The Approximate Sum Capacity of the Symmetric Gaussian $K$-User Interference Channel

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The symmetric Gaussian 2-user IC: channel model

\[
y_k = x_k + g x^*_k + z_k
\]

- Channel is static and real valued.
- Gaussian noises \( z_k \) are of zero mean and variance 1.
- All users are subject to the power constraint \( \|x_k\|^2 \leq n\text{SNR} \).
- Define INR \( \triangleq g^2\text{SNR} \) and \( \alpha \triangleq \frac{\log(\text{INR})}{\log(\text{SNR})} \).

Channel is symmetric:
sum capacity = 2 × symmetric capacity
GDoF of symmetric Gaussian 2-user IC

- Symmetric capacity is known to within $1/2$ bit (Etkin et al. 08).
- DoF for each user is $1/2$.
- GDoF gives more refined view

$$d(\alpha)$$
Symmetric Gaussian 2-user IC

Noisy interference regime

- Treat interference as noise

\[ d(\alpha) \]
Symmetric Gaussian 2-user IC

Weak interference regime

- Jointly decode intended message and part of interference (Han-Kobayashi).

\[ d(\alpha) \]
Symmetric Gaussian 2-user IC

Strong interference regime

- Jointly decode intended message and interference

\[ d(\alpha) \]

\[ \frac{1}{2}, \frac{2}{3}, 1, 2 \]
Symmetric Gaussian 2-user IC

Very strong interference regime

- Decode interference and then successively decode intended message

\[
d(\alpha)
\]

Approx. Sum Capacity of the Symmetric Gaussian \( K \)-User IC
The symmetric Gaussian $K$-user IC : channel model

\[ \mathbf{y}_k = \mathbf{x}_k + g \sum_{\mathbf{m} \neq k} \mathbf{x}_m + \mathbf{z}_k \]

- INR $\triangleq g^2 \text{SNR}$ and $\alpha \triangleq \frac{\log(\text{INR})}{\log(\text{SNR})}$. 

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Approx. Sum Capacity of the Symmetric Gaussian $K$-User IC
DoF is discontinuous at the rationals (Etkin and E. Ordentlich 09, Wu et al. 11).

GDoF of the symmetric $K$-user IC is independent of $K$, except for discontinuity at $\alpha = 1$ (Jafar and Vishwanath 10).

\[
d(\alpha) \approx \frac{1}{K}
\]
What about finite SNR?

- Adding interference cannot increase capacity
  - → **Outer bounds** for $K = 2$ remain valid for $K > 2$. 

![Graph](image-url)
The symmetric Gaussian $K$-user IC: what do we know?

What about finite SNR?

- Can always use time-sharing
  \[ C_{\text{SYM}} > \frac{1}{2K} \log(1 + K \text{SNR}). \]
The symmetric Gaussian $K$-user IC: what do we know?

What about finite SNR?

- Can treat interference as noise
  → achieves the approximate capacity for noisy interference regime
The symmetric Gaussian $K$-user IC: what do we know?

- For the other regimes lattice codes are useful.
- Closed under addition
  \[ \implies K - 1 \text{ interferers folded to one effective interferer.} \]
- Each receiver sees a $K$-user MAC

\[
y_k = x_k + g \sum_{m \neq k} x_m + z_k,
\]
The symmetric Gaussian $K$-user IC: what do we know?

- For the other regimes lattice codes are useful.
- Closed under addition
  \[ \implies K - 1 \] interferers folded to one effective interferer.
- Assume $x_1, \ldots, x_K \in \Lambda$.
  \[ \implies \] Effective 2-user MAC at each receiver
  \[ y_k = x_k + g x_{\text{int},k} + z_k, \]
  where $x_{\text{int},k} = \sum_{m \neq k} x_m \in \Lambda$. 

Approx. Sum Capacity of the Symmetric Gaussian $K$-User IC
The symmetric Gaussian $K$-user IC: what do we know?

- For the other regimes lattice codes are useful.
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Assume $x_1, \ldots, x_K \in \Lambda$.
$\implies$ Effective 2-user MAC at each receiver

$$y_k = x_k + g x_{\text{int},k} + z_k,$$

where $x_{\text{int},k} = \sum_{m \neq k} x_m \in \Lambda$.

How to decode $x_k$?
The symmetric Gaussian $K$-user IC: what do we know?

For large $g$, can decode sum of interferences, subtract and decode desired codeword (Sridharan et al. 08)

$X_k$  

$X_{int,k}$
The symmetric Gaussian $K$-user IC: what do we know?

For large $g$, can decode sum of interferences, subtract and decode desired codeword (Sridharan et al. 08)
The symmetric Gaussian $K$-user IC: what do we know?

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\( x_k \rightarrow x_{\text{int},k} \)
The symmetric Gaussian $K$-user IC: what do we know?

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The symmetric Gaussian $K$-user IC: what do we know?

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Decode $x_{\text{int},k}$
The symmetric Gaussian $K$-user IC: what do we know?

For large $g$, can decode sum of interferences, subtract and decode desired codeword (Sridharan et al. 08)

Decode $x_{\text{int},k}$
The symmetric Gaussian $K$-user IC: what do we know?

For large $g$, can decode sum of interferences, subtract and decode desired codeword (Sridharan et al. 08)

Cancel $x_{\text{int},k}$
The symmetric Gaussian $K$-user IC: what do we know?

For large $g$, can decode sum of interferences, subtract and decode desired codeword (Sridharan et al. 08)

Decode $x_k$
The symmetric Gaussian $K$-user IC: what do we know?

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The symmetric Gaussian $K$-user IC: what do we know?

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Decide $x_k$
The symmetric Gaussian $K$-user IC: what do we know?

For large $g$, can decode sum of interferences, subtract and decode desired codeword (Sridharan et al. 08)

$x_k \rightarrow$  

$\sum x_{\text{int},k}$  

$\sum x_k \rightarrow z$  

$y \rightarrow$  

Decode $x_k$
The symmetric Gaussian $K$-user IC: what do we know?

What about finite SNR?

- Successive decoding is optimal in the very strong interference regime.
The symmetric Gaussian $K$-user IC: strong interference

\[ y_k = x_k + g x_{\text{int},k} + z_k, \quad x_k, x_{\text{int},k} \in \Lambda \]

- Assume strong interference: $g > 1$ but not $\gg 1$.
- For 2-user IC jointly decoding intended message and interference is optimal.
- For $K$-user IC jointly decoding $x_k, x_{\text{int},k}$ seems like a good idea.

Question

What rates are achievable?
The symmetric Gaussian $K$-user IC: strong interference

$$y_k = x_k + g x_{\text{int},k} + z_k, \quad x_k, x_{\text{int},k} \in \Lambda$$

- Assume strong interference: $g > 1$ but not $\gg 1$.
- For 2-user IC jointly decoding intended message and interference is optimal.
- For $K$-user IC jointly decoding $x_k, x_{\text{int},k}$ seems like a good idea.

MAC capacity theorem does not hold when both transmitters use the same lattice codebook
$$\implies$$ Need a new coding theorem.
What’s the problem with using the same lattice code?

Assume there is no noise at all
What’s the problem with using the same lattice code?

AMBIGUITY!
Decoding the two lattice points directly is difficult. Instead...

### New decoder based on compute-and-forward

Decode two equations with integer coefficients and solve for desired codeword.

\[ y_k = x_k + g x_{\text{int},k} + z_k, \]
Decoding the two lattice points directly is difficult. Instead...

New decoder based on compute-and-forward

Decode two equations with integer coefficients and solve for desired codeword.

\[
\begin{bmatrix}
\tilde{y}_1^k \\
\tilde{y}_2^k
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
x_k \\
x_{\text{int, } k}
\end{bmatrix} +
\begin{bmatrix}
z_{\text{eff, } 1} \\
z_{\text{eff, } 2}
\end{bmatrix}
\]
Decoding the two lattice points directly is difficult. Instead...

New decoder based on compute-and-forward

Decode two equations with integer coefficients and solve for desired codeword.

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\begin{bmatrix}
x_k \\
x_{\text{int},k}
\end{bmatrix}
+ \begin{bmatrix}
z_{\text{eff},1} \\
z_{\text{eff},2}
\end{bmatrix}
\]

Main result

We use this approach to obtain the approximate symmetric capacity region of the $K$-user symmetric IC up to an outage set.
The symmetric Gaussian $K$-user IC: new inner bounds

3-user IC @ SNR=20dB

Approx. Sum Capacity of the Symmetric Gaussian $K$-User IC
The symmetric Gaussian $K$-user IC: new inner bounds

3-user IC @ SNR=35dB

![Diagram showing symmetric rates vs. channel use for 3-user IC at SNR=35dB.](image-url)
The symmetric Gaussian $K$-user IC: new inner bounds

3–user IC @ SNR=50dB

Symmetric rate [bits/channel use]

3–user IC @ SNR=50dB

Symmetric rate [bits/channel use]

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Approx. Sum Capacity of the Symmetric Gaussian $K$-User IC
The symmetric Gaussian $K$-user IC: new inner bounds

3-user IC @ SNR=65dB

Approx. Sum Capacity of the Symmetric Gaussian $K$-User IC
Theorem - Nazer-Gastpar 11

For the channel \( y = \sum_{k=1}^{K} h_k x_k + z \) the equation \( \sum_{k=1}^{K} a_k x_k \) with
\[
a = [a_1 \cdots a_K] \in \mathbb{Z}^K
\]
can be decoded reliably as long as the rates of all users satisfy

\[
R < \frac{1}{2} \log \left( \frac{\text{SNR}}{\text{SNR} \| \beta h - a \|^2 + \beta^2} \right)
\]

for some \( \beta \in \mathbb{R} \).
Main tool: compute-and-forward

**Theorem - Nazer-Gastpar 11**

For the channel $y = \sum_{k=1}^{K} h_k x_k + z$ the equation $\sum_{k=1}^{K} a_k x_k$ with $a = [a_1 \ \cdots \ a_K] \in \mathbb{Z}^K$ can be decoded reliably as long as the rates of all users satisfy

$$R < \frac{1}{2} \log \left( \frac{\text{SNR}}{\text{SNR} \| \beta h - a \|^2 + \beta^2} \right)$$

for some $\beta \in \mathbb{R}$.

Use one channel output to decode two equations

$$y_k = x_k + g x_{\text{int},k} + z_k,$$
Main tool: compute-and-forward

**Theorem - Nazer-Gastpar 11**

For the channel $\mathbf{y} = \sum_{k=1}^{K} h_k \mathbf{x}_k + \mathbf{z}$ the equation $\sum_{k=1}^{K} a_k \mathbf{x}_k$ with $\mathbf{a} = [a_1 \cdots a_K] \in \mathbb{Z}^K$ can be decoded reliably as long as the rates of all users satisfy

$$R < \frac{1}{2} \log \left( \frac{\text{SNR}}{\text{SNR} \| \beta \mathbf{h} - \mathbf{a} \|^2 + \beta^2} \right)$$

for some $\beta \in \mathbb{R}$.

Use one channel output to decode two equations

$$\begin{bmatrix} \tilde{y}_k^1 \\ \tilde{y}_k^2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{\text{int},k} \end{bmatrix} + \begin{bmatrix} \mathbf{z}_{\text{eff},1} \\ \mathbf{z}_{\text{eff},2} \end{bmatrix}$$
Main tool: compute-and-forward

- Decoding two equations is not very effective when channel gains are close to integers.
- This causes the notches in the achievable rate region.
- Fortunately, this rarely happens...

![Graph showing normalized computation rate vs. g for First Equation, Second Equation, and their sum.]
Main tool: compute-and-forward

PROMO
To hear more about this come to "The Compute-and-Forward Transform" tomorrow at 15:20.

![Graph showing normalized computation rate vs g](image)

- First Equation
- Second Equation
- Sum

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Approx. Sum Capacity of the Symmetric Gaussian K-User IC
## Compute-and-forward for the symmetric $K$-user IC

<table>
<thead>
<tr>
<th>Transmit</th>
<th>Equations Decoded by Receivers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$a_{11}x_1 + a_{12} \sum_{\ell \neq 1} x_\ell$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$a_{11}x_2 + a_{12} \sum_{\ell \neq 2} x_\ell$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$x_K$</td>
<td>$a_{11}x_K + a_{12} \sum_{\ell \neq 1} x_\ell$</td>
</tr>
</tbody>
</table>

- From one real equation decode two linearly independent equations with integer coefficients.
- Corresponding computation rates are $R_{\text{comp},1}$, $R_{\text{comp},2}$.

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Approx. Sum Capacity of the Symmetric Gaussian $K$-User IC
Approx. symmetric capacity: strong interference regime

\[ C_{\text{SYM}} \geq R_{\text{comp},2} \]

- \( R_{\text{comp},2} \) is the solution to an integer-least squares optimization problem.
- Inner bound can be found numerically and plotted.
Approx. symmetric capacity: strong interference regime

\[ C_{\text{SYM}} \geq R_{\text{comp},2} \]

- \( R_{\text{comp},2} \) is the solution to an integer-least squares optimization problem.
- Inner bound can be found numerically and plotted.

![Graph representing 3-user IC at SNR=30dB](image)
Approx. symmetric capacity: strong interference regime

\[ C_{\text{SYM}} \geq R_{\text{comp},2} \]

- \( R_{\text{comp},2} \) is the solution to an integer-least squares optimization problem.
- Inner bound can be found numerically and plotted.

**Question**

For \( c > 0 \) bits, what is the fraction of channel gains \( g \) for which

\[ \text{outer bound} - \text{inner bound} > c \text{ bits?} \]
Outage set

3-user IC @ SNR=30dB

Strong interference regime

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Approx. Sum Capacity of the Symmetric Gaussian $K$-User IC
Outage set

3-user IC @ SNR=30dB

c = 0.25 bits
Outage set

48% outage for $c = 0.25$ bits
Outage set

3-user IC @ SNR=30dB

- 22% outage for $c = 0.5$ bits
Outage set

11% outage for $c = 0.75$ bits
The symmetric capacity of the symmetric Gaussian $K$-user IC is lower bounded by

$$C_{\text{SYM}} \geq \frac{1}{4} \log^+(\text{INR}) - \frac{c}{2} - 3$$

for all values of $1 \leq g^2 < \text{SNR}$ except for an outage set whose measure is a fraction of $2^{-c}$ of the interval $1 \leq |g| < \sqrt{\text{SNR}}$, for any $c > 0$. 
The symmetric capacity of the symmetric Gaussian $K$-user IC is lower bounded by

$$C_{\text{SYM}} \geq \frac{1}{4} \log^+(\text{INR}) - \frac{c}{2} - 3$$

for all values of $1 \leq g^2 < \text{SNR}$ except for an outage set whose measure is a fraction of $2^{-c}$ of the interval $1 \leq |g| < \sqrt{\text{SNR}}$, for any $c > 0$.

- Outage set approach appeared first in Niesen and Maddah-Ali 11 (next talk)
- The outage set phenomena seems inherent to the problem (Etkin and E. Ordentlich 09).
Weak interference regime: Lattice Han-Kobayshi

- Similar approach works for the weak interference regime.
- Just choose public and private codewords from lattice codebooks.
- Decoding is done using compute-and-forward.
- Achievable rate is the solution to integer least-squares optimization problem.
- Can be shown to be within a constant gap from outer bound (except for an outage set).
Summary: new inner bounds

- New inner bound for strong interference regime.
  - Constant gap from outer bound except for outage set.

### Graph

3-user IC @ SNR=50dB

- Symmetric rate [bits/channel use]
- Log scale on the x-axis
- Log scale on the y-axis

Graph shows the comparison between various bounds and the symmetric rate for different values of $g$. The graph illustrates the constant gap from the outer bound except for the outage set.
New inner bound for moderately weak interference regime.
- Constant gap from outer bound except for outage set.
Summary: new inner bounds

- New inner bound for weak interference regime.
  - Constant gap from outer bound for all channel gains.

3-user IC @ SNR=50dB

Symmetric rate [bits/channel use]

- Approx. Sum Capacity of the Symmetric Gaussian K-User IC