The Posterior Matching Approach in Feedback Communication

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Preliminaries
Feedback Communication Setting

\[ X_n = g_n(\Theta_0, Y^{n-1}) \quad \xrightarrow{\quad P_{Y|X} \quad} \quad Y_n \quad \xrightarrow{\Delta_n(Y^n)} \]

- Memoryless channel \( P_{Y|X} \)
- Instantaneous noiseless feedback
- Message point representation \( \Theta_0 \sim \text{Unif}(0, 1) \)
- A general transmission scheme:
  - Transmission functions \( \{g_n : (0,1) \times \mathbb{R}^{n-1} \mapsto \mathbb{R}\}_{n=1}^{\infty} \)
  - Decoding rules \( \{\Delta_n : \mathbb{R}^n \mapsto \{(a,b) | (a,b) \subseteq (0,1)\}\}_{n=1}^{\infty} \)
Definitions

- **Error probability** \( p_e(n) = \mathbb{P}(\Theta_0 \notin \Delta_n(Y^n)) \)
- **Instantaneous rate** \( R_n = -\frac{1}{n} \log |\Delta_n(Y^n)| \)
- A transmission scheme achieves a rate \( R \) (possibly within input constraints \((\eta, \mu)\)) if

\[
\lim_{n \to \infty} \mathbb{P}(R_n < R) = 0, \quad \lim_{n \to \infty} p_e(n) = 0
\]

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \eta(X_k) < \mu \quad \text{a.s.}
\]

- Implies achievability in the “standard sense”
Optimal Decoding Rules

- Calculate the posterior distribution $P_{\Theta_0|Y^n}$

- **Fixed rate**
  - Set the desired rate $R_n = R$
  - Select $|\Delta_n| = 2^{-nR}$ to maximize $P_{\Theta_0|Y^n}(\Delta_n|Y^n)$

- **Variable rate**
  - Set a target error probability $p_e(n) = \varepsilon_n \to 0$
  - Select $\Delta_n$ with minimal size such that $P_{\Theta_0|Y^n}(\Delta_n|Y^n) \geq 1 - \varepsilon_n$
Background
Feedback – What is it Good for?

- Feedback cannot increase capacity of memoryless channels
  [Shannon’56] [Kadota&Ziv’71]

- Nevertheless, feedback can sometimes
  - Boost reliability
  - Allow rate adaptation to cope with unknown channels
  - Significantly reduce complexity, attain capacity “without coding”
Feedback cannot increase capacity of memoryless channels

[Shannon’56] [Kadota&Ziv’71]

Nevertheless, feedback can sometimes

- Boost reliability
- Allow rate adaptation to cope with unknown channels
- *Significantly reduce complexity, attain capacity “without coding”*
Several “no coding” feedback schemes have been suggested for specific channels

The Horstein Scheme (1963)
- Conjectured to achieve the BSC capacity $1 - h_b(p)$
- Send 0/1 according to whether $\Theta_0$ lies to the left/right of the posterior’s median

The Schalkwijk-Kailath (SK) Scheme (1966)
- Achieves the AWGN capacity $C = \frac{1}{2} \log(1 + \text{SNR})$
- Send $\Theta_0$, receive $\Theta_0$ with a Gaussian bias $Z$
- Find MMSE estimate for $Z$, send amplified error term
- Repeat last step
Horstein/SK share many similarities!
- Message point representation
- Simple, sequential, “no coding” schemes
- “Steering” the receiver in the right direction

However...
- Precise correspondence never established
- No generalization ever provided

Can feedback facilitate achieving capacity without coding in general?
Main Results

[Shayevitz & Feder ’07, ’08, ’09]

- Formalize the common underlying posterior matching principle
- Devise a generic feedback scheme
  - Suitable for any memoryless channel $P_{Y|X}$ and any desired input distribution $P_X$
  - Achieves any rate $R < I(X; Y)$ under general conditions
  - Simple, sequential, no coding
  - Horstein & SK are special cases
- Corollary: The Horstein scheme achieves the BSC capacity (verifying a longstanding conjecture)
- Error exponent and model mismatch analysis
- Applications to joint source-channel coding
Posterior Matching
The Basic Principle

- Say the receiver has observed the output sequence $Y^n$
- What information is it still missing?
- A reasonable answer: any r.v. which
  - is statistically independent of previous outputs $Y^n$
  - Together with $Y^n$ uniquely determines $\Theta_0$
- However...
  - Many possible distributions
  - Channel input may have constraints (e.g., power, discreteness)
- *Match* the distribution to the channel!
The Posterior Matching (PM) Scheme

- Set some input distribution $P_X$
- The next channel input is given by

$$X_{n+1} = F_X^{-1} \circ F_{\Theta_0|Y^n}(\Theta_0|Y^n)$$

- $F_X, F_{\Theta_0|Y^n}$ are c.d.f.'s
- $X_{n+1} \sim P_X$ and independent of $Y^n$
- $Y^n$ is i.i.d. with the “correct” marginal
- A two step procedure
  - Zoom in on the posterior
  - Match to the channel
The Posterior Matching (PM) Scheme

- Set some input distribution $P_X$
- The next channel input is given by

\[ X_{n+1} = F_X^{-1} \circ \mu \circ F_{\Theta_0|Y^n}(\Theta_0|Y^n) \]

- $F_X, F_{\Theta_0|Y^n}$ are c.d.f.’s
- $X_{n+1} \sim P_X$ and independent of $Y^n$
- $Y^n$ is i.i.d. with the “correct” marginal
- A two step procedure
  - Zoom in on the posterior
  - Match to the channel
Lemma: If $P_{XY}$ has a p.d.f., then the transmission scheme is given by

\[ X_1 = F_X^{-1}(\Theta_0) \]
\[ X_{n+1} = F_X^{-1} \circ F_{X|Y}(X_n|Y_n) \]

- $F_X^{-1} \circ F_{X|Y}(\cdot|\cdot)$ is called the PM kernel
- The next input is generated by applying the PM kernel to the last input/output pair only – simple!
- $(X_n, Y_n)$ constitutes a Markov chain over $\mathbb{R}^2$
- By construction, $P_{XY}$ is an invariant distribution
Let $P_{Y|X}$ be an AWGN channel with an input power constraint $P$.

Set $P_X = \mathcal{N}(0, P)$ (capacity achieving).

PM kernel is linear, and yields

$$X_{n+1} = \sqrt{1 + \text{SNR}} \left( X_n - \frac{\text{SNR}}{1 + \text{SNR}} Y_n \right)$$

Precisely the SK scheme!

MMSE error term – *uncorrelated* with previous outputs

Mind the difference – In the PM scheme transmission is *independent* of previous outputs, coincides only in the Gaussian case.
The BSC

- Set $P_X = \text{Bern} \left( \frac{1}{2} \right)$ (capacity achieving)

- $P_X$ has no proper p.d.f. – recursion rule is invalid!

- Nevertheless, the PM scheme coincides with Horstein’s median rule

\[
X_{n+1} = F_X^{-1} \circ F_{\Theta_0|Y^n}(\Theta_0|Y^n) = \begin{cases} 
0 & \Theta_0 < \text{median} \left\{ f_{\Theta_0|Y^n}(\Theta_0|Y^n) \right\} \\
1 & \text{o.w.} 
\end{cases}
\]

- $F_X^{-1}$ quantizes above/below $\frac{1}{2}$

- Can we find a simple recursion rule nonetheless?
A “message point” channel, $\Theta_n, \Phi_n \sim \text{Unif}(0, 1)$

Preserves mutual information, common framework for discrete/continuous/mixed alphabet input distributions/channels

PM scheme recursive representation

$$\Theta_1 = \Theta_0, \quad \Theta_{n+1} = F_{\Theta|\Phi}(\Theta_n | \Phi_n)$$

$(\Theta_n, \Phi_n)$ constitute a Markov chain over $(0, 1)^2$, having $P_{\Theta\Phi}$ as an invariant distribution
The PM kernel for the normalized BSC is given by

\[ F_{\Theta|\Phi}(\theta|\phi) \]

- Equivalent to the original Horstein scheme (set \( X_n = F_X^{-1}(\Theta_n) \))
Exponential Noise with a Mean Input Constraint

- Let $P_{Y|X}$ be an additive noise $\sim \text{Exp}(b)$ channel.
- Impose a mean input constraint $E(X_n) = a$.
- Capacity achieving distribution [Verdú'96]

$$f_X(x) = \frac{b}{a+b} \delta(x) + \frac{a}{(a+b)^2} e^{-\frac{x}{a+b}}, \quad C = \log \left( 1 + \frac{a}{b} \right)$$

- Corresponding (normalized) PM scheme given by

$$\Theta_{n+1} = \begin{cases} \frac{a+b}{b} \cdot \Theta_n \cdot (1 - \Phi_n) \frac{a}{b} & \Theta_n \leq \frac{b}{a+b} \\ \left( \frac{a}{a+b} \cdot \frac{1-\Phi_n}{1-\Theta_n} \right) \frac{a}{b} & \Theta_n > \frac{b}{a+b} \end{cases}$$

- The actual input is $X_n = (a + b) \ln \left( \frac{a}{(a+b)(1-\Theta_n)} \right) 1_{\left[ \frac{b}{a+b}, 1 \right]}(\Theta_n)$
Theorem: Under some general conditions on the input/channel pair \((P_X, P_{Y|X})\), the corresponding PM scheme achieves any rate \(R < I(X; Y)\), with either a fixed/variable optimal decoding rule, within an input constraint \((\eta, \mathbb{E}\eta(X))\).

Specifically, the Theorem holds for
- Discrete memoryless channels (up to some small issues)
- Corollary: The Horstein scheme achieves capacity
- Input/channel pairs \((P_X, P_{Y|X})\) with a joint p.d.f. \(P_{XY}\) continuous over a convex support, and not “too wild”
Proof Outline
Achievability Conditions

- **(REG)** \(I(X; Y) < \infty\), and some mild regularity conditions

- **(ERG)** The invariant distribution \(P_{\Theta \Phi}\) for the Markov chain \((\Theta_n, \Phi_n)\) is ergodic.

- **(FIX)** The (normalized) posterior matching kernel has no universal fixed points, in the sense that

\[
P(F_{\Theta|\Phi}(\theta|\Phi) = \theta) < 1
\]

for any \(\theta \in (0,1)\)
Iterated Function System (IFS)

- An IFS \( \{S_n(s_0)\}_{n=1}^{\infty} \) over a measurable space \( \mathcal{F} \) is generated by a measurable function \( \omega_\phi(s) \) mapping \( \mathcal{F} \) to itself, as follows:

\[
S_1 = s_0, \quad S_{n+1}(s_0) = \omega_{\Phi_n} \circ \omega_{\Phi_{n-1}} \circ \cdots \circ \omega_{\Phi_1}(s)
\]

where \( \{\Phi_n\}_{n=1}^{\infty} \) is an i.i.d control sequence, and \( s_0 \in \mathcal{F} \) is the initial point.

- \( \{S_n(s_0)\}_{n=1}^{\infty} \) is a Markov chain over \( \mathcal{F} \).

- Let \( \lambda : \mathcal{F} \mapsto \mathbb{R}^+ \), let \( \xi : [0, 1) \mapsto [0, 1) \) be \( \cap \)-convex and \( \xi(x) < x \) over \((0, 1) \) (a contraction).

- **Lemma**: If \( \mathbb{E}(\lambda(\omega_{\Phi_1}(s))) \leq \xi(\lambda(s)) \) for any \( s \in \mathcal{F} \), then \( \lambda(S_n(s_0)) \to 0 \) in probability.
The receiver tracks the posterior $P_{\Theta_0|\Phi^n}$

A recursive representation for the posterior c.d.f.

$$F_{\Theta_0|\Phi^{n+1}}(\theta|\Phi^{n+1}) = F_{\Theta|\Phi}(\cdot|\Phi_{n+1}) \circ F_{\Theta_0|\Phi^{n+1}}(\theta|\Phi^n)$$

Recall $\{\Phi_n\}_{n=1}^{\infty}$ is i.i.d

Thus the posterior c.d.f. is an IFS

- Evolves over a function space $\mathfrak{F}_c$ of c.d.f.-like functions
- Generated by the PM kernel (via function composition)
- Initialized at $F_{\Theta_0}(\theta) = \theta$
- Controlled by the channel output sequence $\{\Phi_n\}_{n=1}^{\infty}$
Under assumptions (REG)+(FIX), we can find a length function \( \lambda : \mathcal{F}_c \mapsto \mathbb{R}^+ \) and a contraction \( \xi(\cdot) \) such that

\[
\mathbb{E} \lambda (F_{\Theta|\Phi}(\cdot|\Phi) \circ s) \leq \xi(\lambda(s)), \quad s \in \mathcal{F}_c
\]

Loosely speaking, \( \lambda(s) \approx 0 \) implies that \( s \) “close” a unit step function

Therefore,

- The posterior c.d.f. “tends” to a unit step function about \( \Theta_0 \)
- Any fixed interval containing \( \Theta_0 \) will be reliably decoded eventually
- \( R = 0 \) is achievable!
Achievability of $R < I(X; Y)$

- Expand the posterior p.d.f. at the message point (Bayes rule, memoryless channel, i.i.d. output)

$$\frac{f_{\Theta_0|\Phi^n}(\Theta_0|\Phi^n)}{f_{\Theta_0|\Phi^{n-1}}(\Theta_0|\Phi^{n-1})} = \frac{f_{\Phi^n|\Theta_0,\Phi^{n-1}}(\Phi_n|\Theta_0,\Phi^{n-1})}{f_{\Phi^n|\Phi^{n-1}}(\Phi_n|\Phi^{n-1})} = f_{\Phi|\Theta}(\Phi_n|\Theta_n)$$

- Taking the logarithm, applying the recursion $n$ times and using (ERG)+SLLN, we get

$$\frac{1}{n} \log f_{\Theta_0|\Phi^n}(\Theta_0|\Phi^n) = \frac{1}{n} \sum_{k=1}^{n} \log f_{\Phi|\Theta}(\Phi_k|\Theta_k) \to \mathbb{E} \left( \log f_{\Phi|\Theta}(\Phi|\Theta) \right)$$

$$= I(\Theta; \Phi) = I(X; Y) \quad \text{a.s.}$$

- Roughly, $f_{\Theta_0|\Phi^n}(\Theta_0|\Phi^n) \approx 2^n I(X; Y)$
Achievability of $R < I(X; Y)$

Assume that for all $k \in [n]$

$$F_{\Theta_0|\Phi^n}(\Theta_0 + 2^{-nR}|\Phi^n) - F_{\Theta_0|\Phi^n}(\Theta_0 - 2^{-nR}|\Phi^n) < \varepsilon$$

$F_{\Theta_0|\Phi^n}(\theta|\Phi^k)$ is what the input to the normalized channel at time $k + 1$ would have been, had the message point been $\theta$

Hence, the assumption implies that the input is insensitive to a $2^{-nR}$ perturbation in the message point

Using (REG)+(ERG)+SLLN again, this can be roughly translated into

$$f_{\Theta_0|\Phi^n}(\Theta_0 \pm 2^{-nR}|\Phi^n) \approx 2^n(I(X;Y) - \delta)$$

where $\varepsilon \to 0$ implies $\delta \to 0$
Achievability of $R < I(X;Y)$

- If $R < I(X;Y) - \delta$ we get a contradiction
  \[ \int f_{\Theta_0|\Phi^n}(\theta|\Phi^n) d\theta \approx 2^n(I(X;Y)-R-\delta) \to \infty \]

- Hence, with high probability there exists $k_0 \in [n]$ so that
  \[ F_{\Theta_0|\Phi^n}(\Theta_0 + 2^{-nR}|\Phi^n) - F_{\Theta_0|\Phi^n}(\Theta_0 - 2^{-nR}|\Phi^n) \geq \varepsilon \]

- Due to the repetitive nature of the scheme, one can imagine transmission to have started at time $k_0$ with the message point $\Theta_{k_0}$

- $2^{-nR}$ neighborhood of $\Theta_0 \Rightarrow \varepsilon$-neighborhood of $\Theta_{k_0}$

- Invoking zero rate result, this $\varepsilon$-neighborhood can be decoded in sublinear time $\to$ Achievability proved!
Examples Revisited

- BSC – Verifying the Horstein scheme achieves capacity
- AWGN channel – reconfirming the SK achieves capacity
- Exponential noise, mean constraint – The explicit PM scheme described achieves capacity!
Further Results

- Error probability analysis, providing closed form error exponent expressions for a range of rates (sometimes strictly below capacity)

- Channel model mismatch
  - Scheme designed for \((P_X, P_{Y|X})\)
  - True channel is \(P_{Y^*|X^*}\)
  - Scheme induces some stationary input distribution \(P_{X^*}\)
  - Penalty in rate relative to \(I(X^*; Y^*)\) is

\[
D(P_{Y^*|X^*} \parallel P_{Y|X} \mid P_{X^*}) - D(P_{Y^*} \parallel P_Y)
\]

- Robustness of the SK scheme
Application to Joint-Source Channel Coding (JSCC) with Feedback
A Well Known Gaussian Example

- Gaussian source \( A \sim \mathcal{N}(0, P_s) \)
- AWGN channel, input power constraint \( P \)
- Scalar linear transmission scheme ("uncoded"):

\[
\begin{align*}
A & \rightarrow X \\
\sqrt{\frac{P}{P_s}} & \\
\rightarrow Y & \rightarrow \hat{A} \\
Z & \sim \mathcal{N}(0, N) \quad \frac{\sqrt{P \cdot P_s}}{P + N}
\end{align*}
\]

- Achieves optimal performance under quadratic distortion!

\[
D = \mathbb{E}(A - \hat{A})^2 = P_s \cdot \left(1 + \frac{P}{N}\right)^{-1} \quad \Rightarrow \quad R(D) = C
\]
Suppose $m$ AWGN channels uses per source sample available

Optimal distortion given by

$$R(D) = mC \quad \Rightarrow \quad D = P_s \cdot \left(1 + \frac{P}{N}\right)^{-m}$$

Decays exponentially with the \textit{bandwidth expansion factor} (BEF) $m$

Optimal performance cannot generally be attained by a scalar $1 : m$ \textit{joint source-channel coding (JSCC)} scheme

Assume an instantaneous noiseless feedback link is available

The SK scheme achieves optimal performance!
What About Non Gaussian Settings?

- For a finite BEF, can optimal performance be achieved via feedback for other sources/channels/distortion measures?
- In general no, unless some unique relation is satisfied [Gastpar '02]
- So what can we do anyway?
  - Apply the PM principle
  - Show that the resulting PM-JSCC scheme has "good performance"
Problem Setting

- Source $A \sim P_A$ over an alphabet $\mathcal{A} \subseteq \mathbb{R}$
- Memoryless channel $P_{Y|X}$ with feedback, $\text{BEF} = m$
- A general $1 : m$ JSCC transmission scheme:

\[
X_k = g_k(A, Y^{k-1}) \quad \xrightarrow{P_{Y|X}} \quad Y_k \xrightarrow{\hat{A}} \Delta(Y^m)
\]

- General $(\eta, u)$ input constraint: $\mathbb{E}(\eta(X_k)) \leq u$ for $k = 1, \ldots, m$
- General distortion measure $d : \mathcal{A}^2 \mapsto \mathbb{R}^+$
- Performance measured by the average distortion $D = \mathbb{E}(d(A, \hat{A}))$
The PM-JSCC Scheme

- Set some channel input distribution $P_X$ (design parameter)
- The transmission functions are defined for $k = 0, 1, \ldots m - 1$ by

$$X_{k+1} = F_X^{-1} \circ F_{A|Y^k}(A|Y^k)$$

- Again, $X_{k+1} \sim P_X$ independent of $Y^k$, $Y^m$ is i.i.d. with marginal $P_Y$, and we have a recursive representation:

$$X_1 = F_X^{-1} \circ F_A(A), \quad X_{k+1} = F_X^{-1} \circ F_{X|Y}(X_k|Y_k)$$

- Seems to satisfy Gastpar’s optimality conditions when possible
- Optimal exponential decay for a quadratic distortion measure
- What happens otherwise (which is usually the case..)?
Suppose $\hat{X}_m$ is some estimate of $X_m$

Corresponds to a unique estimate $\hat{X}_1$ of $X_1$, given by reversing the transmission scheme

$$\hat{X}_1 = \omega_{Y_1} \circ \ldots \circ \omega_{Y_{m-2}} \circ \omega_{Y_{m-1}}(\hat{X}_m)$$

where $\omega_y(\cdot) = F_{X|Y}^{-1}(\cdot | y) \circ F_X(\cdot)$ is the inverse PM kernel

$\hat{X}_1$ is generated by a time-reversed IFS (RIFS) with kernel $\omega_y(\cdot)$ and an i.i.d control sequence $Y^m$

Now simply $\hat{A} = F_A^{-1} \circ F_X(\hat{X}_1)$
Set a fixed interval $J_m \subseteq \mathcal{X}$ so that $\mathbb{P}(X_m \in J_m) = P_X(J_m) = 1 - \delta$

- Thus $\mathbb{P}(X_1 \in J_1) = 1 - \delta$, where the random interval $J_1$ is generated by running the RIFS over (the edges of) $J_m$
- $\mathbb{P}(A \in J_A) = 1 - \delta$, where $J_A = F^{-1}_A \circ F_X(J_1)$
- Now set $\hat{A}$ to be any point within $J_A$ (suboptimal)

If the RIFS kernel $\omega_y(\cdot)$ is **contractive on the average**, then $J_1$ is exponentially smaller than $J_m$ w.h.p.

If $F^{-1}_A \circ F_X$ is $M$-Lipschitz, $J_A$ is also exponentially small w.h.p.

Two sources of distortion

- $A \in J_A$ and $J_A$ exponentially small (high prob.) $\Rightarrow$ small distortion
- $A \notin J_A$ or $J_A$ large (low prob.) $\Rightarrow$ $\sup_{a,b \in A} d(a, b) = d_{\max} < \infty$
**Theorem:** Let $\omega_y(\cdot)$ be the inverse PM kernel and define

$$r_q \triangleq \sup_{s \neq t \in \text{supp}(X)} \mathbb{E} [D_{s,t}(\omega_Y)]^q, \quad \bar{d}_\varepsilon \triangleq \sup_{(a,b) \subseteq \mathcal{A}, |b-a| \leq \varepsilon} d(a,b)$$

If there exists $q^* \in (0, 1)$ such that $r_{q^*} < 1$, then the PM-JSCC scheme achieves an average distortion upper bounded by

$$D \leq \inf_{0 < q < q^*, \varepsilon, \ell > 0} \left\{ d_{\max} \left( \mathcal{T}_X(\ell) + \frac{(M\varepsilon^{-1}\ell)^q}{1 - \mathcal{T}_X(\ell)} r_m q \right) + \bar{d}_\varepsilon \right\}$$

within any input constraint of the form $(\eta, \mathbb{E}\eta(X))$.

**Corollary:** If $P_X$ has a polynomially (or faster) decaying tail and $d(a, b) = |a - b|^\gamma$, the distortion decays exponentially with the BEF $m$. 
Uniform Source, Uniform Noise

- $A \sim \text{Unif}(0, 1)$, $P_{Y|X}$ is an additive channel with noise $\sim \text{Unif}(0, 1)$
- Set (for example) $P_X = \text{Unif}(0, 1)$
- The PM-JSCC scheme is given by

$$X_1 = A, \quad X_{k+1} = \frac{X_k}{Y_k} \cdot 1_{(0,1]}(Y_k) + \frac{X_k - Y_k + 1}{2 - Y_k} \cdot 1_{(1,2]}(Y_k)$$

- Very simple interpretation:
  - Begin by transmitting the uncoded source $X_1 = A$
  - Given $Y_1$ find the interval of feasible inputs, and an affine transformation that stretches this interval to $(0, 1)$
  - Generate $X_2$ by applying the transformation to $X_1$
  - Repeat the two steps above for $X_k, Y_k$
The inverse PM kernel is

\[ \omega_y(s) = sf_Y(y) + (y - 1) \cdot 1_{(1,2)}(y) \]

The contraction factor is given by

\[ r_q = \sup_s \mathbb{E} \left( \frac{\partial}{\partial s} \omega_Y(s) \right)^q = \mathbb{E}(f_Y(Y))^q = \left( 1 + \frac{q}{2} \right)^{-1} < 1 \]

Assume a distortion measure \( d(a, b) = |a - b|^{\gamma} \), bounded over \((0, 1)\)

The conditions of the Theorem hold, and we get

\[ D \leq \inf_{q > 0} \left( \left( \frac{\gamma}{q} \right)^{\frac{q}{q+\gamma}} + \left( \frac{q}{\gamma} \right)^{\frac{\gamma}{q+\gamma}} \right) \cdot \left( \frac{1}{1 + \frac{q}{2}} \right)^{\frac{m\gamma}{q+\gamma}} \]
The distortion decays exponentially with the BEF:

\[
\lim_{m \to \infty} - \frac{1}{m} \log D \geq \sup_{q > 0} \gamma q + \gamma \log \left( 1 + \frac{q}{2} \right)
\]

For a quadratic distortion measure (\(\gamma = 2\)) the exponent attained by the PM-JSCC scheme is lower bounded by \(\frac{\log e}{e}\).

This should be contrasted with the exponent promised by the separation principle, which is \(\log e\).
Further Research

- **JSCC setting:**
  - Obtain tighter bounds by “measuring” contraction “relative” to the distortion measure
  - Finite block optimality of the PM-JSCC scheme?
  - Sensitivity to channel variations (graceful degradation)
  - relation to non-feedback JSCC schemes?

- **Communication setting:**
  - Channels with memory, achieve the *directed information rate* – requires a different interpretation of the PM principle
  - Multi-terminal channels
  - Noisy feedback