1. Complete exercises $10 - 11$ from assignment 1.

2. (a) Show that for an ensemble of mixed states $\{p_i, \rho_i\}$

$$S\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i S(\rho_i) + H(\{p_i\})$$

with equality if and only if ρ_i are orthogonal. *Hint:* Prove the result for an ensemble of pure states, by purifying the mixed state representing the ensemble. Now use this to prove the general case.

   (b) Show that discarding a subsystem does not increase the mutual information, i.e., $S(A;B) \leq S(A;BC)$.

3. (a) Use the monotonicity of the relative entropy to prove the strong subadditivity property. *Hint:* Consider the relative entropy $S(\rho^{ABC} \| \rho^A \otimes \rho^{BC})$.

   (b) Show that $S(A) + S(B) \leq S(A,C) + S(B,C)$ is implied by the strong subadditivity property.

   (c) Show that the concavity of the conditional entropy $S(A|B)$ is implied by the strong subadditivity property.

4. For a general tripartite system ABC, show that

   (a) $|S(A|B)| \leq \log \dim A$

   (b) $S(A;B|C) \leq 2 \log \min(\dim A, \dim B)$

   (c) $S(AB;C) \leq S(B;AC) + 2 \log \dim A$

   Now assume that the system is in a state of the form $\rho^{ABC} = \sum_j p_j |j\rangle\langle j|^A \otimes \sigma_j^{BC}$, where $|j\rangle^A$ are orthonormal. This sort of state is known as a *classical-quantum* state, since it represents an effectively classical system A correlated with a quantum system BC. Show that

   (d) $S(A;B|C) \leq \log \dim A$

   (e) $S(AB;C) \leq S(B;AC) + \log \dim A$

   Note that the factor of two disappeared in the classical-quantum setting. This roughly corresponds to the fact that quantum systems support superdense coding while classical systems do not.

5. *Measurements and entropy*

   (a) Assume a state ρ is measured using the operators $M_1 = |0\rangle\langle 0|$ and $M_2 = |0\rangle\langle 1|$ (with an unknown result), ending up in the state ρ'. Show that in this case $S(\rho') < S(\rho)$ is possible.

   (b) Assume a state ρ is measured using some general set of measurement operators, and denote by ρ_m the state obtained given the result m. Show that the average entropy is not increased, i.e.,

$$\mathbb{E}S(\rho_m) \leq S(\rho)$$

   where the expectation is taken w.r.t. the index m. *Hint:* Perform purifications.

   (c) Suppose that given a state ρ, we measure the observable $A = \sum y |\psi_y\rangle\langle\psi_y|$, where y is the corresponding result. Show that the Shannon entropy of the outcome random variable Y satisfies $H(Y) \geq S(\rho)$, with equality if and only if A and ρ commute. Physically, that means that the most predictable outcomes correspond to measurements of observables that commute with the density matrix.

6. Use the Holevo bound to argue that n qubits cannot carry more than n classical bits.

7. (a) Show that a PGM for an ensemble of linearly-independent pure states is always an orthogonal measurement.
   (b) Show that a PGM for an ensemble of orthogonal mixed states is the optimal measurement.

8. *The Peres-Wooters example.* Alice prepares one of the following states with uniform probability, and sends it to Bob:

$$|\varphi_0\rangle = |0\rangle\,, \quad |\varphi_1\rangle = \frac{1}{2}\left(-|0\rangle + \sqrt{3}|1\rangle\right)\,, \quad |\varphi_2\rangle = -\frac{1}{2}\left(|0\rangle + \sqrt{3}|1\rangle\right)$$

   (a) For detection, Bob uses the POVM $E_m = \alpha(I - |\varphi_m\rangle\langle\varphi_m|)$. Find α.
   (b) Convince yourself that due to symmetry, this POVM is optimal for detection (this can be made rigorous).
   (c) Compute the mutual information corresponding to this POVM (which is in fact the accessible information) and compare it with the Holevo bound.
   (d) Suppose Alice can now send *two qubits*. Argue that sending one of the nine options $|\varphi_i\rangle|\varphi_j\rangle$ with uniform probability results in the same accessible information per qubit.
   (e) Alice decides to send two qubits, but limits herself to a uniform selection between one the the following three states:

$$|\Phi_m\rangle = |\varphi_m\rangle|\varphi_m\rangle\,, \quad m = 0, 1, 2$$

   Express the density matrix $\rho = \frac{1}{3}\sum|\Phi_m\rangle\langle\Phi_m|$ in terms of the Bell basis, and compute $S(\rho)$.
   (f) Construct the PGM for the three vectors $|\Phi_m\rangle$, by expanding them in the Bell basis. The PGM in this case is an orthogonal measurement (why?) - express its basis elements in terms of the Bell basis elements.
   (g) Compute the mutual information corresponding to the PGM, and compare it to the Holevo bound in this case. Show that the mutual information gain per qubit now exceeds the accessible information of section (8d), despite the fact that Alice used less encoding states. Explain!