On Coding Without Restrictions for the AWGN Channel
Gregory Poltyrev

Abstract—Many coded modulation constructions, such as lattice codes, are visualized as restricted subsets of an infinite constellation (IC) of points in the n-dimensional Euclidean space. We shall regard an IC as a code without restrictions employed for the AWGN channel. For an IC the concept of coding rate is meaningless and we shall use, instead of coding rate, the normalized logarithmic density (NLD). The maximum value \(C_\infty\) such that, for any NLD less than \(C_\infty\), it is possible to construct an IC with arbitrarily small decoding error probability, will be called the generalized capacity of the AWGN channel without restrictions. We derive exponential upper and lower bounds for the decoding error probability of an IC, expressed in terms of the NLD. The upper bound is obtained by means of a random coding method and it is very similar to the usual random coding bound for the AWGN channel. The exponents of these upper and lower bounds coincide for high values of the NLD, thereby enabling derivation of the generalized capacity of the AWGN channel without restrictions. It is also shown that the exponent of the random coding bound can be attained by linear IC’s (lattices), implying that lattices play the same role with respect to the AWGN channel as linear codes do with respect to a discrete symmetric channel.

Index Terms—Infinite constellation, lattice, error probability, random coding exponent, AWGN channel.

I. INTRODUCTION

The classical approach to coding for continuous channels involves some restrictions on the set of possible codewords. The most usual examples are codes with restricted average and/or peak power. For a given restriction on the set of codewords, one can calculate (at least in principle) the capacity of the channel in terms of the restriction. For many practically interesting models of continuous channels, the capacity without any restrictions is equal to infinity, thereby it is a meaningless feature of the channel.

Recently, several efficient codes for the channel with additive white Gaussian noise (AWGN) were constructed by means of coded modulation methods. Many coded modulation constructions, for example lattice codes [1]–[3], were obtained by restricting an initially infinite constellation (IC) of points in the \(n\)-dimensional Euclidean space. The volume and the shape of the resulting subset of points are determined by the restrictions, as well as by the complexity of realization of the coding and decoding algorithms. Different restrictions yield, for the same IC, different codes. Obviously, a good code can be obtained only from a good IC. Furthermore, the decoding error probability of a code is often estimated by means of the parameters of the IC from which that code is obtained [1]–[3]. Examples of such parameters are the packing radius (the minimum Euclidean distance), the \(\theta\)-series (for lattices), the volume and other characteristics of the Voronoi cell.

We shall regard an IC as a code without restrictions for the AWGN channel. For any IC and fixed intensity of the channel noise it is possible to calculate the error probability of maximum likelihood decoding (the exact definitions will be provided in the next section). In the case of codes of finite size, one considers a code as more efficient than another code of the same size and length (i.e., of the same coding rate) if the decoding error probability for the first code is less than that for the second code. But for an IC the concept of coding rate is meaningless. Nevertheless, we wish to have some characteristic of an IC which assumes, for the purpose of comparing efficiencies, the role of the coding rate in the case of finite constellations. Such a characteristic is the normalized logarithmic density (NLD) of the IC (see the next section). The maximum value \(C_\infty\) such that, for any NLD less than \(C_\infty\), it is possible to construct an IC with arbitrarily small decoding error probability, will be called the generalized capacity of the AWGN channel without restrictions.

In this paper we derive exponential upper and lower bounds for the decoding error probability of an IC in terms of its NLD. The upper bound is obtained by means of a random coding method, and it is very similar to the familiar random coding bound for the AWGN channel [4], [5]. The exponents of the upper and lower bounds coincide for high values of the NLD, thereby allowing us to derive the generalized capacity of the AWGN channel without restrictions. We show also that the exponent of the random coding bound can be attained by linear IC’s (lattices). We conclude from this fact that lattices play the same role in regard to the AWGN channel as linear codes do with respect to a discrete symmetric channel.

The outline of the paper is as follows. In Section II some important parameters of infinite constellations are presented and, subsequently, employed to express upper and lower bounds on the decoding error probability of a fixed constellation. The random coding bound on the error probability of an IC is derived in Section III. The ensemble of linear IC’s (lattices)
is investigated in Section IV. A discussion on the results is contained in Section V.

II. SOME FEATURES OF INFINITE CONSTELLATIONS

Any countable set \( S = \{ s_1, s_2, \ldots \} \) of points in the \( n \)-dimensional Euclidean space \( E_n \) will be called an infinite constellation (IC) of length \( n \). Let \( C_b(a) = \{ x : x = (x_1, x_2, \ldots, x_n), a/2 < x_i \leq a/2 \} \), a cube in the \( n \)-dimensional Euclidean space \( E_n \). Let \( M(S, a) = \| S \cap C_b(a) \| \), where \( \| \cdot \| \) stands for cardinality. The upper limit of the ratio of \( M(S, a) \) and the cube's volume

\[
\lim_{a \to \infty} \frac{M(S, a)}{a^n} = \gamma
\]

is called the density of \( S \). The normalized logarithmic density \( \delta \) of \( S \) is defined by \( \delta = n^{-1} \ln \gamma \).

Let \( V(r, s) \subseteq E_n \) be the \( n \)-dimensional sphere of radius \( r \) centered at the point \( s \). Denote \( V(r) = V(r, 0) \).

The largest number \( d_p \) such that \( V(d_p, s) \cap V(d_p, s_j) = \emptyset \) for all pairs \( i, j, i \neq j \), is called the packing radius of \( S \). The following upper limit

\[
\lim_{a \to \infty} \frac{M(S, a)}{a^n} = \gamma |V(d_p)| = \gamma_p \leq 1
\]

where \( |V(d_p)| \) is the volume of the sphere \( V(d_p) \), is called the packing density of \( S \).

The smallest number \( d_c \) such that \( \cup_{s \in S} V(d_c, s) = E_n \) is called the covering radius of \( S \). The upper limit

\[
\lim_{a \to \infty} \frac{M(S, a)}{a^n} = \gamma |V(d_c)| = \gamma_c \geq 1
\]

is called the covering density of \( S \).

Let us assume that some \( s \in S \) is transmitted through an AWGN channel. Denote the noise vector by \( z \). The conditional probability of decoding error, given that the codeword \( s \in S \) was transmitted, is given by

\[
\lambda(s) = \Pr\{ d(s, s+z) \geq d(s', s+z) \text{ for some } s' \in S, s' \neq s \}
\]

where \( d(x, y) \) is the Euclidean distance between \( x \) and \( y \). We shall call

\[
\lambda(S) = \lim_{a \to \infty} M^{-1}(S, a) \sum_{s \in C_b(a)} \lambda(s)
\]

the average probability of error for \( S \).

For a point \( s \in S \) the Voronoi polyhedron (Voronoi cell) \( W(s) \) is defined as follows: for any \( x \in W(s) \), \( d(x, s) \leq d(x, s') \) for all \( s' \in S \), \( s' \neq s \), and for any \( x \notin W(s) \) there is an \( s' \in S \), \( s' \neq s \) such that \( d(x, s') = d(x, s) \). Each side of the polyhedron \( W(s) \) is a part of the hyperplane which, for some \( s' \in S \), is orthogonal to the straight line \( (s, s') \) and intersects it at its middle. The points \( s' \neq s \) corresponding to the sides of \( W(s) \) are named "the face-defining neighbors" of \( s \).

Let \( d_c(s) \) be the smallest number such that \( W(s) \subseteq V(d_c(s), s) \). Obviously the covering radius of \( S \) is equal to

\[
d_c = \sup_{s \in S} d_c(s).
\]
Proposition 2: Let \( d_r^2(s) \) be the radius of the sphere whose volume is equal to the volume of the Voronoi region \( W(s) \), i.e., \( d_r^2(s) \) is the root of the following equation
\[
|V(r)| = |W(s)|.
\]
Then
\[
d_r(s) < d_r^2(s).
\]

Proof: The statement follows by the definitions of \( d_r \) and \( d_r^2 \).

We proceed now to derive lower and upper bounds on the conditional decoding error probability \( \lambda(s) \) in terms of \( d_r^2 \) and \( d_r^* \).

Proposition 3: For any IC employed for transmitting information through the AWGN channel, the conditional error probability \( \lambda(s) \) satisfies the following inequalities
\[
Pr(|z| > d_r^2(s)) < \lambda(s) \leq \sum_{s' : s' \in \Omega(s), w(s') \leq 2d_r^2(s)} Pr(z + s \in D(d_r^*(s), w(s'))
+ \Pr(|z| > d_r^*(s))
\]
where \( D(r, w) \) is the section of the sphere \( V(r, s) \) which is cut off by the hyperplane that orthogonally intersects a radius of \( V(r, s) \) at distance \( w/2 \) from the center, \( z \) is the noise vector, and \(|z|\) is its Euclidean norm.

Proof: The lower bound follows by (11), since the value of the probability density of the AWGN at any point of \( V(r) \) is larger than it is at any point outside \( V(r) \).

Now denote by \( \Omega(s) \) the set of all face-defining neighbors of \( s \). Then
\[
\lambda(s) \leq \sum_{s' : s' \in \Omega(s), w(s') \leq 2d_r(s)} Pr(z + s \in D(d_r^*(s), w(s'))
+ \Pr(|z| > d_r^*(s)) \leq \sum_{s' : s' \in \Omega(s), w(s') \leq 2d_r(s), s' \neq s} Pr(z + s \in D(d_r^*(s), w(s'))
+ \Pr(|z| > d_r^*(s)).
\]

It follows by Proposition 3 that only a finite number of neighboring points affect the value of the upper bound on the conditional probability \( \lambda(s) \).

For any fixed point \( s \in S \) we define the distance distribution \( \phi(s, \cdot) \), or spectrum, of \( S \) with respect to \( s \) as follows: \( \phi(s, w) \) is the number of points \( s' \in S \) such that \( d(s, s') = w \). Obviously \( \phi(s, \cdot) \) has nonzero values only on a countable set \( \Psi(s) \subset [0, \infty) \). We can rewrite the upper bound (13) in the following way
\[
\lambda(s) \leq \sum_{w \in \Psi(s), w > 2d(r^2(s))} \phi(s, w)
\cdot Pr(z + s \in D(d_r^*(s), w)) + \Pr(|z| > d_r^*(s)).
\]

Now define the average spectrum \( \bar{\phi}(S, \cdot) \) of \( S \) as follows
\[
\bar{\phi}(S, w) = \lim_{n \to \infty} M^{-1}(S, a) \sum_{s \in C(b(a))} \phi(s, w).
\]

The spectrum \( \bar{\phi}(S, w) \) also has nonzero values only on a countable set \( \Psi(S) \subset [0, \infty) \).

Let \( r = d_r^2 \) be the root of the following equation
\[
\sum_{w \in \Psi(S) \cap [0, 2d_r^2]} \bar{\phi}(S, w) \int_{\theta(w)}^{\pi/2} \sin^{n-2} \theta d\theta = \frac{\sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}
\]
where \( \theta(w) = \arccos(w/2r) \).

Proposition 4: For any IC employed for transmitting information through the AWGN channel, the average decoding error probability \( \lambda(S) \) satisfies the following inequality
\[
\lambda(S) \leq \sum_{w \in \Psi(S) \cap [0, 2d_r^2]} \bar{\phi}(S, w) \Pr(z + s \in D(d_r^*(s), w)) + \Pr(|z| > d_r^*(s))
\]
and let \( \Psi_a(S) \subset [0, \infty) \) be the countable set specified by the condition that \( \bar{\phi}_a \neq 0 \) iff \( w \in \Psi_a(S) \).

Denote \( \Psi^*_a(S) = (\Psi_a(S) \cup \Psi(S)) \cap (0, 2d_r^2) \) and \( \bar{\phi}_a(S, w) = \max\{\bar{\phi}_a(S, w), \bar{\phi}(S, w)\} \). Since both inequalities (15) and (14) hold with \( d_r^*(s) \) replaced by any positive value \( d \), e.g., \( d_r^* \), we obtain
\[
M^{-1}(S, a) \sum_{s \in C(b(a))} \lambda(s) \leq \sum_{w \in \Psi^*_a(S)} \bar{\phi}_a(S, w)
\cdot Pr(z + s \in D(d_r^*(s), w)) + \Pr(|z| > d_r^*(s)),
\]
for any \( a > 0 \).

The result now follows by (5).

III. A RANDOM CODING BOUND ON THE ERROR PROBABILITY FOR IC

The bounds derived in the previous section hold for any IC. However, for evaluation of the upper bounds knowledge of \( \{w(s)\} \) is required. In this section we turn our attention to constructing an upper bound on the error probability \( \lambda(S) \) by means of a random coding method [4], [5]. We shall obtain an exponential bound, in which the exponent is a function of the noise variance \( \sigma^2 \) and the normalized logarithmic density (NLD) of the IC, given by \( \delta = n^{-1} \ln \gamma \).

Firstly, we shall define an ensemble of codes with codewords belonging to a cube and derive an upper bound on the average, over this ensemble, of the decoding error probability. Thereafter, we shall construct an IC for which the decoding error probability satisfies the same bound.

Let the length \( n \) and the NLD \( \delta \) be fixed. For some fixed \( a \), consider a code \( G \) with \( N = \lfloor e^{n\delta}a^n \rfloor \) codewords belonging to \( C(b(a)) \). Assume that the codewords of \( G \) are chosen independently of each other according to the uniform distribution on \( C(b(a)) \). The ensemble of all such codes will be denoted by \( J(n, \delta, a) \).
Let $G$ be some code belonging to $\mathcal{I}(n, \delta, a)$. For $\Delta > 0$ and $s \in G$, let $M_i(s)$ be the number of points $s' \in G$ such that $(i - 1)\Delta < d(s, s') \leq i\Delta$. Replacing in (15) $d^2(s)$ by an arbitrary positive number $d$, the following inequality is obtained for the conditional decoding error probability $\lambda_0(s)$:

$$\lambda_0(s) \leq \sum_{i=1}^{\left\lfloor \frac{2d}{\Delta} \right\rfloor} M_i(s) \Pr(z + s \in D(d, (i - 1)\Delta)) + \Pr(|z| > d).$$  

Now we proceed to bound the conditional average, over $\mathcal{I}(n, \delta, a)$, of $M_i(s)$ given the point $s$. It follows by the definition of $\mathcal{I}(n, \delta, a)$ that for any fixed $s \in E_n$ the average number $\overline{M}(s)$ of points within the shell $C_i(s) = V(i\Delta, s) \setminus V((i - 1)\Delta, s)$ satisfies

$$\overline{M}(s) \leq \gamma |C_i(s) \cap Cb(a)| \leq \gamma |V(i\Delta)| \Delta = \frac{\gamma \pi \Delta^2}{\Gamma(\frac{3}{2} + 1)} (i\Delta)^{n-1} \Delta.$$

Averaging (21) over $\mathcal{I}(n, \delta, a)$ and by letting $\Delta \rightarrow 0$ in (21) and (22), we obtain the following bound for the error probability $\overline{\lambda}_0(s)$:

$$\overline{\lambda}_0(s) \leq \frac{\gamma \pi \Delta^2}{\Gamma(\frac{3}{2} + 1)} \int_0^{2d} w^{n-1} \Pr(z + s \in D(d, w)) dw + \Pr(|z| > d).$$

The bound (23) holds for any positive number $d$. Let $d_\delta(n, \delta)$ be the root of the following equation,

$$\frac{\gamma \pi \Delta^2}{\Gamma(\frac{3}{2} + 1)} \int_0^{2d} w^{n-1} \sin^{n-2} \theta \, d\theta = \frac{\sqrt{\pi} \Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}.$$

(24)

It is possible to show, by means of direct computation, that $d_\delta(n, \delta)$, (23) is the right side of the right side of (23).

Since neither (24) nor the right side of (23) depend on the point $s$, (23) with $d = d_\delta(n, \delta)$ serves also as a bound on $\overline{\lambda}_0(s)$, the average [over $\mathcal{I}(n, \delta, a)$] decoding error probability. We wish to investigate the asymptotical behavior of the exponent of this bound.

**Lemma 1:** Let $\rho_0 = \lim_{n \rightarrow \infty} d_\delta^2(n, \delta)/n$. Then

$$\rho_0 = \frac{1}{2\pi e^{25/4} - 1}.$$  

The proof of this lemma is deferred to Appendix A.

**Theorem 1:** For an AWGN channel with noise variance $\sigma^2$

$$\overline{\lambda}_0 \leq e^{-\rho_0 E_{rec}(\delta, \sigma^2)} - o(1),$$

where

$$E_{rec}(\delta, \sigma^2) = \begin{cases} \frac{1}{2} \ln \frac{1}{\sqrt{\pi} \sigma} - \delta, & \delta < \delta_c, \\ \frac{1}{2} \ln \frac{1}{\sqrt{\pi} \sigma^2} + \frac{\delta^2}{4\pi \sigma^2} - \frac{1}{2} \ln \frac{1}{\sqrt{\pi} \sigma^2}, & \delta_c < \delta < \delta^*, \\ 0, & \delta^* < \delta, \end{cases}$$

(27)

The proof of this theorem is provided in Appendix B.

By Theorem 1, there exists a code $G \in \mathcal{I}(n, \delta, a)$ for which the decoding error probability satisfies the bound (26).

Clearly, $G \subset Cb(a) \subset Cb(a + \sqrt{n} \rho_0)$. Consider the infinite constellation $S$ obtained by tiling the whole space $E_n$ with translations of the cube $Cb(a + \sqrt{n} \rho_0)$. For a sufficiently large value of the cube side $a$, i.e., $a = n$, the density of $S$ is equal to $e^6\rho_0 = \rho_n$ and the average decoding error probability $\overline{\lambda}(S)$ of this IC satisfies the bound (26). (See Appendix C for substantiation of this procedure.)

The exponent $E_{rec}(\delta, \sigma^2)$ can be improved for small values of $\delta$ by means of the following expurgation method.

Let, for some $\rho_0 > 0$, $A_{\rho_0}$ and $\overline{A}_{\rho_0}$ be the events of incorrect decoding into a point $s' \in S$ such that $d^2(s, s') < n\rho_0$ and $d^2(s, s') > n\rho_0$, respectively, where $s$ is the transmitted codeword of the code $G$. Denote $\overline{\lambda}_0(s) = \Pr(A_{\rho_0})$ and $\lambda_1(s) = \Pr(\overline{A}_{\rho_0})$. Then we have, analogously to (23),

$$\overline{\lambda}_0(s) = \overline{\lambda}_0(s') + \lambda_1(s') \leq \overline{\lambda}_0(s).$$

By Theorem 1, there exists a subcode $G_0 \subset G$ such that $\overline{\lambda}_0(s') < \lambda_{\rho_0}(s')$. (See Appendix C for substantiation of this procedure.)

The average, over $\mathcal{I}(n, \delta, a)$, of the number of points $M_{\rho_0}(s)$ with Euclidean distance less than $\sqrt{n} \rho_0$ apart from $s$ satisfies the following inequality

$$\overline{M}_{\rho_0}(s) \leq \gamma |V(\sqrt{n\rho_0})| = e^{\rho_0 25/4} (\sqrt{n\rho_0})^n \equiv \overline{M}_{\rho_0}.$$  

(29)

Evidently, the right side of (29) does not depend on $s$. Choosing $\rho_0$ to be the root of the equation $\overline{M}_{\rho_0} = c$, where $c$ is a constant, we obtain by (29)

$$\rho_0 = \frac{1}{2\pi e^{25/4} - 1} - o(1).$$

(30)

Let $\rho_0$ be the solution of the equation $\overline{M}_{\rho_0} = 0.05$. By Chebyshev's inequality we conclude that $\lambda_{\rho_0} \leq 4\lambda_1$ and $M_{\rho_0} \leq 0.2$ for at least half of the codes in $\mathcal{I}(n, \delta, a)$. Let $G$ be one of these codes. Similar reasoning based on Chebyshev's inequality yields that there is a subcode $G_0$ of $G$ with $|G_0| \geq 0.5|G|$ such that $\lambda_{\rho_0}(s') \leq 16\lambda_1$ and $M_{\rho_0}(s') \leq 0.8$ for any $s' \in G_0$. It follows by the last inequality that for $G_0$ we have $M_{\rho_0} = 0$, hence also $\lambda_{\rho_0} = 0$. (Details of this expurgation procedure are provided in Appendix C.)

The expurgation procedure together with Theorem 1 and Proposition 3 enable us to prove the following theorem.

**Theorem 2:** For the AWGN channel with noise variance $\sigma^2$, the following statements hold.

i) There exists a sequence of infinite constellations $S_n$, $n = 1, 2, \ldots$ (where $n$ is the dimension of the Euclidean space) such that the average decoding error probability of $S_n$ satisfies the following asymptotic inequality

$$\overline{\lambda}(S_n) \geq \frac{1}{n} E_U(\delta, \sigma^2) - o(1)$$

(31)

where

$$E_U(\delta, \sigma^2) = \begin{cases} \frac{1}{2} \ln \frac{1}{\sqrt{\pi} \sigma} - \delta, & \delta \leq \delta_c, \\ \frac{1}{2} \ln \frac{1}{\sqrt{\pi} \sigma} - \frac{\delta^2}{4\pi \sigma^2} - \frac{1}{2} \ln \frac{1}{\sqrt{\pi} \sigma}, & \delta_c < \delta < \delta^*, \\ 0, & \delta^* < \delta, \end{cases}$$

(32)
ii) For any infinite constellation \( S_n \),
\[
\frac{1}{n} \ln \lambda(S_n) \leq E_L(\delta, \sigma^2) + o(1), \quad \delta < \delta^* \tag{33}
\]
where
\[
E_L(\delta, \sigma^2) = \frac{1}{4\pi e^{2\phi+1}\sigma^2} + \delta + \frac{1}{2} \ln 2\pi \sigma^2. \tag{34}
\]

iii) \( \lim_{n \to \infty} \lambda(S_n) \geq 0.5 \) for any sequence of infinite constellations \( S_n, n = 1, 2, \ldots \) with NLD \( \delta > \delta^* \).

The proof of this Theorem is presented in Appendix C.

Some remarks:
1) We define the generalized capacity \( C_{\infty} \) of the AWGN channel without restrictions as the largest number such that for any NLD \( \delta < C_{\infty} \) there exists, for a sufficiently large \( n \), an IC with arbitrarily small decoding error probability. It follows by Theorem 2 that \( C_{\infty} = 1/2 \ln \frac{1}{2\pi} \).

2) It is well known [5] that for the customary codes of finite size (i.e., codes with restrictions) the exponent of the upper bound on the decoding error probability obtained by the random coding method consists of the following three regions: "sphere packing bound" (sp), \( R_c < R \leq C \); "straight line bound" (sl), \( R_{sp} < R \leq R_c \); "expurgated bound" (ep), \( 0 < R \leq R_{sp} \), where \( C \) is the capacity of the channel, and \( R_c, R_{sp} \) are two critical rates corresponding to the boundaries of the sp and sl regions. Theorem 2 implies that in the case of an IC (i.e., a code without restrictions) the exponent \( E_U(\delta, \sigma^2) \) consists, similarly, of three analogous regions: sp, \( \delta_c < \delta \leq C_{\infty} \); sl, \( \delta_{sp} < \delta \leq \delta_c \); ep, \( \delta \leq \delta_{sp} \). It also follows also by Theorem 2 that within the sp region the exponent \( E_U(\delta, \sigma^2) \) is asymptotically tight, as in the case of restricted codes.

3) In the proof of Theorem 2 (see Appendix C) we have constructed a code with maximum decoding error probability satisfying the bound (31). This implies that the generalized capacity \( C_{\infty} \) of the AWGN channel without restrictions applies not only to the average but also to the maximum decoding error probability, analogously to the conventional capacity.

4) Any IC in \( E_n \) can be considered as a packing of \( n \)-dimensional Euclidean spheres. It is easy to check that the packing density of the code constructed in the proof of Theorem 2 satisfies asymptotically tight the Minkowski-Hlawka bound.

5) Let
\[
\mu = \gamma^{-\frac{3}{2}} \sigma^2 = e^{-2\phi}. \tag{35}
\]

It is possible to interpret \( \gamma - 2/n \) as the normalized average volume per point of the IC, and \( \mu \) may be regarded as the generalized signal to noise ratio (GSNR). Remarkably, the exponents \( E_U(\delta, \sigma^2) \) and \( E_L(\delta, \sigma^2) \) depend only on \( \mu \) and can be expressed as follows
\[
E_U(\delta, \sigma^2) = E_U(\mu) = \begin{cases} 
\frac{\gamma_0}{\sqrt{2\pi} \sigma^2} e^{-2\phi} \gamma \sqrt{e^{2\phi+1}\sigma^2}, & 8\pi e \leq \mu, \\
\frac{\gamma_0}{\sqrt{2\pi} \sigma^2} e^{-2\phi} \gamma \sqrt{e^{2\phi+1}\sigma^2}, & 4\pi e \leq \mu < 8\pi e, \\
\frac{\gamma_0}{\sqrt{2\pi} \sigma^2} e^{-2\phi} \gamma \sqrt{e^{2\phi+1}\sigma^2}, & 2\pi e \leq \mu < 4\pi e, \\
\frac{\gamma_0}{\sqrt{2\pi} \sigma^2} e^{-2\phi} \gamma \sqrt{e^{2\phi+1}\sigma^2}, & \mu \leq 2\pi e,
\end{cases} \tag{36}
\]
and
\[
E_L(\delta, \sigma^2) = E_L(\mu) = \frac{\mu}{4\pi e^2} - \frac{1}{2} \ln \frac{\mu}{2\pi}, \quad 2\pi e \leq \mu. \tag{37}
\]

The functions \( E_U(\mu) \) and \( E_L(\mu) \) are plotted in Fig. 1.

IV. LINEAR INFINITE CONSTELLATIONS (LATTICES)

In this section we shall consider linear IC's (i.e., lattices). A lattice is an IC which is defined by means of some basis of \( n \) linearly independent vectors \( g_i, i = 1, \ldots, n \), belonging to \( E_n \). The matrix \( T \), whose rows are \( g_i \), \( i = 1, \ldots, n \), is called a generator matrix of the lattice (we shall use \( T \) also for denoting the lattice defined by the matrix \( T \)). Any linear combination of the basis vectors, with integer coefficients, is a point of the lattice and the lattice consists of only such points. A lattice possesses the following two important properties (e.g., see [1]).

1) The density \( \gamma \) of the lattice \( T \) is equal to
\[
\gamma = (\det T)^{-1}. \tag{38}
\]

2) The Voronoi cells of all lattice points are congruent, i.e.,
\[
W(s) = W(0) + s, \quad \text{for all } s \in T. \tag{39}
\]

It follows, by the second property that the conditional probability of error \( \lambda(s) \) does not depend on \( s \) and coincides with the average (over the lattice) probability of error \( \lambda(T) \). For any lattice the spectrum \( \phi(s, w) \) is also independent of \( s \) and coincides with the average spectrum \( \phi(T, w) \). This implies that for any lattice \( d_0(s) = d_0 \); \( d_1(s) = d_0 \); \( d^*_2(s) = d^*_2 \); \( d^*_3(s) = d^*_3 \). Thus it is sufficient to calculate only the probability \( \lambda(0) \).

For some fixed lattice \( T \) and positive number \( \Delta \), let \( M_i \) be the number of points \( s \in T \) such that \( (i - 1)\Delta < |s| \leq i\Delta \). Let \( B_i = \{ x: x \in E_n, (i - 1)\Delta < |x| \leq i\Delta \} \) and \( \chi_i(x) \) be the characteristic function of \( B_i \). Then
\[
M_i = \sum_{s \in T, |s| \neq 0} \chi_i(s). \tag{40}
\]
For any fixed \( d \) and \( \Delta \) we have, analogously to (21), the following upper bound for \( \lambda(0) \)

\[
\lambda(0) \leq \sum_{i=1}^{[d/\Delta]} M_i \Pr(x \in D(d, (i-1)\Delta)) + \Pr(|x| > d)
\]

\[
= \sum_{s \in T} \sum_{i=1}^{[d/\Delta]} \chi_i(s) \Pr(x \in D(d, (i-1)\Delta)) + \Pr(|x| > d)
\]

\[
\equiv \sum_{s \in T} f(s) + \Pr(|x| > d).
\] (41)

Let us turn our attention to the construction of an upper bound on the average probability of error over an ensemble of lattices with fixed density (i.e., with fixed determinant). For this purpose we shall employ the Siegel version of the Minkowski–Hlawka theorem (see [6, p. 205, Theorem 5]), reformulated here as follows.

**Theorem 3 (Minkowski–Hlawka–Siegel):** On the set of all the lattices of density \( \gamma \) in \( \mathbb{E}^n \) there exists a probability measure \( \xi \) such that, for any Riemann integrable function \( f(x) \) which vanishes outside some bounded region

\[
E_\xi \sum_{s \in T} f(s) = \gamma \int_{\xi \in T} f(x) d(x)
\] (42)

where \( E_\xi \) stands for expectation with respect to the measure \( \xi \).

By applying Theorem 3 to the function \( f(x) \) defined by (41), and considering the limit \( \Delta \to 0 \), we deduce that the average probability of error \( \lambda(0) \) for an ensemble of lattices with density \( \gamma \) satisfies the inequality (23). Therefore Theorem 1 holds also for the ensemble of lattices.

We can improve also in the case of lattices the error probability exponent of Theorem 1 by means of an expurgation method. Let \( \chi_{\rho_0}(x) \) be the characteristic function of the set \( \{x: |x| < \sqrt{n/\rho_0}\} \). Then, by applying Theorem 3 to \( \chi_{\rho_0} \), we conclude that the average \( \overline{M}_{\rho_0} \), taken with respect to the measure \( \xi \), of the number of lattice points with Euclidean norm less than \( \sqrt{n/\rho_0} \) is bounded by the right side of (29).

By choosing \( \rho_0 \) to be the root of the equation \( \overline{M}_{\rho_0} = 0.2 \) and applying Chebyshev’s inequality, we conclude that there is a subset of lattices of density \( \gamma \), with measure at least 0.5, such that the number of lattice points \( M_{\rho_0} \) with norm less than \( \sqrt{n/\rho_0} \) does not exceed 0.8 and \( \lambda_1 \leq 4\lambda_1 \) (see [28]). Since \( M_{\rho_0} \) is an integer, this implies that \( M_{\rho_0} = 0 \). The number of points \( M_{\rho_0}(s) \) at distance \( w \) from any given lattice point \( s \) is the same for all lattice points and is equal to the number of points with norm \( w \). Hence \( M_{\rho_0}(s) = M_{\rho_0} = 0 \) for any lattice from the previously mentioned subset. Thus, analogously to Theorem 2, the following theorem can be proved.

**Theorem 4:** Consider an AWGN channel with noise variance \( \sigma^2 \). There is a sequence of lattices \( T_n, n = 1, 2, \cdots \) (where \( n \) is the dimension of the Euclidean space) such that the decoding error probability \( \lambda(0) \) of \( T_n \) satisfies the following asymptotic inequality:

\[
\frac{1}{n} \ln \lambda(0) \geq E_{\xi}(\delta, \sigma^2) + o(1)
\] (43)

where \( E_{\xi}(\delta, \sigma^2) \) is given by (32) and \( \delta = 1/n \log(\det T_n) \).

V. CONCLUSION

The main results obtained in this paper are the following: 1) For any GSNR [see (36)] \( \mu > 2\pi \) there exists an IC with exponentially decreasing probability of decoding error and the exponent is equal to \( E_{\xi}(\mu) \) [see (37)]. 2) The exponent \( E_{\xi}(\mu) \) is asymptotically tight in the region \( 2\pi \leq \mu \leq 4\pi \). The exponent \( E_{\xi}(\mu) \) is attained also by linear IC’s (lattices). Some further aspects of our results are presented in the following remarks.

For \( 2\pi \leq \mu \leq 4\pi \), the exponent of the error probability for an "average" lattice is asymptotically equal to the probability that the noise vector \( x \not\in V(d) \). It follows by Lemma 1 that for an "average" lattice \( G \) the value of \( d^* \) approaches asymptotically, as \( n \to \infty \), the root \( d^* \) of the equation \( |V(d)| = \det G \). It is easy to see that \( d^* \) is the packing (and also the covering) radius for a hypothetical, nonexisting, dense packing. It is shown in [7] that the values \( d^* \) and \( d^* \) are surprisingly close to each other for most good lattices of dimensions 16 to 48.

Assume that we use for transmitting information only such points of (a possibly shifted version of) the lattice which belong to the sphere \( V(\sqrt{n} \bar{P}) \), where \( P \) is the restriction on the average power of every codeword. Then the number of codewords will be \( M = \gamma |V(\sqrt{n} \bar{P})| \) and for the coding rate \( R \) we have

\[
R = \frac{\ln M}{n} = n^{-1} \ln \left( \gamma \sqrt{n} \bar{P}^n \right) \left( \frac{n}{n+1} \right) \approx \frac{1}{2} \ln 2\pi e P\delta.
\] (44)

By setting \( \delta = \delta^* = 1/2 \ln \frac{1}{2\pi e \rho} \) in (44) (\( \delta^* \) is the generalized capacity of the AWGN channel), we conclude that for any coding rate \( R < 1/2 \ln P/\sigma^2 = C(P) \) the error probability of the code decreases exponentially with the length \( n \). The right side of the latter inequality is less than \( C(P) = 1/2 \ln (1 + P/\sigma^2) \)—the capacity of the AWGN channel with the restriction \( P \) on average power. However, the difference between \( C(P) \) and \( C(P) \) is negligible for large values of \( P \), i.e., for large values of the volume occupied by the code.

De Buda showed [8] that it is possible to construct codes, the words of which have equal energy and coincide with the points of some shifted lattice, such that not only the capacity \( C(P) \) but also the random coding exponent of Shannon for the error probability are achieved. The main reason that explains the improvement of the error probability for codes with words lying on the surface of a sphere consists of the fact that the decision regions of a code with words of fixed energy have a conical, rather than spherical, symmetry.

The more conventional coded modulation constructions based on lattices employ as codewords those points of a (possibly shifted) lattice which belong to some cube. If 0 is the center of such cube and \([-\alpha, \alpha] \) is its side, then the number of codewords is \( M = \gamma/2\alpha^2 \) and the coding rate is \( R = \ln 2\alpha + \delta \). Using \( \delta = \delta^* \), we deduce that for any coding rate \( R < 1/2 \ln 2\alpha^2/\sigma^2 \) the error probability of the code decreases exponentially with the length \( n \). Thus we conclude that the capacity \( C(\alpha) \) of the AWGN channel with restriction \( \alpha \) on the peak power satisfies \( C(\alpha) \geq 1/2 \ln 2\alpha^2/\sigma^2 \). The average power of the points belonging to the cube centered...
at 0 and with side $[-a, a]$ is equal to $a^2/3$. Hence
$$C(\alpha) = \overline{C} + 1/2 \ln 6/\pi e.$$

It is well known [5] that, in the case of discrete symmetric
channels, the random coding exponents for the ensemble of
all codes and for the ensemble of linear codes coincide. As
we have seen, the same situation prevails for AWGN channels
employing codes without restrictions. The question, whether
the random coding exponents for the ensemble of all codes and
for the ensemble of linear codes coincide also for all additive
noise channels, remains open.

VI. APPENDIX A

Proof of Lemma 1: For the inner integral in (24) we have
\begin{equation}
\int_0^{\theta(\omega)} \sin^{n-2} \theta d\theta = \frac{\sin^{n-1} \theta}{(n-1) \cos \theta} (1 + o(1)) = \frac{(1 - \frac{\omega^2}{4 \theta^2})^{n-1}}{(n-1) \omega^2} (1 + o(1)).
\end{equation}
(Al)

Substitution of (Al) into the outer integral of (24) yields
\begin{equation}
\frac{1}{n-1} \int_0^{2d} 2d\omega^{n-2} \left( 1 - \frac{\omega^2}{4 \theta^2} \right)^{n-1} \omega^2 d\omega
= \frac{2^{n-1} d^n}{n-1} \int_0^1 \frac{u^{n-1}}{n-1} (1 - u) \frac{1}{2^n} \frac{1}{(n-1) \Gamma(n)}
= \frac{2^{n-1} d^n \Gamma(n)}{n-1} \frac{1}{\Gamma(n)}
= \frac{2^{n-1} d^n \Gamma(n)}{n^2}.
\end{equation}
(A2)

where $B(p, q)$ is the beta function [9]. It follows by (Al),
(A2), and (24) that
\begin{equation}
\frac{\gamma \pi^{\frac{n-1}{2}} 2^{n-1} d^n \frac{1}{\Gamma(n)}}{n^2} = 1 + o(1)
\end{equation}
(A3)

where we have used the property that $\Gamma(x+1) = x \Gamma(x)$. The
proof is completed by applying Stirling’s formula to $\Gamma(x)$ in
(A3).

VII. APPENDIX B

Proof of Theorem 1: Let $z_1, z_2, \cdots, z_n$ be $n$ iid Gaussian
random variables with zero expected value and variance $\sigma^2$.
We shall use the following inequalities, which can be obtained
by means of the Crâmer-Chernoff bound (see [10])
\begin{equation}
\Pr \left( \sum_{i=1}^n z_i^2 \leq n\rho \right)
\leq \begin{cases} 
\exp \left( -n \left( \frac{\sigma^2}{2\rho^2} - 0.5 \ln \frac{\sigma^2}{\rho^2} \right) \right), & \rho \leq \sigma^2, \\
1, & \rho \geq \sigma^2,
\end{cases}
\end{equation}
(B1)

and
\begin{equation}
\Pr \left( \sum_{i=1}^n z_i^2 \geq n\rho \right)
\leq \begin{cases} 
\exp \left( -n \left( \frac{\sigma^2}{2\rho^2} - 0.5 \ln \frac{\sigma^2}{\rho^2} \right) \right), & \rho \geq \sigma^2, \\
1, & \rho \leq \sigma^2.
\end{cases}
\end{equation}
(B2)

We have used the property that $\Gamma(x+1) = x \Gamma(x)$. The
proof is completed by applying Stirling’s formula to $\Gamma(x)$ in
(A3).

\begin{align}
&\Pr \left( z_1 \geq \sqrt{n\omega}, \sum_{i=1}^n z_i^2 \leq n\rho \right) \\
&\leq \begin{cases} 
\exp \left( -n \left( \frac{\sigma^2}{2\rho^2} - 0.5 \ln \frac{\sigma^2}{\rho^2} \right) \right), & \rho \leq \sigma^2, \\
\exp \left( -n \frac{\sigma^2}{2\rho^2} \right), & \rho \geq \sigma^2.
\end{cases}
\end{align}
(B3)

Let $I = I_1 + I_2$ be the integral in the first term of (23), where
\begin{equation}
I_1 = \int_0^{2\sqrt{d^2 - n\sigma^2}} w^{n-1} \Pr \left( z_1 \geq \frac{w}{2}, \sum_{j=1}^n z_j^2 \leq d \right) dw
\end{equation}
(B4)

and
\begin{equation}
I_2 = \int_0^{2\sqrt{d^2 - n\sigma^2}} w^{n-1} \Pr \left( z_1 \geq \frac{w}{2}, \sum_{j=1}^n z_j^2 \leq d \right) dw
\end{equation}
(B5)

For $I_1$ we have
\begin{equation}
I_1 \leq \int_0^{2\sqrt{d^2 - n\sigma^2}} w^{n-1} \Pr \left( z_1 \geq \frac{w}{2} \right) dw
\end{equation}
\begin{equation}
\leq \int_0^{2\sqrt{d^2 - n\sigma^2}} w^{n-1} e^{-\frac{w^2}{8\sigma^2}} dw
= \sigma^2 2\sqrt{d^2 - n\sigma^2} \int_0^{\frac{\sigma^2}{2\rho^2}} 2d^2 \Gamma \left( \frac{n}{2} \right) \sigma^2 e^{-\frac{w^2}{8\sigma^2}} dw
= \sigma^2 2\sqrt{d^2 - n\sigma^2} \int_0^{\frac{n}{2}} \Pr \left( \sum_{j=1}^n z_j^2 \leq d^2 - n\sigma^2 \right)
\end{equation}
(B6)

Let $\omega = w^2/4n$ and $\rho = d^2/n$. By (B3) we deduce that
\begin{equation}
I_2 \leq \exp \left( -n \left( \frac{\rho}{2\sigma^2} - 0.5 \ln \frac{\sigma^2}{\rho^2} \right) \right)
\cdot \int_0^{2\sqrt{d^2 - n\sigma^2}} w^{n-1} \left( d^2 - \frac{w^2}{4} \right)^{\frac{n}{2}} dw
= 2n-1 \cdot \frac{\frac{n}{2}}{\Gamma \left( \frac{n}{2} \right)} \frac{1}{\Gamma \left( \frac{n}{2} \right)}
= 2n-1 \cdot \frac{\frac{n}{2}}{\Gamma \left( \frac{n}{2} \right)} \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n}{2} \right)}
= 2n-1 \cdot \frac{\frac{n}{2}}{\Gamma \left( \frac{n}{2} \right)} \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n}{2} \right)}
= \frac{\frac{n}{2}}{\Gamma \left( \frac{n}{2} \right)} \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n}{2} \right)}
\end{equation}
(B7)

The inequality (23) holds for any $\delta$. Set $d = \sqrt{n\rho}$ [see (25)].
By (23), (B1), (B6), (B7), (B2) and Stirling’s formula for $\Gamma(x)$
the following inequality is obtained:
\begin{equation}
-\frac{1}{n} \ln \lambda \geq \min \{ E_1(\delta, \sigma^2), E_2(\delta, \sigma^2), E_3(\delta, \sigma^2) \} + o(1)
\end{equation}
(B8)

where by (E6), (B1), and (23) (with $d = \sqrt{n\rho}$)
\begin{equation}
E_1(\delta, \sigma^2)
= \begin{cases} 
\frac{1}{2} \ln \frac{1}{\delta^2} - \delta, & \sigma^2 \leq \frac{\delta^2}{4\ln 2}, \\
\frac{1}{2} \ln \frac{1}{\delta^2} + \frac{\delta^2}{2\sigma^2} - 0.5 \ln \frac{\delta^2}{\sigma^2} - \delta, & \sigma^2 \geq \frac{\delta^2}{2}.
\end{cases}
\end{equation}
(B9)
and by (23), (B7), and (B2) \(d = \sqrt{n\rho_6}\)

\[
E_2(\delta, \sigma^2) = E_3(\delta, \sigma^2) = \left\{ \begin{array}{ll} e^{-2\delta} + \delta + 0.5 \ln 2\pi \sigma^2, \\ 0, \quad \sigma^2 \geq \rho_6, \end{array} \right. \quad (B10)
\]

It is easy to check that \(E_1(\delta, \sigma^2) \leq E_2(\delta, \sigma^2),\) if \(\sigma^2 \leq \rho_6/2,\) and \(E_1(\delta, \sigma^2) \geq E_2(\delta, \sigma^2),\) if \(\rho_6/2 \leq \sigma^2 \leq \rho_6.\) Thus [see (28)]

\[
\min \{E_1(\delta, \sigma^2), \ E_2(\delta, \sigma^2), \ E_3(\delta, \sigma^2)\} = E_{ee}(\delta, \sigma^2)
\]

whereby the theorem is proved.

VIII. APPENDIX C

Proof of Theorem 2: We shall show first that in the ensemble \(\mathcal{J}(n, \delta, a)\) there is a code \(G\) whose average decoding error probability satisfies the upper bound (31). Analogously to (B6) in the proof of Theorem 1, we obtain the following inequality

\[
\text{Pr} \left( \frac{n\rho_6}{4} \leq \sum_{i=1}^{n} z_i^2 \leq \sigma^2 - n\sigma^2 \right) \leq \frac{n}{2} \Gamma \left( \frac{n}{2} \right)
\]

By (B1), (B2), and (C1) we conclude (see the proof of Theorem 1) that the exponent of \(\lambda_1(s)\) [the second term in (28)] does not depend on \(s\) and it is not less than \(E_2(\delta, \sigma^2).\) Now let \(\rho_6\) be the solution of the equation \(M_{\rho_6} = 0.05.\) It follows by Chebyshev’s inequality that at least 3/4 of the codes in \(\mathcal{J}(n, \delta, a)\) have the average (taken over codewords) value of \(\lambda_1 \leq 4\lambda_1.\) We obtain by Chebyshev’s inequality also that at least 3/4 of the codes in \(\mathcal{J}(n, \delta, a)\) have the average (over codewords) value of \(M_{\rho_6} \leq 4M_{\rho_6} = 0.2.\) Hence \(\lambda_1 \leq 4\lambda_1\) and \(M_{\rho_6} \leq 0.2)\) for not less than 0.5 of the codes in \(\mathcal{J}(n, \delta, a).\) Let \(G\) be one of these codes. By Chebyshev’s inequality we obtain, in a way analogous to the previous one, that there is a subcode \(G_0\) with the number of codewords \(|G_0| \geq 0.5|G|\) such that \(\lambda_1(s) \leq 16\lambda_1\) and \(M_{\rho_6}(s) \leq 0.8\) for all \(s \in G_0.\) It follows by the last inequality that for the code \(G_0\) the probability \(\lambda_0 = 0.\) Therefore the decoding error probability of the code \(G_0\) satisfies the upper bound (31).

Now we shall construct an IC based on the code \(G_0.\) Let \(I\) be the set of all \(n\)-dimensional vectors with integer components. And let \(S_i(G_0) = G_0 + i(a + \sqrt{n\rho_6}), \ i \in I.\) We define the infinite constellation \(S\) as follows \(S = \cup_{i \in I} S_i(G_0).\) The decoding error probability \(\lambda(s)\) for any point \(s = s_0 + i(a + \sqrt{n\rho_6}) \in S, s_0 \in G_0,\) satisfies the inequality \(\lambda(s) \leq \lambda_0(s_0) + \text{Pr}(|z| > \sqrt{n\rho_6}),\) where \(z\) is the noise vector. Therefore the decoding error probability of the infinite constellation \(S\) satisfies the upper bound (31). Clearly, the NLD of \(S\) is equal to \(n^{-1} \text{ln}(|G_0|/(a + \sqrt{n\rho_6})) = \delta + \ln(a/a + \sqrt{n\rho_6}) - \ln 2/n.\) Setting \(a = n\sqrt{\rho_6},\) the NLD of \(S\) becomes \(\delta - o(1).\) This completes the proof of statement i) of the theorem.

Let us turn now to statement ii). By the definition of \(d^*(s),\) for any \(S\) with NLD \(\delta\) there exists in \(S\) at least one point \(s_0\) such that

\[
|V(d^*(s_0))| = \frac{\pi^{\frac{n}{2}}}{{\Gamma}((\frac{n}{2}) + 1)} \leq \delta^{-1}
\]

implying

\[
\rho_p(s_0) = \frac{d^2(s_0)}{n} \leq \frac{1}{2\pi e^{2\delta/1} + o(1)}.
\]

By the lower bound of Proposition 3

\[
\sup \lambda(s) \geq \text{Pr} \left( \sum_{i=1}^{n} z_i^2 \geq n\rho_p(s_0) \right).
\]

Let \(\rho'_p = \rho_p(s_0) + 1/n\) and let the noise variance satisfy \(\sigma^2 \leq \rho_p(s_0).\) For the right side of (C4) we then have

\[
\text{Pr} \left( \sum_{i=1}^{n} z_i^2 \geq n\rho_p(s_0) \right) \leq \int_{n\rho_p(s_0)}^{\infty} y^{\frac{n}{2} - 1} e^{-\frac{y^2}{2}} dy \leq \frac{n}{2\pi e^{2\delta/1} + o(1)}.
\]

Suppose now that for some \(\Delta > 0\) there is an \(S_0\) with NLD \(\delta\) such that

\[
\lambda(S_0) = e^{-nE_L(\delta, \sigma^2) + \Delta}.
\]

Let \(\rho_\Delta, a_1,\) and \(a_2\) be reals which satisfy the following conditions

\[
\text{Pr} (|z| \geq \sqrt{n\rho_\Delta}) = E_L(\delta, \sigma^2) + \Delta,
\]

\[
n^{-1} \ln \left( \frac{M(S, a_1)}{|V(a_1)|} \right) \geq \delta - \frac{1}{n}
\]

and

\[
-n^{-1} \ln \left( \frac{M^{-1}(s, a_2)}{\sum_{a_2 \in V(a_2)} \lambda(s)} \right) \geq E_L(\delta, \sigma^2) + \Delta - \frac{1}{n},
\]

Let \(r_0 = \max \{a_1, a_2, n\rho_\Delta\} and G_{r_0} = S \cap Cb(r_0).\) It follows by (C9) that by expurgating the worst half of the codewords of \(G_{r_0},\) we obtain a code \(G'_r\) for which the decoding error probability \(\lambda(s)\) satisfies the following inequality

\[
\lambda(s) \leq e^{-nE_L(\delta, \sigma^2) + \Delta - \frac{1}{n} \ln n},
\]

for all \(s \in G'_r.\)

By translating the code \(G'_r\) over the whole space \(E_n\) we obtain, analogously to the proof of i), an infinite constellation \(S_0 = \cup_{i \in I} S_i(G'_r), S_i(G'_r) = G'_r + i(r_0 + \sqrt{n\rho_6}).\)
with NLD $\delta - o(1)$ and maximum decoding error probability satisfying the following inequality

$$\sup_S \lambda(s) \leq e^{-n\left(\epsilon_L(\delta, \sigma^2) + \Delta - \frac{\ln(n)}{n}\right)}.$$  

(C11)

This contradicts (C4) and (C5), hence ii) is proved.

It follows by (C2)-(C4) that if $\delta > \delta^*$ then $\lim_{n \to \infty} \sup_{S \in S_n} \lambda(s) = 1$ for any sequence of infinite constellations $S_n$. Analogously to the proof of ii) we can show that $\lim_{n \to \infty} \lambda(S_n) \geq 0.5$ for any sequence $S_n$, $n = 1, 2, \ldots$ with NLD $\delta > \delta^*$. This completes the proof of iii).

ACKNOWLEDGMENT

The author wishes to thank J. Snyders for a series of very useful remarks and suggestions which undoubtedly improved the content of this paper. The author is also grateful to the anonymous reviewers for the extensive comments that helped to improve the exposition. In particular, I wish to thank the reviewers for pointing out some inaccuracies in the first version of the paper.

REFERENCES