The Posterior Matching Approach in Feedback Communication

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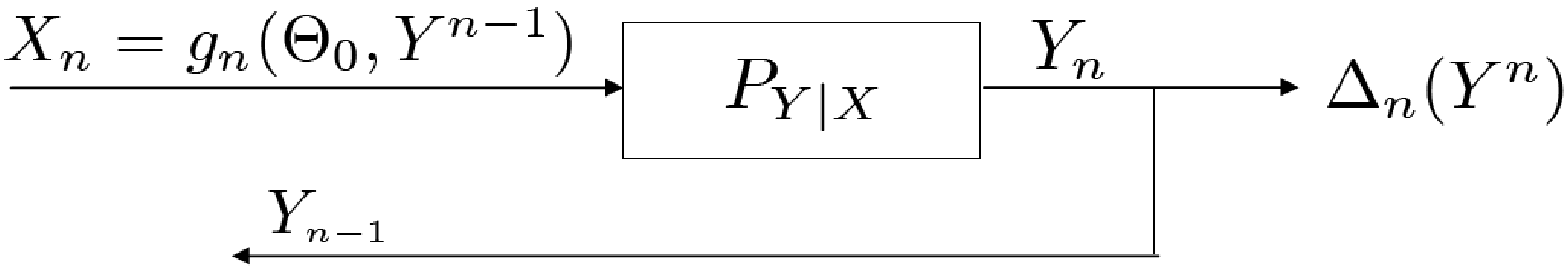
Preliminaries

Feedback Communication Setting



- Memoryless channel $P_{Y|X}$
- Instantaneous noiseless feedback
- Message point representation $\Theta_0 \sim \mathrm{Unif}(0,1)$
- A general *transmission scheme*:
  - *Transmission functions* $\{g_n : (0,1) \times \mathbb{R}^{n-1} \mapsto \mathbb{R}\}_{n=1}^{\infty}$
  - *Decoding rules* $\{\Delta_n : \mathbb{R}^n \mapsto \{(a,b) | (a,b) \subseteq (0,1)\}\}_{n=1}^{\infty}$

Definitions

- *Error probability* $p_e(n) = \mathbb{P}(\Theta_0 \notin \Delta_n(Y^n))$
- *Instantaneous rate* $R_n = -\frac{1}{n}\log|\Delta_n(Y^n)|$
- A transmission scheme *achieves* a rate R (possibly within input constraints (η, μ)) if

$$\lim_{n\to\infty} \mathbb{P}(R_n < R) = 0\,, \qquad \lim_{n\to\infty} p_e(n) = 0$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \eta(X_k) < \mu \quad \text{a.s.}$$

- Implies achievability in the “standard sense”

Optimal Decoding Rules

- Calculate the posterior distribution $P_{\Theta_0|Y^n}$
- *Fixed rate*
  - Set the desired rate $R_n = R$
  - Select $|\Delta_n| = 2^{-nR}$ to maximize $P_{\Theta_0|Y^n}(\Delta_n|Y^n)$
- *Variable rate*
  - Set a target error probability $p_e(n) = \varepsilon_n \to 0$
  - Select Δ_n with minimal size such that $P_{\Theta_0|Y^n}(\Delta_n|Y^n) \geq 1 - \varepsilon_n$

Background

Feedback – What is it Good for?

- Feedback cannot increase capacity of memoryless channels

  [Shannon'56] [Kadota&Ziv'71]

- Nevertheless, feedback can sometimes
  - Boost reliability
  - Allow rate adaptation to cope with unknown channels
  - Significantly reduce complexity, attain capacity "without coding"

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Previous Work

- Several “no coding” feedback schemes have been suggested for specific channels
- *The Horstein Scheme* (1963)
  - Conjectured to achieve the BSC capacity $1 - h_b(p)$
  - Send $0/1$ according to whether Θ_0 lies to the left/right of the posterior’s median
- *The Schalkwijk-Kailath (SK) Scheme* (1966)
  - Achieves the AWGN capacity $C = \frac{1}{2}\log(1 + \mathrm{SNR})$
  - Send Θ_0, receive Θ_0 with a Gaussian bias Z
  - Find MMSE estimate for Z, send amplified error term
  - Repeat last step

Motivation for this Work

- Horstein/SK share many similarities!
  - Message point representation
  - Simple, sequential, “no coding” schemes
  - “Steering” the receiver in the right direction
- *However...*
  - Precise correspondence never established
  - No generalization ever provided
- Can feedback facilitate achieving capacity without coding in general?

Main Results

[Shayevitz&Feder '07,'08,'09]

- Formalize the common underlying *posterior matching principle*
- Devise a generic feedback scheme
  - Suitable for any memoryless channel $P_{Y|X}$ and any desired input distribution P_X
  - Achieves any rate $R < I(X;Y)$ under general conditions
  - Simple, sequential, no coding
  - Horstein & SK are special cases
- Corollary: The Horstein scheme achieves the BSC capacity (verifying a longstanding conjecture)
- Error exponent and model mismatch analysis
- Applications to joint source-channel coding

Posterior Matching

The Basic Principle

- Say the receiver has observed the output sequence Y^n
- What information is it still missing?
- A reasonable answer: any r.v. which
  - is statistically independent of previous outputs Y^n
  - Together with Y^n uniquely determines Θ_0
- However...
  - Many possible distributions
  - Channel input may have constraints (e.g., power, discreteness)
- *Match* the distribution to the channel!

The Posterior Matching (PM) Scheme

- Set some input distribution P_X
- The next channel input is given by

$$X_{n+1} = F_X^{-1} \circ F_{\Theta_0|Y^n}(\Theta_0|Y^n)$$

- $F_X, F_{\Theta_0|Y^n}$ are c.d.f.'s
- $X_{n+1} \sim P_X$ and independent of Y^n
- Y^n is i.i.d. with the "correct" marginal
- A two step procedure
  - Zoom in on the posterior
  - Match to the channel

The Posterior Matching (PM) Scheme

- Set some input distribution P_X
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- $X_{n+1} \sim P_X$ and independent of Y^n
- Y^n is i.i.d. with the "correct" marginal
- A two step procedure
  - Zoom in on the posterior
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Recursive Representation

Lemma: If P_{XY} has a p.d.f., then the transmission scheme is given by

$$X_1 = F_X^{-1}(\Theta_0)$$

$$X_{n+1} = F_X^{-1} \circ F_{X|Y}(X_n|Y_n)$$

- $F_X^{-1} \circ F_{X|Y}(\cdot|\cdot)$ is called the *PM kernel*
- The next input is generated by applying the PM kernel to the last input/output pair only – simple!
- (X_n, Y_n) constitutes a Markov chain over $\mathbb{R}^2$
- By construction, P_{XY} is an invariant distribution

The AWGN Channel

- Let $P_{Y|X}$ be an AWGN channel with an input power constraint P
- Set $P_X = \mathcal{N}(0, \mathrm{P})$ (capacity achieving)
- PM kernel is linear, and yields

$$X_{n+1} = \sqrt{1+\mathrm{SNR}}\left(X_n - \frac{\mathrm{SNR}}{1+\mathrm{SNR}} Y_n\right)$$

- Precisely the SK scheme!
- MMSE error term – *uncorrelated* with previous outputs
- Mind the difference – In the PM scheme transmission is *independent* of previous outputs, coincides only in the Gaussian case

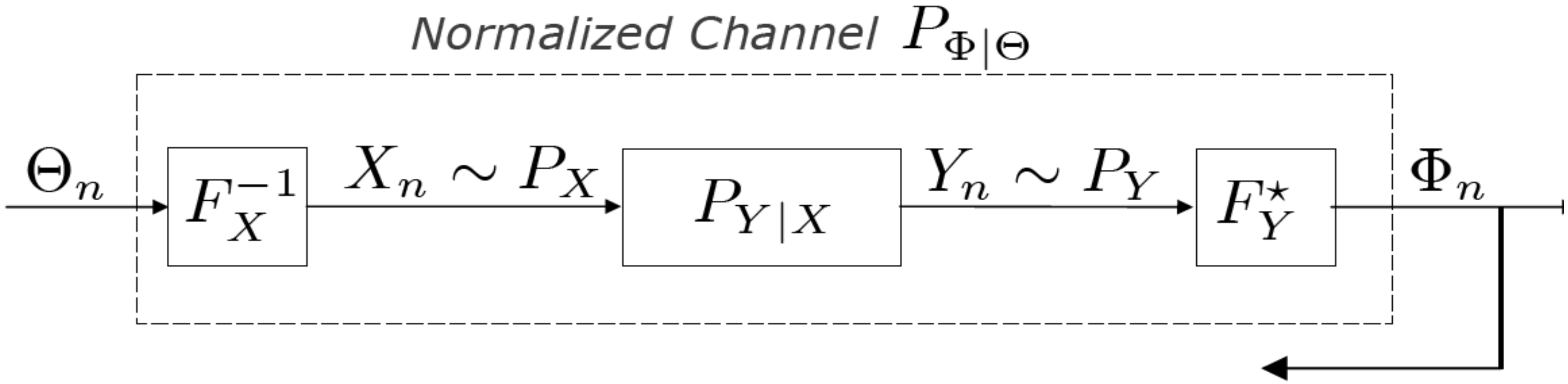
The BSC

- Set $P_X = \mathrm{Bern}\left(\frac{1}{2}\right)$ (capacity achieving)
- P_X has no proper p.d.f. – recursion rule is invalid!
- Nevertheless, the PM scheme coincides with Horstein's median rule

$$X_{n+1} = F_X^{-1} \circ F_{\Theta_0|Y^n}(\Theta_0|Y^n) = \begin{cases} 0 & \Theta_0 < \mathrm{median}\left\{f_{\Theta_0|Y^n}(\circ|Y^n)\right\} \\ 1 & \text{o.w.} \end{cases}$$

- F_X^{-1} quantizes above/below $\frac{1}{2}$
- Can we find a simple recursion rule nonetheless?

Normalized Recursive Representation



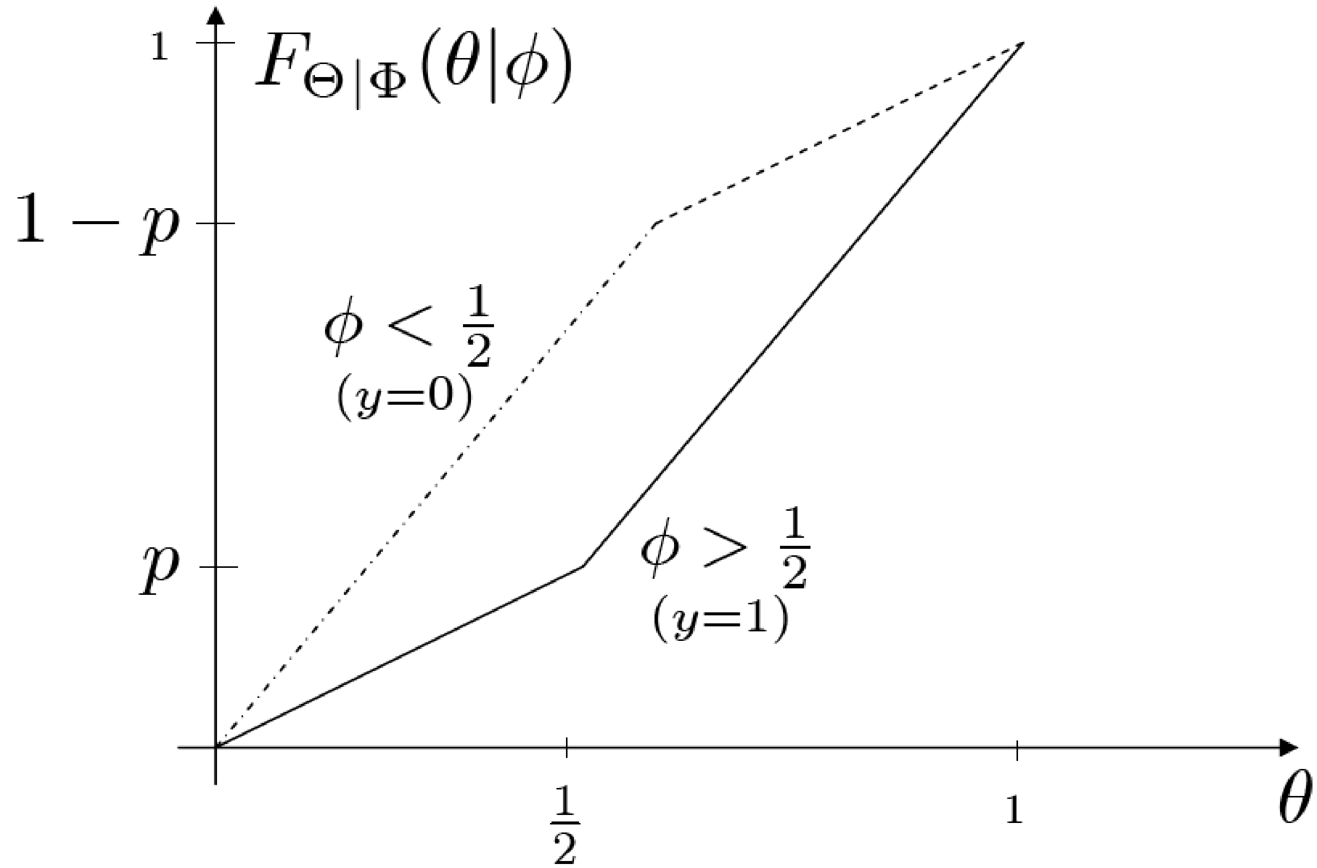
- A “message point” channel, $\Theta_n, \Phi_n \sim \mathrm{Unif}(0,1)$
- Preserves mutual information, common framework for discrete/continuous/mixed alphabet input distributions/channels
- PM scheme recursive representation

$$\Theta_1 = \Theta_0\,, \quad \Theta_{n+1} = F_{\Theta|\Phi}(\Theta_n|\Phi_n)$$

- (Θ_n, Φ_n) constitute a Markov chain over $(0,1)^2$, having $P_{\Theta\Phi}$ as an invariant distribution

The Normalized BSC

- The PM kernel for the normalized BSC is given by



- Equivalent to the original Horstein scheme (set $X_n = F_X^{-1}(\Theta_n)$)

Exponential Noise with a Mean Input Constraint

- Let $P_{Y|X}$ be an additive noise $\sim \mathrm{Exp}(b)$ channel
- Impose a mean input constraint $\mathbb{E}(X_n) = a$
- Capacity achieving distribution [Verdú'96]

$$f_X(x) = \frac{b}{a+b}\,\delta(x) + \frac{a}{(a+b)^2}\, e^{-\frac{x}{a+b}}\,, \quad C = \log\left(1 + \frac{a}{b}\right)$$

- Corresponding (normalized) PM scheme given by

$$\Theta_{n+1} = \begin{cases} \frac{a+b}{b} \cdot \Theta_n \cdot (1 - \Phi_n)^{\frac{a}{b}} & \Theta_n \leq \frac{b}{a+b} \\ \left(\frac{a}{a+b} \cdot \frac{1-\Phi_n}{1-\Theta_n}\right)^{\frac{a}{b}} & \Theta_n > \frac{b}{a+b} \end{cases}$$

- The actual input is $X_n = (a+b)\ln\left(\frac{a}{(a+b)(1-\Theta_n)}\right) \mathbb{1}_{[\frac{b}{a+b},1)}(\Theta_n)$

Achieving the Mutual Information

Theorem: Under some general conditions on the input/channel pair $(P_X, P_{Y|X})$, the corresponding PM scheme achieves any rate $R < I(X;Y)$, with either a fixed/variable optimal decoding rule, within an input constraint $(\eta, \mathbb{E}\eta(X))$

- Specifically, the Theorem holds for
  - Discrete memoryless channels (up to some small issues)
    - Corollary: The Horstein scheme achieves capacity
  - Input/channel pairs $(P_X, P_{Y|X})$ with a joint p.d.f. P_{XY} continuous over a convex support, and not "too wild"

Proof Outline

Achievability Conditions

- (REG) $I(X;Y) < \infty$, and some mild regularity conditions
- (ERG) The invariant distribution $P_{\Theta\Phi}$ for the Markov chain (Θ_n, Φ_n) is ergodic.
- (FIX) The (normalized) posterior matching kernel has no universal fixed points, in the sense that

$$\mathbb{P}(F_{\Theta|\Phi}(\theta|\Phi) = \theta) \; < \; 1$$

for any $\theta \in (0,1)$

Iterated Function System (IFS)

- An IFS $\{S_n(s_0)\}_{n=1}^{\infty}$ over a measurable space $\mathfrak{F}$ is generated by a measurable function $\omega_\phi(s)$ mapping $\mathfrak{F}$ to itself, as follows:

$$S_1 = s_0\,, \qquad S_{n+1}(s_0) = \omega_{\Phi_n} \circ \omega_{\Phi_{n-1}} \circ \cdots \circ \omega_{\Phi_1}(s)$$

  where $\{\Phi_n\}_{n=1}^{\infty}$ is an i.i.d control sequence, and $s_0 \in \mathfrak{F}$ is the initial point

- $\{S_n(s_0)\}_{n=1}^{\infty}$ is a Markov chain over $\mathfrak{F}$
- Let $\lambda : \mathfrak{F} \mapsto \mathbb{R}^+$, let $\xi : [0,1) \mapsto [0,1)$ be $\cap$-convex and $\xi(x) < x$ over $(0,1)$ (a *contraction*)
- Lemma: If $\mathbb{E}(\lambda(\omega_{\Phi_1}(s))) \leq \xi(\lambda(s))$ for any $s \in \mathfrak{F}$, then $\lambda(S_n(s_0)) \to 0$ in probability

Receiver Tracking Process as an IFS

- The receiver tracks the posterior $P_{\Theta_0|\Phi^n}$
- A recursive representation for the posterior c.d.f.

$$F_{\Theta_0|\Phi^{n+1}}(\theta|\Phi^{n+1}) = F_{\Theta|\Phi}(\cdot|\Phi_{n+1}) \circ F_{\Theta_0|\Phi^{n+1}}(\theta|\Phi^n)$$

- Recall $\{\Phi_n\}_{n=1}^{\infty}$ is i.i.d
- Thus the posterior c.d.f. is an IFS
  - Evolves over a function space $\mathfrak{F}_c$ of c.d.f.-like functions
  - Generated by the PM kernel (via function composition)
  - Initialized at $F_{\Theta_0}(\theta) = \theta$
  - Controlled by the channel output sequence $\{\Phi_n\}_{n=1}^{\infty}$

Contraction of the Posterior

- Under assumptions (REG)+(FIX), we can find a length function $\lambda : \mathfrak{F}_c \mapsto \mathbb{R}^+$ and a contraction $\xi(\cdot)$ such that
  - $\mathbb{E}\lambda(F_{\Theta|\Phi}(\cdot|\Phi) \circ s) \le \xi(\lambda(s)) \,, \quad s \in \mathfrak{F}_c$
  - Loosely speaking, $\lambda(s) \approx 0$ implies that s “close” a unit step function
- Therefore,
  - The posterior c.d.f. “tends” to a unit step function about Θ_0
  - Any fixed interval containing Θ_0 will be reliably decoded eventually
  - $R = 0$ is achievable!

Achievability of $R < I(X;Y)$

- Expand the posterior p.d.f. at the message point (Bayes rule, memoryless channel, i.i.d. output)

$$\frac{f_{\Theta_0|\Phi^n}(\Theta_0|\Phi^n)}{f_{\Theta_0|\Phi^{n-1}}(\Theta_0|\Phi^{n-1})} = \frac{f_{\Phi_n|\Theta_0,\Phi^{n-1}}(\Phi_n|\Theta_0,\Phi^{n-1})}{f_{\Phi_n|\Phi^{n-1}}(\Phi_n|\Phi^{n-1})} = f_{\Phi|\Theta}(\Phi_n|\Theta_n)$$

- Taking the logarithm, applying the recursion n times and using (ERG)+SLLN, we get

$$\frac{1}{n}\log f_{\Theta_0|\Phi^n}(\Theta_0|\Phi^n) = \frac{1}{n}\sum_{k=1}^{n}\log f_{\Phi|\Theta}(\Phi_k|\Theta_k) \to \mathbb{E}\left(\log f_{\Phi|\Theta}(\Phi|\Theta)\right)$$

$$= I(\Theta;\Phi) = I(X;Y) \qquad \text{a.s.}$$

- Roughly, $f_{\Theta_0|\Phi^n}(\Theta_0|\Phi^n) \approx 2^{nI(X;Y)}$

Achievability of $R < I(X;Y)$

- Assume that for all $k \in [n]$

$$F_{\Theta_0|\Phi^n}(\Theta_0 + 2^{-nR}|\Phi^n) - F_{\Theta_0|\Phi^n}(\Theta_0 - 2^{-nR}|\Phi^n) < \varepsilon$$

- $F_{\Theta_0|\Phi^n}(\theta|\Phi^k)$ is what the input to the normalized channel at time $k+1$ would have been, had the message point been θ
- Hence, the assumption implies that the input is insensitive to a 2^{-nR} perturbation in the message point
- Using (REG)+(ERG)+SLLN again, this can be roughly translated into

$$f_{\Theta_0|\Phi^n}(\Theta_0 \pm 2^{-nR}|\Phi^n) \approx 2^{n(I(X;Y)-\delta)}$$

where $\varepsilon \to 0$ implies $\delta \to 0$

Achievability of $R < I(X;Y)$

- If $R < I(X;Y) - \delta$ we get a contradiction

$$\int f_{\Theta_0|\Phi^n}(\theta|\Phi^n)d\theta \approx 2^{n(I(X;Y)-R-\delta)} \longrightarrow \infty$$

- Hence, with high probability there exists $k_0 \in [n]$ so that

$$F_{\Theta_0|\Phi^n}(\Theta_0 + 2^{-nR}|\Phi^n) - F_{\Theta_0|\Phi^n}(\Theta_0 - 2^{-nR}|\Phi^n) \geq \varepsilon$$

- Due to the repetitive nature of the scheme, one can imagine transmission to have started at time k_0 with the message point Θ_{k_0}
- 2^{-nR} neighborhood of $\Theta_0 \Rightarrow \varepsilon$-neighborhood of Θ_{k_0}
- Invoking zero rate result, this ε-neighborhood can be decoded in sublinear time $\rightarrow$ Achievability proved!

Examples Revisited

- BSC – Verifying the Horstein scheme achieves capacity
- AWGN channel – reconfirming the SK achieves capacity
- Exponential noise, mean constraint – The explicit PM scheme described achieves capacity!

Further Results

- Error probability analysis, providing closed form error exponent expressions for a range of rates (sometimes strictly below capacity)
- Channel model mismatch
  - Scheme designed for $(P_X, P_{Y|X})$
  - True channel is $P_{Y^*|X^*}$
  - Scheme induces some stationary input distribution P_{X^*}
  - Penalty in rate relative to $I(X^*;Y^*)$ is

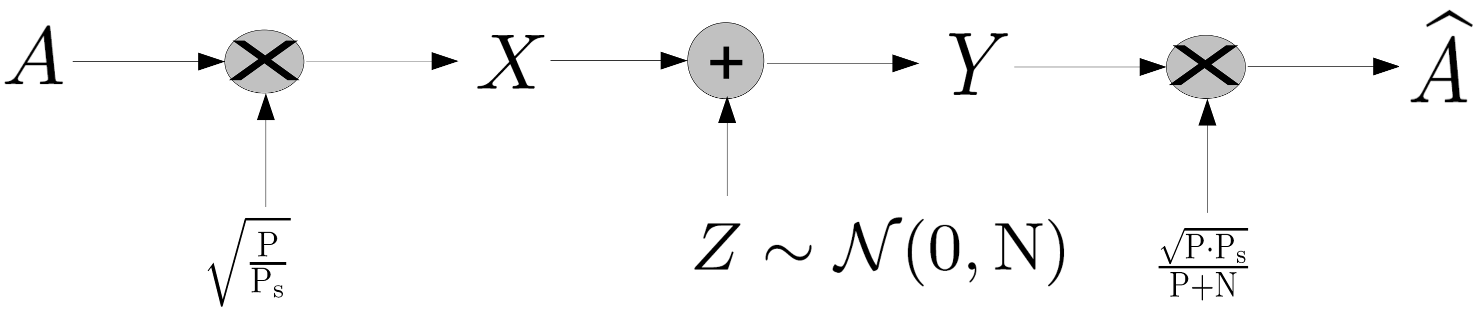
$$D(P_{Y^*|X^*} \| P_{Y|X} \mid P_{X^*}) - D(P_{Y^*} \| P_Y)$$

  - Robustness of the SK scheme

Application to Joint-Source Channel Coding (JSCC) with Feedback

A Well Known Gaussian Example

- Gaussian source $A \sim \mathcal{N}(0, \mathrm{P_s})$
- AWGN channel, input power constraint P
- Scalar linear transmission scheme ("uncoded"):



- Achieves optimal performance under quadratic distortion!

$$D = \mathbb{E}(A - \widehat{A})^2 = \mathrm{P_s} \cdot \left(1 + \frac{\mathrm{P}}{\mathrm{N}}\right)^{-1} \quad \Rightarrow \quad \mathrm{R(D) = C}$$

AWGN with Bandwidth Expansion

- Suppose m AWGN channels uses per source sample available
- Optimal distortion given by

$$R(D) = mC \quad \Rightarrow \quad D = \mathrm{P_s} \cdot \left(1 + \frac{\mathrm{P}}{\mathrm{N}}\right)^{-\mathrm{m}}$$

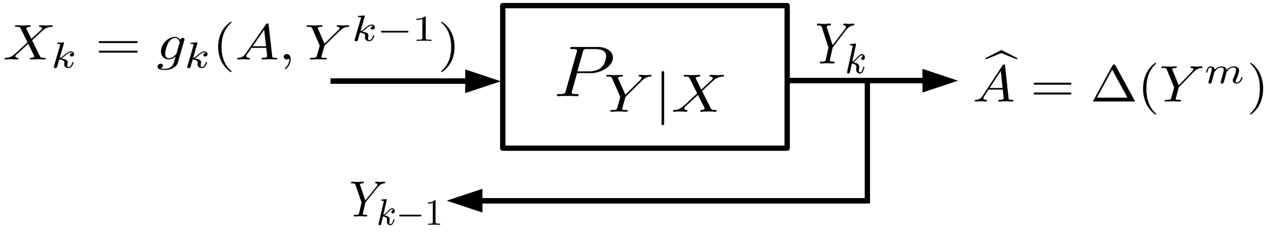
- Decays exponentially with the *bandwidth expansion factor* (BEF) m
- Optimal performance cannot generally be attained by a scalar $1:m$ *joint source-channel coding* (JSCC) scheme
- Assume an instantaneous noiseless feedback link is available
- The SK scheme achieves optimal performance!

What About Non Gaussian Settings?

- For a finite BEF, can optimal performance be achieved via feedback for other sources/channels/distortion measures?
- In general no, unless some unique relation is satisfied [Gastpar '02]
- So what can we do anyway?
  - Apply the PM principle
  - Show that the resulting PM-JSCC scheme has "good performance"

Problem Setting

- Source $A \sim P_A$ over an alphabet $\mathcal{A} \subseteq \mathbb{R}$
- Memoryless channel $P_{Y|X}$ with feedback, BEF $= m$
- A general $1 : m$ JSCC transmission scheme:



- General (η, u) input constraint: $\mathbb{E}(\eta(X_k)) \leq u$ for $k = 1, \ldots, m$
- General distortion measure $d : \mathcal{A}^2 \mapsto \mathbb{R}^+$
- Performance measured by the average distortion $D = \mathbb{E}(d(A, \widehat{A}))$

The PM-JSCC Scheme

- Set some channel input distribution P_X (design parameter)
- The transmission functions are defined for $k = 0, 1, \ldots m-1$ by

$$X_{k+1} = F_X^{-1} \circ F_{A|Y^k}(A|Y^k)$$

- Again, $X_{k+1} \sim P_X$ independent of Y^k, Y^m is i.i.d. with marginal P_Y, and we have a recursive representation:

$$X_1 = F_X^{-1} \circ F_A(A)\,, \quad X_{k+1} = F_X^{-1} \circ F_{X|Y}(X_k|Y_k)$$

- Seems to satisfy Gastpar's optimality conditions when possible
- Optimal exponential decay for a quadratic distortion measure
- What happens otherwise (which is usually the case..)?

Distortion Analysis Outline

- Suppose $\widehat{X}_m$ is some estimate of X_m
- Corresponds to a unique estimate $\widehat{X}_1$ of X_1, given by reversing the transmission scheme

$$\widehat{X}_1 = \omega_{Y_1} \circ \cdot \circ \omega_{Y_{m-2}} \circ \omega_{Y_{m-1}}(\widehat{X}_m)$$

where $\omega_y(\cdot) = F^{-1}_{X|Y}(\cdot|y) \circ F_X(\cdot)$ is the *inverse PM kernel*

- $\widehat{X}_1$ is generated by a time-reversed IFS (RIFS) with kernel $\omega_y(\cdot)$ and an i.i.d control sequence Y^m
- Now simply $\widehat{A} = F_A^{-1} \circ F_X(\widehat{X}_1)$

Distortion Analysis Outline (cont.)

- Set a fixed interval $J_m \subseteq \mathcal{X}$ so that $\mathbb{P}(X_m \in J_m) = P_X(J_m) = 1 - \delta$
  - Thus $\mathbb{P}(X_1 \in J_1) = 1 - \delta$, where the random interval J_1 is generated by running the RIFS over (the edges of) J_m
  - $\mathbb{P}(A \in J_A) = 1 - \delta$, where $J_A = F_A^{-1} \circ F_X(J_1)$
  - Now set $\widehat{A}$ to be any point within J_A (suboptimal)
- If the RIFS kernel $\omega_y(\cdot)$ is *contractive on the average*, then J_1 is exponentially smaller than J_m w.h.p.
- If $F_A^{-1} \circ F_X$ is M-Lipschitz, J_A is also exponentially small w.h.p.
- Two sources of distortion
  - $A \in J_A$ and J_A exponentially small (high prob.) $\Rightarrow$ small distortion
  - $A \notin J_A$ or J_A large (low prob.) – $\sup_{a,b \in \mathcal{A}} d(a,b) = d_{\max} < \infty$

Distortion Analysis Outline (cont.)

Theorem: Let $\omega_y(\cdot)$ be the inverse PM kernel and define

$$r_q \triangleq \sup_{s\neq t\in \mathrm{supp}(X)} \mathbb{E}\left[D_{s,t}(\omega_Y)\right]^q \,, \quad \bar{d}_\varepsilon \triangleq \sup_{(a,b)\subseteq \mathcal{A}\,,|b-a|\leq\varepsilon} d(a,b)$$

If there exists $q^* \in (0,1)$ such that $r_{q^*} < 1$, then the PM-JSCC scheme achieves an average distortion upper bounded by

$$D \leq \inf_{0<q<q^*,\varepsilon,\ell>0} \left\{ d_{\max}\left(\mathscr{T}_X(\ell) + \frac{(M\varepsilon^{-1}\ell)^q}{1-\mathscr{T}_X(\ell)}\, r_q^m\right) + \bar{d}_\varepsilon \right\}$$

within any input constraint of the form $(\eta, \mathbb{E}\eta(X))$.

Corollary: If P_X has a polynomially (or faster) decaying tail and $d(a,b) = |a-b|^\gamma$, the distortion decays exponentially with the BEF m.

Uniform Source, Uniform Noise

- $A \sim \mathrm{Unif}(0,1)$, $P_{Y|X}$ is an additive channel with noise $\sim \mathrm{Unif}(0,1)$
- Set (for example) $P_X = \mathrm{Unif}(0,1)$
- The PM-JSCC scheme is given by

$$X_1 = A\,, \qquad X_{k+1} = \frac{X_k}{Y_k} \cdot \mathbb{1}_{(0,1]}(Y_k) + \frac{X_k - Y_k + 1}{2 - Y_k} \cdot \mathbb{1}_{(1,2)}(Y_k)$$

- Very simple interpretation:
  - Begin by transmitting the uncoded source $X_1 = A$
  - Given Y_1 find the interval of feasible inputs, and an affine transformation that stretches this interval to $(0,1)$
  - Generate X_2 by applying the transformation to X_1
  - Repeat the two steps above for X_k, Y_k

Uniform Source, Uniform Noise (cont.)

- The inverse PM kernel is

$$\omega_y(s) = s f_Y(y) + (y-1)\cdot \mathbb{1}_{(1,2)}(y)$$

- The contraction factor is given by

$$r_q = \sup_s \mathbb{E}\left(\frac{\partial}{\partial s}\omega_Y(s)\right)^q = \mathbb{E}(f_Y(Y))^q = \left(1+\frac{q}{2}\right)^{-1} < 1$$

- Assume a distortion measure $d(a,b) = |a-b|^\gamma$, bounded over $(0,1)$
- The conditions of the Theorem hold, and we get

$$D \leq \inf_{q>0}\left(\left(\frac{\gamma}{q}\right)^{\frac{q}{q+\gamma}} + \left(\frac{q}{\gamma}\right)^{\frac{\gamma}{q+\gamma}}\right)\cdot\left(\frac{1}{1+\frac{q}{2}}\right)^{\frac{m\gamma}{q+\gamma}}$$

Uniform Source, Uniform Noise (cont.)

- The distortion decays exponentially with the BEF:

$$\lim_{m\to\infty} -\frac{1}{m}\log D \;\geq\; \sup_{q>0} \frac{\gamma}{q+\gamma}\log\left(1+\frac{q}{2}\right)$$

- For a quadratic distortion measure ($\gamma = 2$) the exponent attained by the PM-JSCC scheme is lower bounded by $\frac{\log e}{e}$
- This should be contrasted with the exponent promised by the separation principle, which is $\log e$

Further Research

- JSCC setting:
  - Obtain tighter bounds by “measuring” contraction “relative” to the distortion measure
  - Finite block optimality of the PM-JSCC scheme?
  - Sensitivity to channel variations (graceful degradation)
  - relation to non-feedback JSCC schemes?
- Communication setting:
  - Channels with memory, achieve the *directed information rate* – requires a different interpretation of the PM principle
  - Multi-terminal channels
  - Noisy feedback