Mixing between fresh and salt waters at aquifer regional scale and identification of transverse dispersivity

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S U M M A R Y  
In aquifers in which freshwater flows above saltwater, a mixing layer develops between the two water bodies. In a typical regional aquifer, this mixing layer is thin compared to the length scale of the aquifer. Its modeling by available numerical codes is impractical due to the needed fine discretization. Here, an approximate model of the mixing layer in steady state 3D flow is developed, based on the boundary layer approach. At first, mixing is neglected and a sharp interface solution is derived. Subsequently, the flow and mixing equations are rewritten in a curvilinear coordinates system, attached to the sharp interface solution. In line with the boundary layer approximation, only transverse dispersion is considered. A simplified solution for the mixing layer is obtained by assuming similarity and using von Karman integral method. The approach is demonstrated for Yarkon–Taninim basin (Israel), a Karstic aquifer extending over 6000 km². The main aim of the research was to identify the regional scale transverse dispersivity for the aquifer. The determined value was $\zeta_T \approx 0.04$ m. This is an important finding, as it is the first time the parameter is evaluated for an aquifer at regional scale.

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I ntroduction and background  
Modeling of flow of fresh and salt waters in aquifers is a topic of considerable theoretical and practical interest. It is relevant to coastal aquifers, brine bodies in deep aquifers and petroleum reservoirs. Recent applications include CO₂ storage and disposal of brines from desalination plants. We consider here stable configurations in which the denser fluid underlies the lighter ones. Early studies adopted a sharp interface approximation to solve the problem and analytical solutions were obtained for a few cases of homogeneous media and simple boundary conditions. With the development of electronic computers, versatile codes have been developed, permitting solution of complex problems. These codes solve the variable density equations for miscible fluids, leading to the existence of a mixing zone between the two fluid bodies; it is common to designate in such cases the surface of median concentration as “interface”. One of the parameters whose value is needed in order to apply such codes is the pore-scale dispersivity and primarily the transverse component $\zeta_T$, which controls the thickness of the transition zone and the rate of entrainment of the denser fluid in steady or quasi-steady regimes. The values of $\zeta_T$ are not well known, as its identification under field conditions can be carried out only indirectly. A survey of published studies of numerical solutions carried out in the past, revealed that values adopted by modelers were quite arbitrarily large, sometimes dictated by the need to override the ubiquitous numerical dispersion, caused by the coarse grids generally used in numerical codes.

Our work on the subject was motivated by the study of the Yarkon–Taninim aquifer in Israel (Paster et al., 2006). This deep and thick aquifer, extending over an area of around 6000 km² is an important source of freshwater. At its North-West edge the flowing freshwater is underlain by a body of salt water (at concentration close to that of the Mediterranean Sea). Recently, exploratory wells were drilled in the interface zone and it was found the transition layer is surprisingly thin (see Paster et al., 2006), considering the heterogeneous aquifer structure and the large transport scale (of order of tens of km). Recent analysis of tracer transport in field experiments at the Borden Site and Cape Cod aquifers (Fiori and Dagan, 1999) led to identified $\zeta_T$ of 0.5 mm. Though these aquifers are relatively homogeneous and transport occurred at a scale of tens of meters, again $\zeta_T$ is considerably smaller that values usually adopted in the literature.

Since one of our aims was to use the data available for the Yarkon–Taninim aquifer in order to identify the $\zeta_T$ value for a regional scale flow, we needed a model for the mixing layer appropriate to this application. Existing numerical variable density codes are...
extremely complex on one hand, especially in the case of such large scale formations, and are not applicable to flows with thin mixing zones. Thus, we have devoted a few years in order to develop approximate models based on the boundary layer (BL) approach, which was conceived for similar problems in fluid mechanics. The results are summarized in a few papers (Paster and Dagan, 2007, 2008a,b) and the basic idea is to start with a sharp interface model as a first approximation and to develop subsequently a relatively simple BL solution which renders the thickness of the transition zone, the density profile and the rate of entrainment, depending on $\chi$. The aim of the present study is to extend the procedure and apply it to the Yarkon–Taninim aquifer. In the historic period (before 1950), preceding the heavy pumping of the aquifer, flow was steady and the freshwater outlet in the Northern part was the Taninim springs. Data are available on the freshwater discharge and its salinity, i.e. the salt water discharge, prevailing at the time. Using a flow model for the sharp interface approximation and our BL approach we were able to identify the value of the transverse pore-scale dispersivity; as shown in the sequel, the result is $\chi \approx 0.04$ m. We believe this is an important finding, as to the best of our knowledge it is the first time this important parameter was identified for regional scale flow in a heterogeneous aquifer.

In the first stage, of solving flow with a sharp interface by a numerical procedure, we account for the spatial variability of transmissivity by assigning to each element the value identified in previous studies of the freshwater flow in the Yarkon–Taninim aquifer. Variability of hydraulic conductivity at a smaller scale (which is not known) is not considered here. Its presence causes the interface to fluctuate around its mean position and this spreading effect is captured by a macrodispersion coefficient (see for example an analysis of flow in stratified media in Dagan and Zeitoun (1998)). Spreading in itself does not result in mixing across the interface, which is due to transverse pore-scale dispersion, which in turn is responsible for the creation of the boundary layer. The rate of entrainment of salt water, whose computation is the main objective of the present study, depends essentially on the length of streamlines along the interface between the toe and the outlet and the freshwater velocity in their neighborhood. We solve for these without accounting for the aforementioned spreading effect, which is assumed to have a minor effect in view of the large distance between the toe and the Taninim springs.

The plan of the paper is as follows: In ‘Statement of the problem’ the problem at hand is stated and the governing equations are defined. In ‘Zero-order (sharp interface) solution’ the sharp interface solution is discussed, and a first-order correction of Dupuit assumption is presented. The BL approach is developed in ‘Boundary layer approximation for the mixing layer’ and applied to the Yarkon–Taninim aquifer in ‘Application to Yarkon–Taninim aquifer’.

**Statement of the problem**

We consider a regional aquifer, with stable flow of freshwater above a body of saltwater. The location of the lower boundary (bottom) and the upper boundary (top) of the aquifer are given by $z = z_{\text{top}}(x,y)$ and $z = z_{\text{bot}}(x,y)$, respectively, where $z$ is a vertical coordinate and $(x,y) \in \Omega$ (Fig. 1). The domain $\Omega$ is the projection of the aquifer area over the horizontal $(x,y)$ plane. It is assumed that the extent of the aquifer, i.e. $\Omega$, and the stratigraphic data, i.e. $z_{\text{bot}}$ and $z_{\text{top}}$, are known. The bottom is assumed to be impervious.

The flow in the aquifer considered here is confined in some areas and phreatic in the others. The freshwater is replenished by either natural or artificial recharge, and the outlets are pumping wells and springs. The distribution of the net recharge is assumed to be known.

The freshwater flows above a body of saltwater, and a mixing layer, i.e. a transition zone, develops between the two bodies. This layer is thin relative to the aquifer thickness and changes in density take place only in the layer, between the freshwater above it and the saltwater below.

The following assumptions, common to problems involving the mixture of freshwater and seawater, are adopted (Paster and Dagan, 2008a): (i) the density variations are neglected in the equation of the mass conservation of the fluid, i.e. Boussinesq assumption; (ii) the state equation is linear, i.e. the density and the salt concentration are linearly related; and (iii) the dispersion coefficient is that of a tracer.

In this study, only steady state problems are considered. Although this is a limitation, the problem at hand is still an important one and has some applications, e.g. regional aquifers where flow and transport change slowly over time. It is also a first step toward solving more complex, unsteady, problems.

The governing equations are momentum balance (Darcy’s law), conservation of fluid mass, and conservation of salt mass. These are given by (see, e.g. Bear, 1972)

\[ \mathbf{q} = -K_i(\nabla \phi + (\rho / \rho_i - 1) \mathbf{k}) \]  
\[ \nabla \cdot \mathbf{q} = 0 \]  
\[ \nabla \cdot \mathbf{p} = \nabla \cdot \mathbf{F} \]  

where $\mathbf{q}$ is the fluid flux (specific discharge), $K_i = \kappa \rho g / \mu$ is the hydraulic conductivity of the medium filled with freshwater, $\kappa$ is the permeability, $\rho_i$ is the density of freshwater, $\mathbf{g}$ is the gravity acceleration vector, $\mu$ is the coefficient of viscosity, the pressure head $\phi = p / (\rho g) + z$ is the commonly used freshwater equivalent head (the height of the freshwater column in a piezometer implanted at the point of elevation $z$), $p$ is the pressure, $\rho$ is the density, $\mathbf{k}$ is a unit vector pointing upward, and $\mathbf{F}$ is the dispersive flux. The components of the latter are

\[ F_i = n(D_{ij} \delta_{ij} + D_i) \frac{\partial \rho}{\partial x_i} \]  

where $i,j \in \{1,2,3\}$, $n$ is the porosity, $D_i$ is the effective coefficient of molecular diffusion, $D_{ij}$ is the tensor of pore-scale dispersion, $\mathbf{x}$ is a Cartesian coordinates vector, and $\delta_{ij}$ is the Kronecker unit tensor (the index summation rule is adopted). As in the case of a tracer, $D_i$ is given by

\[ nD_{ij} = 2\tau_x \delta_{ij} + (2\tau_t - 2\tau_x) \frac{q_{ji}}{q} \]  

where $\tau_x$ and $\tau_t$ are the transverse and longitudinal dispersivities, respectively, and $q = |\mathbf{q}|$. Eqs. (1)–(5) form a closed system for the dependent variables $\mathbf{q}$, $\phi$ and $\rho$, functions of $\mathbf{x}$.

Next, it is assumed that the effect of molecular diffusion is small in comparison to the mechanical dispersion, and can be neglected (see discussion at Paster and Dagan (2007), Section 4.2). Furthermore, it is assumed that transverse dispersivity can be regarded a constant, i.e. $\tau_x$ represents its average value within the mixing layer.

**Zero-order (sharp interface) solution**

**Dupuit solution**

In many regional aquifers, where freshwater flows above a saltwater body, the thickness of the mixing layer is small compared to the characteristic length scale (e.g. the horizontal extent of the interface zone). Then, the problem can be modeled at zero-order as one of sharp interface flow. Furthermore, in modeling of regional
aquifers, it is common to assume that the flow is shallow and essentially horizontal, i.e. to adopt Dupuit assumption (see, e.g. Bakker, 2003). In addition, the hydraulic conductivity, equal to the transmissivity divided by depth, is also varying in the horizontal plane solely, i.e. $K_z = K(x,y)$, such that the problem becomes a 2D one. Furthermore, for steady state, flow in the saltwater body is typically caused by entrainment of saltwater into the mixing layer, and the magnitude of the specific discharge is usually negligible compared to that of the freshwater. Hence, the saltwater body may be assumed to be in rest in the zero-order (sharp interface) solution. Still, the flow in the salt water body caused by the entrainment in the mixing zone can be modeled as a second-order effect of the boundary layer approximation (Van Duijn and Peletier, 1992; Paster and Dagan, 2007) by solving Laplace equation, provided that the boundary conditions at the contact with the sea are known. This small effect was modeled by Paster and Dagan (2007) for the 2D sea water intrusion problem and is not considered in the present work.

Thus, the zero-order (sharp interface) solution fulfills

$$
\begin{align*}
\rho_0 &= \rho_f, \quad q_{0s} = -K_z \nabla h_{0s} \quad (z > 2z_R) \\
\rho_0 &= \rho_s, \quad q_{0s} = 0.
\end{align*}
$$

where the subscript 0 denotes the zero-order solution, $q_{0s}(x,y)$ and $q_{0s}(z)$ are the head in the freshwater and in the saltwater bodies, respectively, $\nabla z = (\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})$ denotes the horizontal component of $\nabla z_{int}(x,y)$ is the elevation of the sharp interface, $\delta = (\rho_s - \rho_f) / \rho_f$ is the dimensionless density difference and $h_0 = (\rho_f g + z)$ is the water level in a piezometer filled with saltwater which taps the saltwater body. Hence, in the case of a continuous saltwater body considered here, $h_0$ is constant.

Pressure continuity across the sharp interface leads to the Ghyben–Herzberg formula (Bear, 1972, pp. 564)

$$
\frac{h_0}{h_0} = \frac{\rho_f - \rho_s}{\rho_f} \quad (z < 2z_R)
$$

in regions of interface flow.

It is emphasized that in reality, the entrainment of salt from the saltwater body due to mixing must be compensated by a small and diminishing of the saltwater body.

The vertical integration of Eq. (6) leads to the nonlinear equation for $q_{0s}$ (Bear, 1972, pp. 374–378)

$$
\nabla h \cdot (-K_z \nabla q_{0s}) = R
$$

where $\nabla h = (\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y})$ is the freshwater body thickness (Fig. 2), $z_R$ and $z_1$ are the elevations of the upper and lower boundary of the freshwater zone, respectively, and $R(x,y)$ is a source term, i.e. net recharge. In the unconfined portion of the aquifer, $z_R = z_{up}$, whereas in the confined portion, $z_R = z_{np}$. The lower boundary $z_1$ is given by $z_1 = z_{np}$ in the zones of the aquifer where the flow of freshwater is above an interface, and $z_1 = z_{lo}$ otherwise (in the freshwater only zone). Then, Eq. (8) can also be written as

$$
\nabla h \cdot (-T_i \nabla q_{0s}) = R \quad \text{i.e.} \quad \nabla h \cdot \bf{Q_{0s}} = R
$$

where $T_i(x,y) = K_z / L$ is the transmissivity of the freshwater zone of the aquifer and $\bf{Q_{0s}} = q_{0s} \bf{z}$ is the discharge vector, i.e. the vertically integrated specific discharge (e.g. Strack, 1976). The common practice of calibration of a groundwater (freshwater) numerical flow models involves discretization of either $K_z$ or $T_i$ over a 2D numerical grid, in order to match the calculated values of freshwater heads with measured ones, for instance. In this paper it is assumed that such a calibration was conducted and the value of $\zeta$, $q_{0s}$ and $\bf{Q_{0s}}$ are known over a selected numerical grid in the aquifer.

First-order correction to Dupuit solution

To model the mixing layer by a BL approach it is convenient to map the flow domain onto a curvilinear coordinates system that is defined with the aid of a 3D flux field (Paster and Dagan, 2008a). Toward this aim, while maintaining the solutions for $Q_{0s}$ and $\zeta$, the Dupuit based horizontal component $q_{0h} = q_{0s}/\zeta$ is supplemented by a vertical one $q_0$ (see, e.g. Nordbotten and Celia, 2006). For the case of no recharge in the interface zone considered here (i.e. $R = 0$ in Eq. (9)), a streamfunction $\psi(x,y)$ can be defined for the field $Q_{0h}$, such that $Q_{0h} = -\partial \psi / \partial x$.

Rewriting the continuity Eq. (2) as

$$
\nabla h \cdot \frac{\partial q_0}{\partial z} = 0
$$

and using (9) leads to

$$
\nabla h \cdot q_{0h} = \nabla h \cdot \left( \frac{Q_{0h}}{\zeta} \right) = \nabla h \cdot \left( \frac{Q_{0h}}{\zeta} \right) = -\frac{1}{\zeta} Q_{0h} \cdot \nabla^2 \psi
$$

by integration in (10).

This correction leads to an approximate three-dimensional flux field $\bf{q_0}$ ($\bf{q_{0s}}$, $\bf{q_{0h}}$, $\bf{q_0}$), which satisfies exactly the continuity equation, but only approximately Darcy’s law in the vertical direction. Furthermore, $q_{0h}$ is tangent to the upper boundary of the freshwater zone since by (12) $q_{0s}/q_{0h} = \partial z_1 / \partial s$ at the interface $z = z_1$, which leads to $q_{0h} = q_{0s} \frac{\partial z_1}{\partial s} (z - z_1) + q_0 \frac{\partial z_1}{\partial s}$

Boundary layer approximation for the mixing layer

This study models the thin mixing layer that develops at the interface between freshwater and saltwater along the lines of Paster and Dagan (2008a). There, a confined aquifer of constant conductivity and a horizontal, impervious top and bottom were considered. In the present work, the limitation of the existence of confined top removed and the conductivity and the elevation of the top and bottom are varying with $x$ and $y$, as discussed above. Following the procedure of (Paster and Dagan, 2008a) we render the variables dimensionless with respect to a flux $U$ and a length scale $L$ (e.g. the constant conductivity and the thickness of the aquifer in Paster and Dagan, 2008a). The flow equations (Eqs. (1)–(5)), rewritten in terms of dimensionless variables, are given in Appendix A.

The procedure for deriving the boundary layer equations for the more general case discussed here, follows the procedure of Paster and Dagan (2008a). First, a curvilinear coordinates system is derived for the area above the sharp interface (Appendix B). Then, the governing dimensionless equations are written in terms of these coordinates (Appendix C). Following Cole (1968), the fluxes are scaled by $q_0$ and considered as independent variables. Next, inner, BL variables are defined and the BL equations are obtained. These equations are practically the same expressions as Eqs. (53) and (54) in Paster and Dagan (2008a). Then, von Karman integral
approach is adopted, and a similar solution is assumed. The resulting expressions are similar to the ones obtained in Section 6 in Paster and Dagan (2008a), and the final result for the total entrainment of saltwater along a certain area of the interface defined by $\psi_{11} < \psi < \psi_{12}$, $0 < s' < s$ is

$$Q_s(s) = \int \int q_s dA = 2s_2^{1/2} \beta \int_{\psi_{11}}^{\psi_{12}} [K(s, \psi_s)]^{1/2} d\psi_s$$

(13)

where

$$K(s, \psi_s) = \int_{\psi_{11}}^{\psi_{12}} \frac{1}{\xi} ds'$$

(14)

The constant $\beta$ is determined by the choice of the similar solution for the velocity in the BL. Following Paster and Dagan (2008a), a fourth-order polynomial is adopted as the similar solution. This choice leads to $\beta = 0.343$.

Another result is the expression for the thickness of the BL, given for the adopted similar solution by

$$d\zeta \approx 5.84 \times s_2^{1/2} \xi [K(s, \psi_s)]^{1/2}$$

(15)

Hence, the procedure to determine the salt water discharge accumulated over an area of the interface and the BL thickness, consists of the following steps:

- Solving the flow problem and determining the shape of the sharp interface,
- Deriving the streamfunction $\psi_s$,
- Depicting the streamlines and the dependence of $\zeta(s, \psi_s)$ upon the curvilinear coordinate $s$ for a set of streamlines,
- Numerical integration in (14) from the toe downstream to derive the auxiliary function $K$, and
- Integration across streamlines in (13) to obtain the total salt water discharge entrained in the BL on a portion of the interface zone.

**Application to Yarkon–Taninim aquifer**

**General**

The main aim of the paper is to apply the approach to the Yarkon–Taninim formation (Israel), a regional aquifer which extends over 6000 km$^2$ and serves as a major source of freshwater of Israel (for a comprehensive review, see Weinberger et al., 1994). In the Northern part of the aquifer, the freshwater flows above a body of saltwater, over an area of approximately 600 km$^2$. The Cl content in the saltwater body is 19 g/l, slightly lower than the value measured in the Mediterranean Sea (22 g/l). A few recent exploratory wells crossed the transition zone between freshwater and saltwater and found that the thickness of the transition zone is 5–20 m. Thus, the transition zone is thin compared to the thickness of the aquifer of 600–1000 m (Paster et al., 2006).

Until 1950 the pumping from the aquifer was negligible. The northern part of the aquifer was drained by Taninim springs where 100 Mm$^3$ of brackish water were discharged. The average Cl content of the water was around 900–950 mg/l. According to the chemical and isotopic composition, it was a mixture of freshwater and a salty end-member, with salinity close to that of seawater (Schilman and Almogi-Labin, 2003). Thus, the fraction of saltwater end-member in the discharged water is approximately 3.5% (Guttman and Zukerman, 1995). According to the conceptual model developed in Paster et al. (2006) this end-member is originating from the saltwater body and is flushed by the freshwater flow and discharged into the springs. Also, the flow in the saltwater body, resulting in the sea water component of the Taninim springs is assumed to be of significantly smaller magnitude than of the flow in the freshwater body. Hence, the saltwater flow can be assumed to be primarily caused by the entrainment into the mixing layer. Thus, the steady state BL approach can serve to model the flushing during the quasi-steady period before 1950. The aim of this study is to use the BL solution for quantifying the salt water entrainment $Q_s$ in order to identify the field scale average value of the transverse dispersivity $\xi_2$, by matching the calculated and measured $Q_s$.

![Fig. 1. Sharp-interface flow solution in the Northern part of Yarkon–Taninim aquifer.](image-url)
Details of the sharp interface model

The boundary of the aquifer (see Fig. 1), the conductivity of the aquifer, the recharge distribution and location were adopted from a finite-differences freshwater flow model of the aquifer developed in the past (Guttman and Zukerman, 1995). The conductivity and recharge distributions in that model were determined as discrete values at the finite-difference grid cells. The elevations of the top and bottom of the aquifer, at a grid resolution of 250 m × 250 m, were obtained from the Geophysical Institute of Israel (Ben Gai et al., 2007). A no-flow boundary condition was set along the boundary of the aquifer, except a segment in the southern part, where a fixed head was stipulated. This head was determined as 𝜽eff = 25 m, following historical measurements and the model calibration of Guttman and Zukerman (1995). During the historical regime the head was practically not changing in time.

The freshwater head throughout the aquifer was derived by solving numerically Eq. (8) with the aid of a 2D finite elements solver (Matlab’s PDE Toolbox). The conductivity and recharge distributions were interpolated to the triangular cells of the model. As a first guess, the thickness of the freshwater body was assumed to be constant and equal to 700 m over the area of the aquifer. Then, the elevation of the interface ℎint was computed by Eq. (7) with the constant equivalent head of the saltwater body, given by ℎint = −5.6 m (Paster et al., 2006). The freshwater thickness 𝜺 was determined subsequently over the area of the aquifer as explained in ‘Dupuit solution’. As a next step, this value was used in order to solve Eq. (8) again, and the process of solving Eq. (8) and updating 𝜺 was repeated iteratively until convergence was achieved. An additional step of calibration of the conductivity was carried out by requiring the head near Taninim springs, to be equal to the mean observed value, ℎTaninim = 3 m, (Blake and Goldschmidt, 1947). In order to fulfill this requirement, the conductivity was multiplied by a constant close to unity (indicated by ‘factor’ in Table 1).

The sharp interface model results are depicted in Figs. 1–4. Thus, the freshwater flow pattern in the Northern part of the aquifer is illustrated in Fig. 1. A representative W–E cross-section is depicted in Fig. 2. Typical cross-sections along two streamlines are shown in Figs. 3a and 4a. It is seen that in a certain region in the N–W part of the aquifer, the top is so low that there is no freshwater flow there, i.e. 𝜺 = 0, and the saltwater body reaches the top (Fig. 2). The thickness of the freshwater body neighboring this zone from the right is small and the basic assumptions of the BL approach are not valid there. On the other hand, the discharge 𝐒eff is also small and it can be safely assumed that the contribution of this region to the total entrainment of saltwater to the mixing layer is negligible.

Results for the mixing layer

The salt water discharge along the various streamlines was determined by numerical quadratures (Eqs. (13) and (14)), starting from the toe and ending at the Taninim springs outlet zone. The

<table>
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<tr>
<th>Analysis #</th>
<th>Parameter</th>
<th>Original value</th>
<th>New value</th>
<th>𝜺 (m)</th>
<th>Factor</th>
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<tr>
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<td>GII data</td>
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<td>0.94</td>
</tr>
<tr>
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<td>GII data − 50 m</td>
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<tr>
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</tr>
</tbody>
</table>

Fig. 2. A typical W–E cross-section of the Northern part of the aquifer (A–A’, see Fig. 1). The aquifer is confined except at the Eastern part. The saltwater body fills the aquifer in the Western part of the cross-section.
relationship \(d\psi_s = Q_{0f} \, dn\) was used, where \(n\) is a curvilinear coordinate normal to lines of constant \(w_s\) (see Eq. (B.4)). Then, by (13)

\[
\alpha^2 = \frac{Q_{0f} \tan}{2\mu \int K_{\text{tan}}(\psi_s) \, \frac{1}{2} \, Q_{0f} \, dn} \tag{16}
\]

Here, \(Q_{0f} \tan = 3.5 \times 10^6 \, \text{m}^3/\text{y}\) is the known component of salt-water discharge in Taninim springs resulting from the measured total discharge and salinity (Paster et al., 2006), whereas \(K_{\text{tan}}(\psi_s)\) is the computed value of \(K(s, \psi_s)\) at the springs. The integration \(\int \) denotes integration over all the streamlines that reach the springs. Since there are neither sources nor sinks in the sharp interface area (except Taninim springs), the value of \(Q_{0f} \, dn\) between adjacent streamlines can be computed along an arbitrary curve in the \((x, y)\) plane that connects these streamlines in the sharp interface zone. Hence, the value of the integral in (16) was approximated by

\[
\sum_{i} [K_{\text{tan}}(\psi_{s_i})]^{1/2} (Q_{0f} \, dn)_i
\]

where \(i\) denotes a streamline number \((i = 1, \ldots, 40\) in our case), and \((Q_{0f} \, dn)_i\) is calculated by averaging \(Q_{0f} \, dn\) between streamlines \(i - 1\) and \(i + 1\). The values of \([K_{\text{tan}}(\psi_{s_i})]^{1/2} (Q_{0f} \, dn)_i\) are represented in Fig. 6. It is emphasized that along some of the streamlines the interface is located very close to the top of the aquifer (e.g. streamline no. 1), thus leading to \(\zeta \to 0\) and \(K_{\text{tan}}(\psi_s) \to \infty\), while \((Q_{0f} \, dn)_i \to 0\), and consequently the contribution of this streamline was neglected in the summation (17).

The central result of the analysis is the calibrated value in (16)

\[
\alpha_T \simeq 0.04 \, \text{m}
\]

Using this value, it is now possible to calculate the thickness of the mixing layer by (15). The results along some of the streamlines are depicted in Figs. 3d and 4d. It is seen that in an area close to the springs outlet, the ratio between the thickness of the mixing layer

\[
\begin{align*}
\text{Fig. 3.} & \text{ Streamsurface #5 (the projection of the streamsurface on the } (x, y) \text{ plane is shown in Fig. 1). (a) Cross-section along the streamsurface. (b) Thickness of freshwater, } \zeta. \\
& \text{(c) The result of integration of (14) along the streamline, } K. \text{ (d) The thickness of the mixing layer, } \Delta. \text{ (e) The ratio of } \Delta \text{ and } \zeta. \\
\end{align*}
\]

\[
\begin{align*}
\text{Fig. 4.} & \text{ Similar to Fig. 3, for streamline #25.} \\
\end{align*}
\]
and the total thickness, $A$, is reaching values of 0.2–0.4 (Figs. 3e and 4e). These relatively high values may be beyond the range of validity of the BL approximation. However, this area has a small contribution to the total saltwater entrainment compared to the contribution from the total area and can therefore be neglected.

**Sensitivity analysis**

A sensitivity analysis was conducted in order to verify the robustness of the model. The solution of the flow problem and computation of $x_T$ was repeated with a change in a key model parameter each time (see Table 1). These parameters are: (i) the saltwater body head, (ii) the conductivity distribution, (iii) the total discharge at Taninim springs, (iv) the density difference, (v) the location of the top and bottom of the aquifer, (vi) the total discharge at Taninim springs, (vii) the assumed head near Taninim springs. The new value of each of the changed variables was chosen to reflect a reasonable error of the variable and the results are given in Table 1. It is observed that $x_T$ is dependent upon the various variables in a complex manner. For example, when the head of the saltwater body is lowered by 1 m, the sharp interface model yields a lower interface, extending over a smaller area. Thus, it is expected that $K_T$ will become smaller and $x_T$ will increase, as indeed is the case. However, the assumption of constant conductivity all over the aquifer is nonrealistic and it may lead to zones of relatively high freshwater velocity. Nevertheless, the variation of $x_T$ is rather limited, except for case 3. Thus, it can be concluded that the model is quite robust under this test. Hence, the calculated value of $x_T = 0.04$ m can be regarded as representative for Yarkon–Taninim aquifer.

**Summary and conclusions**

In the present study the previous work by Paster and Dagan (2008a) was extended and applied on modeling steady mixing in a variable density aquifer flow. The 3D simplified BL solution was generalized to a 3D problem of a regional aquifer of an impervious bottom and spatially variable conductivity. Assuming that the mixing layer is thin compared to the typical length scale of the aquifer, the sharp interface solution was adopted as a first approximation of the problem. The interface flow was solved by adopting Darcy’s assumption. Then, a pseudo-orthogonal curvilinear coordinate system was developed along the streamsurfaces of the sharp interface solution. Expressions for the saltwater entrainment into the mixing zone, and the thickness of the mixing zone, were obtained by using the BL approach and the von Karman integral method, with an assumed similarity profile for the specific discharge distribution in the mixing zone.

The approach was applied to the Yarkon–Taninim aquifer, where a thin mixing layer separates a deep saltwater body from the overlying flowing freshwater. Due to the small thickness of the mixing layer on one hand, and the large extent of the aquifer on the other hand, the BL approach is applicable for modeling the mixing problem in the quasi-steady period before 1950, prior to the beginning of the intensive pumping of the aquifer. The outlet of the Northern part of the aquifer are the saline Taninim springs, where the total amount of discharged saltwater was measured in that period. The unknown transverse dispersivity was determined by integration of the entrainment in the springs capture zone and by equating the calculated and measured salt water discharges. The result was $x_T = 0.04$ m, and a sensitivity analysis showed this value to be quite robust. The thickness of the mixing layer was computed as well, based on this value of $x_T$.

This finding is important for applications, since to the best of our knowledge this is the first time that $x_T$ is identified at a regional aquifer scale. The value is surprisingly low in view of the fractured and heterogeneous nature of the Yarkon–Taninim aquifer. However, it is in line with other recent findings (Fiori and Dagan, 1999) of field scale values of $x_T$. Previous studies, which used numerical density driven groundwater models for modeling regional aquifers, assumed unrealistic values of $x_T$, larger by 1–2 orders of magnitude than the value identified here.

Future studies may assess the accuracy of these results by comparison with numerical density driven groundwater models. Also, further research may extend the approach to time-dependent problems. Furthermore, the present approach may be extended to cases of saltwater which is not stagnant in the zero-order (sharp interface) approximation, e.g. when significant pumping of the saltwater occurs.

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**Appendix A. Governing equations in terms of dimensionless variables**

In the present study, we assume that the dispersivities $x_T$ and $x_L$ are constant. Subsequently, we define $U$, a representative velocity of the problem (e.g. a characteristic $K_T$ value) and $L$, a representative length of the problem (e.g. the average thickness of the aquifer). We introduce the dimensionless variables

$$\begin{align*}
x' &= x/L, \quad q' = q/U, \quad F = F/[U(\rho_k - \rho_L)], \\
\phi' &= \phi/L, \quad K' = K/U, \quad \rho' = (\rho - \rho_L)/(\rho_k - \rho_L), \\
\varepsilon &= Pe^{-1} = x_T/L, \quad \lambda = x_L/x_T, \quad D_0 = nD_0/[UL] \quad (A.1)
\end{align*}$$

where $Pe$ denotes Peclet number, describing the ratio of transverse dispersion and advection. The substitution of (A.1) in (1)–(4) leads to the following dimensionless equations

$$\begin{align*}
f' &= -K' \nabla \phi' + \delta \rho' \nabla z' \quad (A.2) \\
\nabla \cdot q' &= 0 \quad (A.3) \\
q' \cdot \nabla \rho' &= \nabla \cdot \left[ (D_0 + \varepsilon q') \nabla \rho' + \varepsilon (\lambda - 1) \frac{q'}{q} (q' \cdot \nabla \rho') \right] \quad (A.4)
\end{align*}$$

Unlike the case of a tracer, the solution of the specific discharge $q'$ cannot be separated from that of the density, due to the presence of $\rho'$ in (A.2). Furthermore, the system is nonlinear, due to both advective and dispersive fluxes in (A.4). The parameters $D_0$ and $\varepsilon$ are typically much smaller than unity, calling for a boundary layer solution of this system.

**Appendix B. Development of the curvilinear coordinates system above the sharp interface**

Along the same lines of Paster and Dagan (2008a), the flow domain is mapped over a curvilinear coordinates system. This mapping is obtained by defining a streamfunction $\psi'$ and a pseudo-potential $\phi'$ along a streamsurface of $\psi'(x', y', z') = \psi_U/(UL) = \text{const}$, in terms of non dimensional variables,
The mapping onto \((\phi', \psi')\) plane of the boxed area in Fig. 3a. Orthogonality prevails exactly at the sharp interface \((z = z_1, \psi' = 0)\) and at the aquifer top \((z = z_2, \psi' = 1)\), and approximately elsewhere. The interface toe is mapped on \(\psi' = 0\).

\[
\psi'(s', z', \psi') = \frac{z' - z_1}{z' - z_2} \quad \text{(B.1)}
\]

\[
\phi'(s', z, \psi) = \left(\frac{z - z_1}{z - z_2}\right)^2 \frac{\partial \phi}{\partial \psi} \frac{\partial z}{\partial \psi} + \frac{z - z_1}{z - z_2} \frac{\partial \phi}{\partial z} \frac{\partial z'}{\partial \psi} \frac{\partial z}{\partial z'} + \frac{o}{\partial z} \left(1 + \left(\frac{\partial \phi}{\partial z}\right)^2\right) ds'' \quad \text{(B.2)}
\]

These definitions are obtained in a similar manner to the procedure described in Paster and Dagan (2008a, Appendix B), and the basic properties of these variables are similar. First, \(\psi' = 0\) and \(\psi'' = 1\) at the lower and upper boundary of the freshwater domain, respectively, while \(\partial \phi'/\partial z' = -q_{\text{in}}/Q_0\) and \(\partial \psi'/\partial z' = q_{\text{in}}/Q_0\) everywhere. The non-dimensional variables are \(q_{\text{in}} = q_{\text{in}}/U, q_{\text{in}} = q_{\text{in}}/U\) and \(Q_0 = Q_{\text{in}}/(UL)\). Lines \(\phi', \psi' = \text{const}\) are illustrated in Fig. 5 for one streamline.

The pseudo-potential \(\phi'\) satisfies \(\partial \phi'/\partial \psi' = q_{\text{in}}/Q_0\) and \(\partial \phi'/\partial z' = q_{\text{in}}/Q_0\) along the lower boundary of the freshwater \(z' = z_1\) and at the upper boundary \(z' = z_2\). It is seen therefore that lines \(\phi' = \text{const}\) and \(\psi' = \text{const}\) are orthogonal in the \((s', z')\) plane at \(z' = z_1\) and \(z' = z_2\), respectively, since \(\nabla \phi' \cdot \nabla \psi' = 0\) there. This property is illustrated in Fig. 5 which also shows that in the given example these lines are close to orthogonality everywhere.

Finally, the vector \(\nabla \psi' = (-q_{\text{in}}/Q_0, q_{\text{in}}/Q_0)\) is also orthogonal to \(\nabla \psi\) since the vector \(q_{\text{in}}\) is tangent to streamsurfaces \(\psi_{\text{in}} = \text{const}\). Thus, the functions \(\phi, \psi, \psi_{\text{in}}\) define a curvilinear orthogonal system in the neighborhood of the interface \(\psi' = 0\) and on the top \(\psi' = 0\), and almost orthogonal everywhere else (Fig. 5).

The metric coefficients for the curvilinear coordinates system are approximated, along the same line of Paster and Dagan (2008a, Appendix C) by

\[
h_1 = \frac{ds'}{d\phi'} = \frac{Q_{\text{in}}}{Q_0} \quad \text{(B.3)}
\]

\[
h_2 = \frac{dn'}{d\psi'} = 1 \quad \text{(B.4)}
\]

\[
h_3 = \frac{ds'}{d\psi'} = \frac{Q_{\text{in}}}{Q_0} \quad \text{(B.5)}
\]

where \(ds', dn', ds''\) are spatial infinitesimal displacements corresponding to infinitesimal increment of \(\phi', \psi', \psi''\), respectively (Batchelor, 1967, Appendix 2).

Appendix C. Rewriting the equations of flow and transport in the curvilinear coordinates system (see, e.g. Appendix 2 in Batchelor, 1967)

The components of the fluid flux in the \((\phi, \psi, \psi_{\text{in}})\) system are \(q_{\alpha}, q_{\beta}\), while \(q_{\gamma} = 0\) (note that in this appendix, the primes are omitted for the sake of simplicity). In the \((\phi, \psi, \psi_{\text{in}})\) system, the
continuity Eq. (A.3) becomes, with the use of the metric coefficients (B.3)--(B.5)

\[
\frac{\partial}{\partial \varphi} \left( \frac{q_0}{Q_{0\varphi}} \right) + \frac{\partial}{\partial \psi} \left( \frac{q_0}{Q_{0\psi}} \right) = 0
\]  
(C.1)

In a similar way, Darcy's law (A.2) leads to

\[
q_s = -K_r \left( \frac{q_0}{Q_{0S}} \frac{\partial \varphi}{Q_0} + \delta \rho \sin \theta_0 \right)
\]  
(C.2)

\[
q_s = -K_r \left( \frac{q_0}{Q_{0S}} \frac{\partial \psi}{Q_0} + \delta \rho \cos \theta_0 \right)
\]  
(C.3)

where \( \varphi \) is the head of the variable density fluid and \( \theta_0 \) is the angle between the tangent to a streamline and the horizontal direction in the sharp interface solution. The solution of the sharp interface problem is supposed to provide \( q_0, Q_0^s \) and \( \theta_0 \) as functions of \( (\varphi, \psi, \zeta) \) through the intermediate of \( x, y, z \).

Subsequently, \( \varphi \) is eliminated from Darcy's law in a similar way to the use of the \( \text{rot} \) operator in the 2D problem (Van Duijn and Peletier, 1992). Hence, division of (C.2) and (C.3) by \( K_r q_0/Q_{0S} \) and elimination of the derivatives of \( \varphi \) leads to

\[
\frac{\partial}{\partial \psi} \left( \frac{q_0}{K_r^s q_0} \frac{Q_{0s}}{Q_0} \right) - \frac{\partial}{\partial \varphi} \left( \frac{q_0}{K_r^s q_0} \frac{Q_{0s}}{Q_0} \right) + \frac{\partial}{\partial \psi} \left( \frac{\rho q_0}{Q_{0s}} \sin \theta_0 \right) = -\frac{\partial}{\partial \varphi} \left( \rho Q_{0s} \cos \theta_0 \right)
\]  
(C.4)

The transport Eq. (A.4) renders

\[
\frac{q_s}{Q_s} \frac{\partial \rho}{\partial \psi} + \frac{q_s}{Q_s} \frac{\partial \rho}{\partial \varphi} + \frac{\partial}{\partial \psi} \left( \frac{F_\psi}{Q_0} \right) + \frac{\partial}{\partial \varphi} \left( \frac{F_\varphi}{Q_0} \right) = \frac{\partial^2}{\partial \psi^2} \left( \frac{Q^s_{0s}}{Q_0} F_\psi \right) + \frac{\partial^2}{\partial \varphi^2} \left( \frac{Q^s_{0s}}{Q_0} F_\varphi \right)
\]  
(C.5)

The components of the solute flux, i.e. \( F_\varphi, F_\psi, F_{s\psi} \) are exactly the expressions given by Paster and Dagan (2008a, Eq. (C6)), with \( Q \) replaced by \( Q_{0s} \).

References


