI Rate $\sim \frac{\log n}{\sqrt{n}}$ *Negligible !*

Communicating UI Reliably

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- Positions agreed upon via feedback or common randomness
- Position selection $\sim$ $\log^2 n$ bits per block (negligible)
- Use, e.g., a repetition code for UI (slightly increases UI rate)
- Effective channel for UI is $\sim$ "BSC"

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Mismatch of training estimate and actual channel $\Rightarrow$ **Error !**

Fortunately...

Error probability w.r.t. randomization ***tends to zero*** *as* $n \to \infty$

Final Scheme Outline

Divide into blocks of size $\sqrt{n}$. In each block

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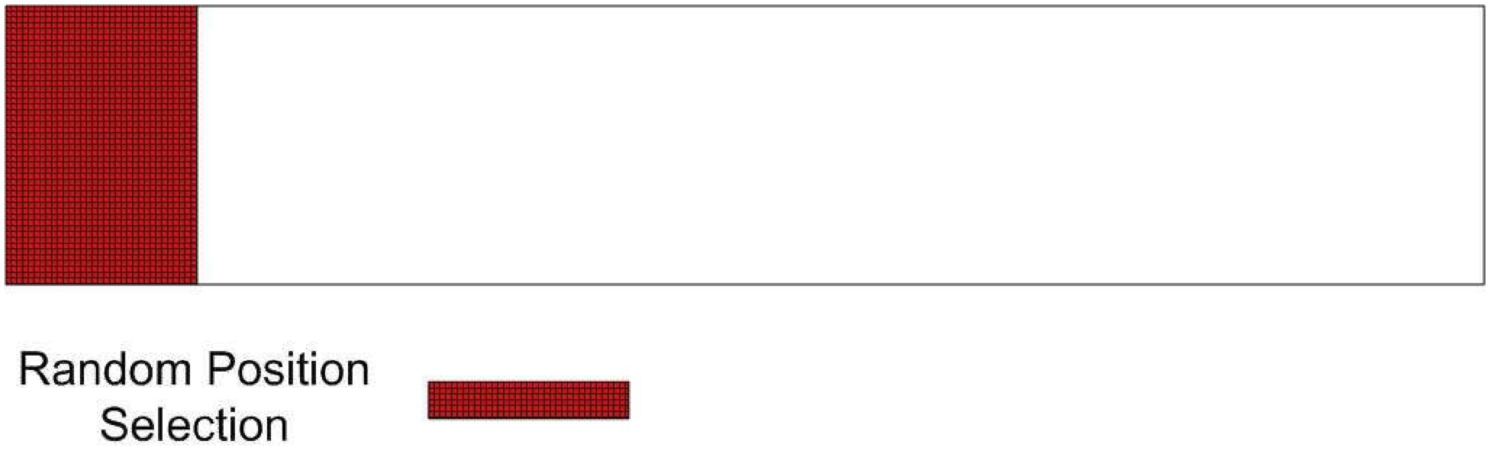

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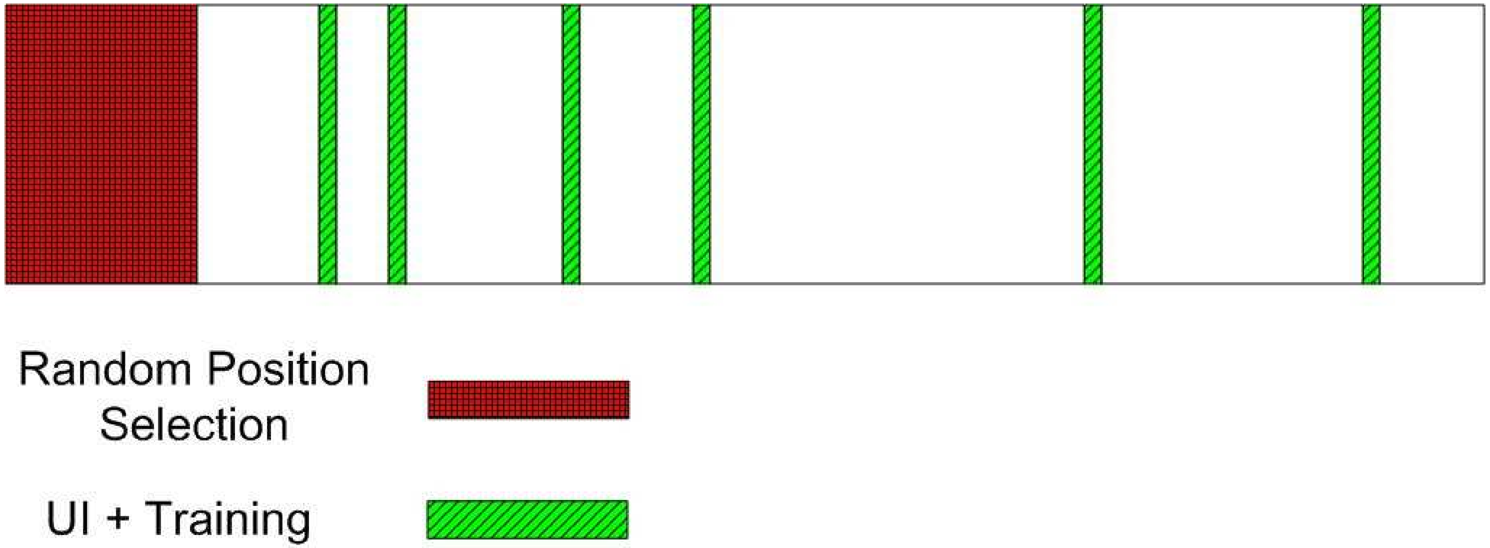

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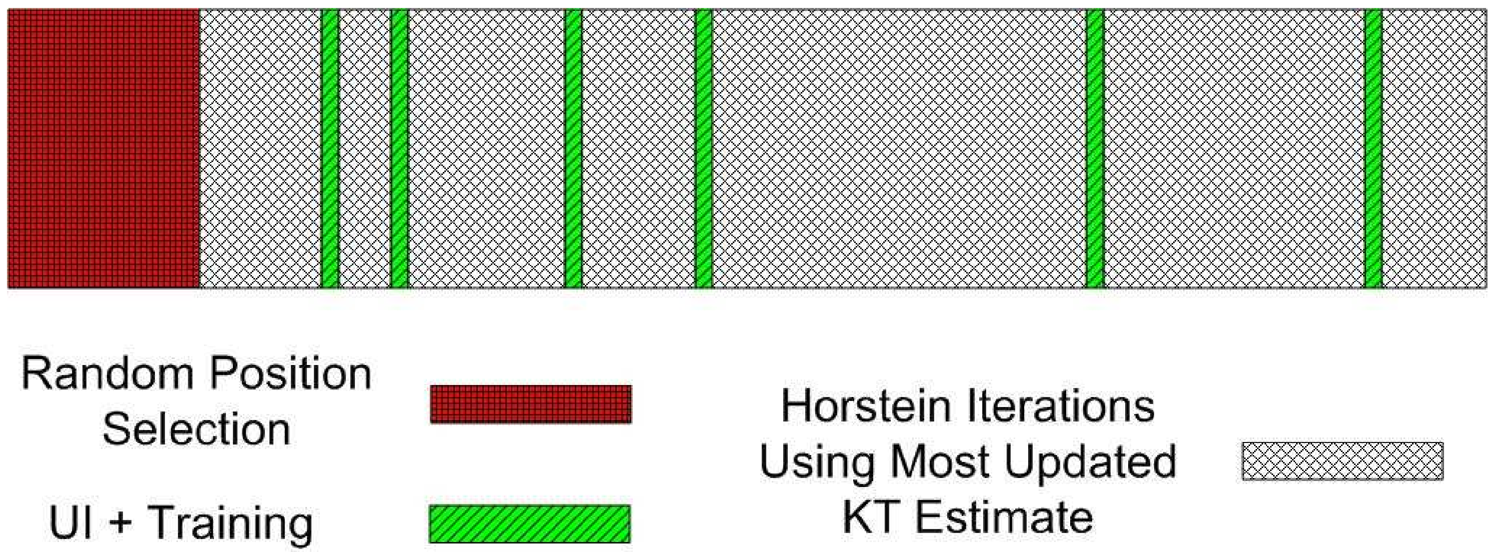

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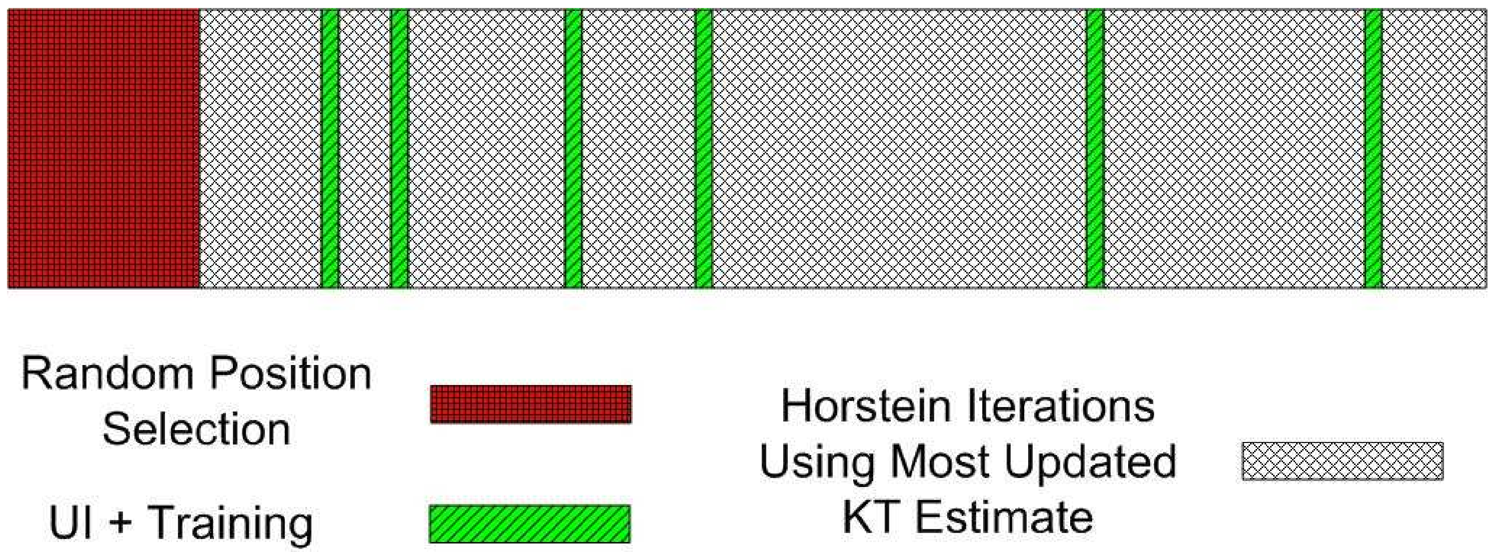

- Use training for UI decoding and block discarding

- Decode the bits representing the last message interval
- UI is correctly decoded with probability $\to 1$
- If UI is correct then the bit decoding rate satisfies

$$\boxed{R \geq 1 - h_b(p) - \delta(n)}$$

with probability $\to 1$

Summary

- Transmission scheme over a binary modulo-additive channel with an individual noise sequence and feedback
- Rate approaching $1 - h_b(p)$ for **any** noise sequence, classical capacity is zero
- Beating the Berlekamp bound without knowing p by using randomization, at a cost of a negligible error probability

Future Research

- By using a universal predictor at the transmitter, may approach $1 - h_b(\pi(\underline{z}))$
- Modify scheme to approach $1 - \rho(\underline{z})$?
- Applications in multi-user communications