1. *Coherent superdense coding and Teleportation.*

   (a) Recall that in the superdense coding protocol, Alice was in possession of two bits $a_1 a_2$ and applied the operation $Z^{a_1} X^{a_2}$ to her part of the shared ebit, before sending it to Bob who then measured the two qubits in the Bell basis to recover $a_1 a_2$. Show that this protocol can be made coherent, by Alice replacing her classically controlled operation with a quantum controlled operation, where the control is the quantum register $|a_1 a_2\rangle$. We thus establish the resource inequality 1 qubit + 1 ebit $\geq$ 2 cobits.

   (b) What happens in (1a) if Alice measures the register $|a_1 a_2\rangle$ in the computational basis before it is applied as a control?

   (c) Consider the teleportation protocol, and assume Alice and Bob can communicate *coherently*. Show that if Alice, instead of measuring her qubits in the computational basis sends a copy of her registers coherently to Bob, and Bob replaces his classically controlled operation with a quantum controlled operation, then we get the resource inequality 2 cobits + 1 ebit $\geq$ 1 qubit + 2 ebits, instead of the 2 cbits + 1 ebit $\geq$ 1 qubit attained by incoherent communication.

   (d) Suppose Alice and Bob can communicate coherently, but have no shared ebits. Show that if they borrow an ebit from the bank then they can perform teleportation, returning the ebit to the bank when completed. This is called *entanglement catalysis*, and thus 2 cobits $\geq$ 1 qubit + 1 ebit *catalytically*. Combined with (1a), deduce that superdense coding and teleportation are (catalytically) reversible in the presence of coherent classical communication.

2. Consider the qubit *amplitude damping channel* $\mathcal{E}_d$, defined by the following two channel operators:

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

   where $0 \leq \gamma \leq 1$.

   (a) Provide an explicit expression for $\mathcal{E}_\gamma(\rho)$, in terms of the matrix ρ elements.

   (b) What happens to the state ρ when transmitted through a concatenation of many such channels?

   (c) We shall now find an expression for the entanglement-assisted capacity of the amplitude damping channel. Convince yourself that due to symmetry, it suffices to consider inputs of the form (can you prove this fact rigorously?)

$$\rho = \begin{pmatrix} 1-x & 0 \\ 0 & x \end{pmatrix}$$

   Use this to show that

$$C_E(\mathcal{E}_\gamma) = \max_x \Big(h(x) + h(x(1-\gamma)) - h(x\gamma) \Big)$$

   where $h(\cdot)$ is the binary entropy function.

3. *Additive gain from entanglement.* Consider a quantum channel $\mathcal{E}$ over a d-dimensional input Hilbert space.

   (a) Show that $C_{CQ}(\mathcal{E} \otimes I_B) = C_{CQ}(\mathcal{E}) + C_{CQ}(I_B)$, where I_B is the identity map over some auxiliary system B.

(b) Use the above to argue that

$$C_E(\mathcal{E}) \le C(\mathcal{E}) + \log d$$

Hint: Introduce a noiseless auxiliary channel for entanglement sharing, as in the derivation of the entanglement assisted capacity.

4. *Coherent information and entanglement.*

   (a) Show that a bipartite system in a pure state $|\psi^{AB}\rangle$ is entangled if and only if $I_c(A\rangle B) > 0$.

   (b) A bipartite mixed state ρ^{AB} is said to be *separable* if it can be expressed as

   $$\rho^{AB} = \sum_i p_i \, \sigma_i^A \otimes \tau_i^B$$

   for some ensemble $\{p_i \, , \sigma_i^A \otimes \tau_i^B\}$. A state is said to be *entangled* if it is not separable, thereby generalizing the notion of entanglement to mixed states. Show that if ρ^{AB} is separable, then $I_c(A\rangle B) \le 0$.

5. Consider transmission over a *time sharing* of quantum channels, where the channel is *noisy* with quantum capacity Q_{noisy} a fraction p of the time, and otherwise *noiseless* with quantum capacity Q. Assuming the sharing pattern is a-priori known to both terminals, show that the quantum capacity in this setting equals $p\,Q_{noisy} + (1-p)Q$. *Hint:* The noisy channel can *simulate* the noiseless one.

6. Consider a quantum channel $\mathcal{E}$, and denote by $\mathcal{N}$ the channel between the transmitter and the (a-priori pure) corresponding environment. The channel $\mathcal{E}$ is called *degradable* if there exists a quantum channel $\mathcal{T}$ so that $\mathcal{T} \circ \mathcal{E} = \mathcal{N}$. The channel $\mathcal{E}$ is called *anti-degradable* if $\mathcal{N}$ is degradable w.r.t. $\mathcal{E}$.

   (a) Prove that if $\mathcal{E}$ is anti-degradable, then $Q(\mathcal{E}) = 0$.

   (b) Show that the amplitude damping channel is degradable for $\gamma \le \frac{1}{2}$ and anti-degradable for $\gamma \ge \frac{1}{2}$.

   (c) Prove the subadditivity of the conditional entropy, i.e. $S(AB|CD) \le S(A|C) + S(B|D)$. *Hint:* Apply the strong subadditivity.

   (d) Prove that if $\mathcal{E}$ is degradable, then its quantum capacity is equal to its maximal one-shot coherent information, i.e., $Q(\mathcal{E}) = \max_\rho I_c(\rho, \mathcal{E})$. *Hint:* Provide a system-environment unitary representation for $\mathcal{E}$ and $\mathcal{T}$. Now express the coherent information as the conditional entropy of the output of $\mathcal{T}$ given its respective environment, then use the subadditivity property proved in (6c).

   (e) Give an expression for the quantum capacity $Q(\mathcal{E}_\gamma)$ of the amplitude damping channel. *Hint:* Optimization over the same input as in (2c) is sufficient.

   (f) Give an expression for the entanglement-assisted quantum capacity $Q_E(\mathcal{E}_\gamma)$ of the amplitude damping channel. Demonstrate that $Q_E(\mathcal{E}_\gamma)$ can be positive while $Q(\mathcal{E}_\gamma) = 0$.

   (g) Show that as $\gamma \to 0$, shared entanglement becomes insignificant for the transmission of quantum states over $\mathcal{E}_\gamma$. Explain!

7. We shall prove that the classically-assisted quantum capacity of a quantum p-erasure channel $\mathcal{E}_{er}$ is given by $Q_{\leftrightarrow}(\mathcal{E}_{er}) = 1 - p$, thereby demonstrating that two way classical communication can increase quantum capacity.

   (a) *(Achievability)* Describe a simple protocol attaining a rate of $(1-p)$ qubits per channel use.

   (b) *(Converse)* Assume $Q_{\leftrightarrow}(\mathcal{E}_{er}) > 1 - p$. Show that if Alice and Bob share some amount of ebits in advance, they can *simulate* the quantum p-erasure channel and increase the number of their shared ebits using only LOCC, leading to a contradiction.