1. (a) Show that if U_1, U_2 are unitary then $U_1 \otimes U_2$ is also unitary. Find its inverse.

   (b) Show that $(A \otimes C)(B \otimes D) = (AB) \otimes (CD)$ whenever the dimensions are compatible.

   (c) Show that $\langle\psi|A|\psi\rangle = \mathrm{tr}(A|\psi\rangle\langle\psi|)$, where $\mathrm{tr}(\cdot)$ is the *matrix trace* operator.

   (d) Let $\boldsymbol{v}_A, \boldsymbol{v}_B$ be the vectors of eigenvalues of the matrices A, B respectively. Show that $\boldsymbol{v}_A \otimes \boldsymbol{v}_B$ is the vector of eigenvalues of $A \otimes B$.

   (e) Show that $\mathrm{rank}(A \otimes B) = \mathrm{rank}(A) \cdot \mathrm{rank}(B)$ and $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A) \cdot \mathrm{tr}(B)$.

2. (a) A system is to be measured in an orthonormal basis $|v_i\rangle$, but the measuring apparatus works with the orthonormal basis $|w_i\rangle$. Show that the measurement can be performed by applying a unitary transformation U to the system before measuring using the apparatus, followed by applying $U^\dagger$.

   (b) Prove that no general measurement can distinguish between two non-orthogonal states (do not use Neumark's Theorem).

   (c) Assume that M classical messages are to be encoded into M states in a system A. The encoding is *error free* if the encoded messages are distinguishable. Show that an error free encoding exists if and only if $M \leq \dim(A)$. Conclude that a qubit can reliably represent at most one classical bit.

3. (a) Show that applying any unitary operator U to both qubits in a *singlet* $|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ does not change the state (up to a global phase factor).

   (b) Show that $\langle\beta_{ij}|E \otimes I|\beta_{ij}\rangle$ does not depend on i, j for any positive operator E on a single qubit, where β_{ij} is a Bell state. Explain the significance for security in the *superdense coding protocol*.

4. Define a *unitary superdense coding protocol* for sending M classical messages, as a state $|\psi\rangle$ of a bipartite system AB together with a set of M unitary encoding operators $\{U_m\}$ on the system A. The protocol is *error free* if the encoded states $(U_m \otimes I_B)|\psi\rangle$ are distinguishable. Show that the encoded states are contained in a subspace of AB, of dimension at most $(\dim A)^2$. Conclude that for an error free protocol the number of encoding operators must satisfy $M \leq (\dim A)^2$, and that consequently the ratio of two bits per qubit cannot be improved upon by such a protocol (this is true for a non-unitary protocol as well).

5. (a) Consider the POVM given by the operators

$$E_1 = \frac{1}{2}|0\rangle\langle 0|\,, \quad E_2 = \frac{1}{2}|1\rangle\langle 1|\,, \quad E_3 = \frac{1}{2}|+\rangle\langle +|\,, \quad E_4 = \frac{1}{2}|-\rangle\langle -|$$

   Verify that this is an eligible POVM.

   (b) Find a realization of the POVM above as an orthogonal measurement in a two-qubit state space, by adding an ancilla qubit.

6. (a) Show that a density matrix ρ in an n-dimensional state space satisfies $\frac{1}{n} \leq \mathrm{tr}(\rho^2) \leq 1$, and provide necessary and sufficient conditions for equalities.

   (b) Verify that (2b) and (2c) are also valid for mixed states.

7. *Broadcasting a mixed state.* Suppose a system A is in the state ρ_i with probability p_i.

   (a) Show that if ρ_i all commute then it is possible to prepare a state σ_i in a system AB, so that $\mathrm{tr}_A(\sigma_i) = \mathrm{tr}_B(\sigma_i) = \rho_i$. *Hint:* Commuting Hermitian matrices can be jointly diagonalized.

(b) Verify that the above *broadcasting* procedure is *not a cloning* procedure, i.e., show that $\sigma_i \neq \rho_i \otimes \rho_i$. Can you think of a classical analog?

(c) Give an example of non-commuting density matrices where broadcasting is impossible. It is actually true that broadcasting is possible if and only if ρ_i commute!

8. Let ρ^{AB} be the state of a bipartite system AB.

   (a) Prove that applying any unitary transformation U to the system A *alone*, does not change the density operator of system B. Namely, show that $\text{tr}_A(\rho^{AB}) = \text{tr}_A\big((U \otimes I_B)\rho^{AB}(U^\dagger \otimes I_B)\big)$

   (b) Prove that measuring the system A *alone* using any set of measurement operators $\{M_m\}$, does not change the density operator of system B.

   (c) In the *teleportation protocol*, find Bob's density matrix just before Alice performs her measurements. What is his density matrix just after the measurements are performed?

9. A unit vector $|\phi\rangle$ is *closest* to the ensemble $\{p_i, |\psi_i\rangle\}$ if it maximizes the expected projection $e(\phi) = \sum_i p_i \left|\langle\phi|\psi_i\rangle\right|^2$. Show that the closest vector is the eigenvector of the ensemble's density matrix ρ with the maximal eigenvalue. What is the corresponding expected projection?

10. Let $\{p_i, |\psi_i\rangle\}_{i=1}^n$ be an ensemble describing the system A, with a corresponding density matrix ρ. Let R be an auxiliary system of dimension n, with an orthonormal basis $|i\rangle$.

    (a) Show that the state $\sum_i \sqrt{p_i}|\psi_i\rangle|i\rangle$ in AR is a *purification* of ρ.

    (b) Assume we measure R in the basis $|i\rangle$. What is the probability of obtaining the result i? What is the corresponding state of the system A?

    (c) Let the state $|AR\rangle$ be any purification of ρ in AR. Show that there is an orthonormal basis for R so that after measuring w.r.t that basis, the system A behaves according to the original ensemble $\{p_i, |\psi_i\rangle\}$.

11. Let $|\psi\rangle$ and $|\phi\rangle$ be two pure state of a bipartite system AB.

    (a) Show that the Schmidt number of $|\psi\rangle$ is equal to the rank of the reduced density matrix $\rho_A = \text{tr}_B(|\psi\rangle\langle\psi|)$.

    (b) Show that $|\psi\rangle$ and $|\phi\rangle$ are related by local unitary operations, i.e., there exist unitary operators U on A and V on B so that $|\psi\rangle = (U \otimes V)|\phi\rangle$, if and only if $|\psi\rangle$ and $|\phi\rangle$ have the same Schmidt coefficients.