Proceedings of the ASME 2011 International Design Engineering Technical Conferences & Computers and Information in Engineering
IDETC/CIE 2011
August 28-31, 2011, Washington, DC, USA

DETC2011- 48146

The correction to Grubler criterion for calculating the Degrees of Freedoms of Mechanisms

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ABSTRACT

It is well known that the widely used Chybychev-Grubler Kutzbach's criterion does not give the correct mobility for many mechanisms. In this paper we will show that if we add to the known Grubler equation the number of inner forces existing in the mechanism, termed *self-stresses*, we will derive the correct answer. The unique contribution of this paper is that it enables to find this correction on the given mechanism without any need to write the corresponding matrices or any equations.
It should be noted that the correct mobility is obtained from the given configuration of the mechanism; that is the instantaneous mobility and not the global mobility.

Introduction

The problem of identifying the correct mobility of mechanisms has been a source of concern for many researchers, as described for example in [1],[2]. The most simple equation used to compute the mobility of a mechanism is the Chybytchev-Grubler [3] , [4]formula, which for simplicity will be referred to in this paper as the Grubler equation. There are many published works that enable to find the correct DOF for a given mechanism, most of them rely on checking the corresponding matrix of the mechanism, such as the Jacobian matrix [1]and [5]. There are other important works that go in other directions, such as Lie algebra [6] and general views, such as [7]. In this paper we do not give a comprehensive review on the works done in mobility since this work is aimed to enable engineers/students, once they get the mechanism, to calculate the correct DOF(degrees of freedom) only from the topology and configuration of the given mechanism. Our paper shows that if we add a correction to the Grubler equation we will have the correct mobility. This number is proved to be the number of inner self-stresses existing in the mechanism in the specific configuration. Thus, the paper deals with the instantaneous mobility and not the global mobility, although there are cases where it is the same, as shown in the paper. The examples appearing in the paper are for two dimensions, but the method is also applicable to 3d, as it can be followed from the underlying proof of the method.

1.Grubler equation and its correction number

Grubler equation is well known [4], and for 2d it can be written:

$$F = 3*N - 2*P_L - P_H \quad (1)$$

where N denotes the number of links, P_L and P_H denote the number of low and high turning pairs.
As widely reported in many studies, for example [1], this equation sometimes does not give the correct DOF. This paper shows that the difference between the correct mobility and Grubler's result is the number of self-stresses. Thus, the corrected Grubler's equation becomes:

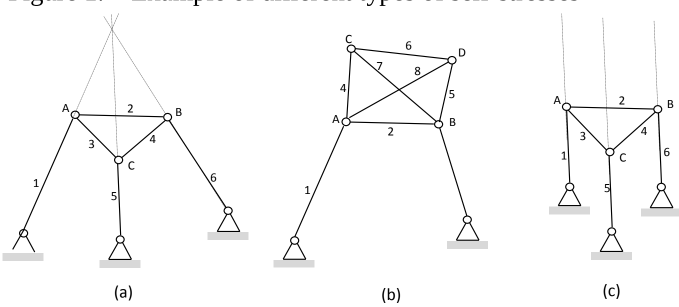
$$F = 3*N - 2*P_L - P_H + SS \quad (2)$$

The focal point of this paper is to show how we find this number, that which we call the Grubler's correction and it is proven in section 4.
For the sake of clarity all the mechanisms in the examples have only low turning pairs.

2.The Self-Stresses in Mechanical Systems

Self-stresses are inner forces that exist in the mechanical elements satisfying the force equilibrium around each joint. There are two types of self-stresses: geometrical (instantaneous) and topological. The former exists only for specific configurations, while the latter exists for almost any geometry for the given mechanism. For example, the determinate truss in Fig. 1a has a self-stress in the rods since it is at a singular position (the continuations of the three ground rods intersect at the same point). Thus, this is a geometrical self-stress and the truss has an infinitesimal motion. Once the truss moves an infinitesimal distance, it goes out of the singular position (the three lines do not intersect), there is no self-stress and the truss therefore becomes immobile.
On the other hand, the truss in Fig. 1b constantly possesses a self-stress in the region {A,B,C,D}. As a result, although Grubler's equation yields zero DOF it constantly has a self-stress and thus has a finite motion.
The truss in Fig. 1c has a special property. It is both a triad, as in Fig. 1a, and is at the singular position, i.e. the continuations of the three ground links intersect at infinity. Due to the parallelism and the equality of the ground rods lengths, it is constantly at the singular for any configuration and it thus possesses a continuous self-stress. Therefore, Gruber's correction remains throughout the entire cycle therefore it has a finite motion.

Figure 1. – Example of different types of self-stresses



a) Geometrical self-stress (Assur graph at the singular position).
b) Topological self-stress.
c) Continuously geometrical self-stress.

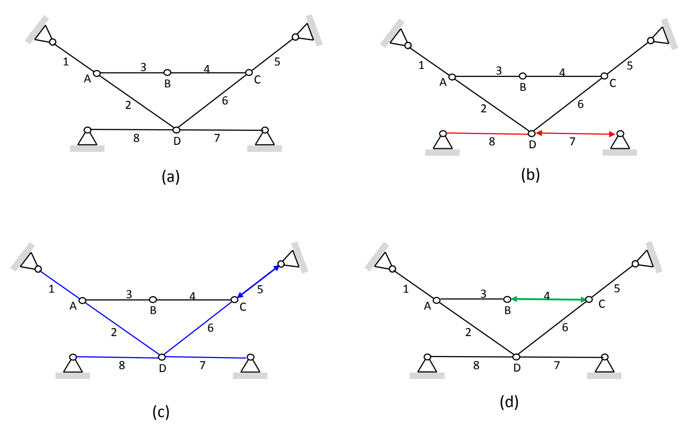
In the following section we introduce two methods for calculating the Grubler's correction and all are performed on the mechanism. The examples are applied to structures which have geometrical self-stresses, i.e., Assur Graphs at the singular configuration.

3.The two main methods for finding the Grubler's Corrections

We start with the method that defines the minimum number of independent inner forces.

3.1 The force method for finding Grubler's Correction

In this method we search for the maximum number of independent forces that can act in the links, such that the force equilibrium around each joint is satisfied. Since each inner force defines a set of links where there are inner forces, termed *self-stress set*, this method searches the maximum independent self-stress sets. This number of independent inner forces is the Grubler's correction.
An example of applying the force method to calculate the exact dof is given in Fig.2. For the system in Fig.2.a, Grubler's equation yields zero dof. In Fig.2b we apply inner force in link 7 that defines the self-stress set – {7,8}. Applying an inner force within another link, not included in the existent self-stress sets (in this case link 5), yields a self-stress in links {5,6,1,2,7,8} defining the second set as shown in Fig.2 c. The only links that do not belong to any set are {3,4}. However, applying an inner force in one of them, for example in link 4, does not define any self-stress set since around joint C there cannot be equilibrium of forces, as shown in Fig. 2d.
In summary, there are two self-stresses, as shown in Fig. 2e. Thus the correction number is two, meaning, there are 2 dof for this specific geometry of the system. In Fig.2f we can see the two independent motions, which in this case are both infinitesimal motions. Joint B is mobile and links {5,4,3,1} can rotate with one dof, thus the exact number of dof for this configuration is two.



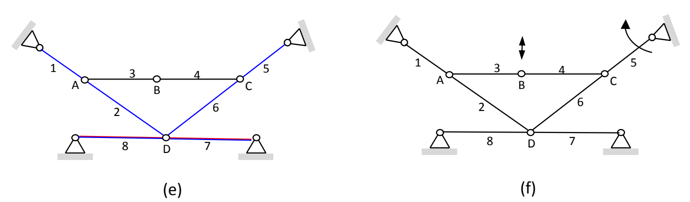


Figure 2 – An example of finding the Grubler's correction using the force method.

The pairs of the following links (1,2), (5,6),(7,8) and (3,4) are collinear

Another method for finding Grubler's correction, i.e., number of self-stresses is based on the assembly process, as given below.

3.2 The Assembly method for finding the Grubler's correction

In this method we start to assemble the system and during the assembly we find the number of self-stresses. The case where a self-stress occurs is as follows:
The assembly self-stress rule : If you have to add a link during the construction of the system in such a way that it is inserted between joints in which at least two of them are immobile, we will have an additional self-stress

For the sake of clarity, we apply the assembly method to the same problem that we had applied to the force *method*, as appears in Fig. 2.
It is easy to verify that there is no problem assembling the first 5 links, (Figs. 3a –3d), since at most one end joint of the links is immobile. The first problem arises in step – 'e' where link 5 has to be added while joint C is restricted to be collinear with link 5, thus it cannot rotate and cannot move since joint D is immobile. The other end joint of link 5 is a pinned joint and thus is of course immobile. Therefore, according to the assembly self-stress rule, a new self-stress has been created. The process continues until step 'i' (Fig.3i) at which the two end joints of link 8 are again immobile, thus the second self-stress is constructed.
Note, any other order of assemble results with the same number of self-stresses.

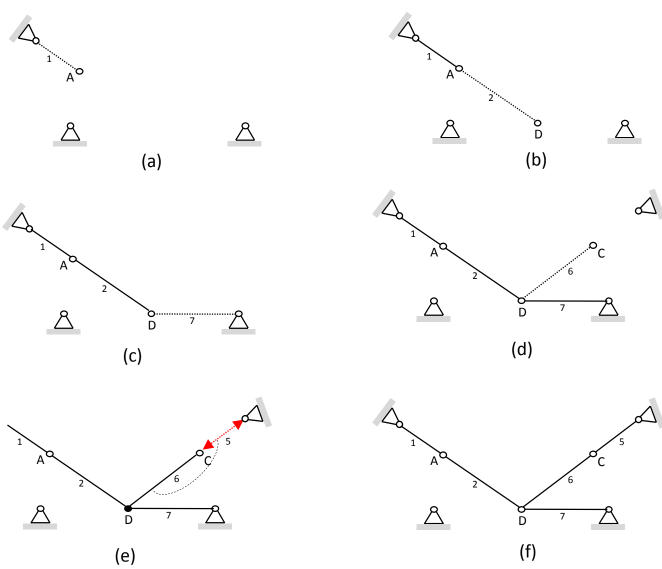
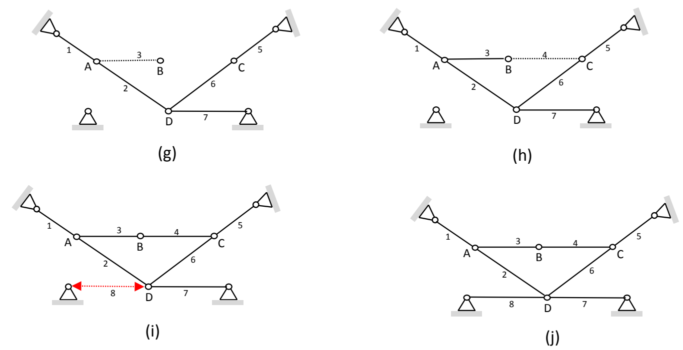


Figure 3 – An example of finding the Grubler's correction using the assembly method.

4. The mathematical proof underlying the proposed method



For sake of convenience, the proof relies on the mathematical foundation developed in rigidity theory[8]. It should be clarified that other proofs, relying on Jacobian matrices and other methods result with the same conclusion.
The idea underlying Grubler's equation is that each link has 'd' dof (d stands for the dimension) and once we connect between two links (i.e, two rigid bodies) we add constraints and thus the dof is reduced. In the rigidity theory community this result is achieved differently. In this community we start with the free joints and at each iteration we add a link, i.e. constraint. In this way we reduce the dof until the actual dof is obtained. For simplicity, let us explain the idea on linkages whose joints are all of type revolute joints and each link is with degree two, i.e. it connects between two joints. In these linkages we start with 2*j dof and adding each link reduces the dof by one. Hence we have the following variant of Grubler equations:

$$F = 2 * J - N. \qquad (3)$$

This relation can be written in matrix form, known as *rigidity matrix* [9],[10], with N rows and 2*j columns. For each link 'i', whose end joints are U and V, should appear with four elements in row 'i' as follows; in column Ux: Ux – Vx and in column Uy: Uy-Vy. The same numbers with opposite signs appear in the two columns Vx and Vy as follows; in column Vx: Vx-Ux and in column Vy: Vy – Uy.

Example of mechanism, four bar link, for which we construct its corresponding rigidity matrix appears in Fig 4.

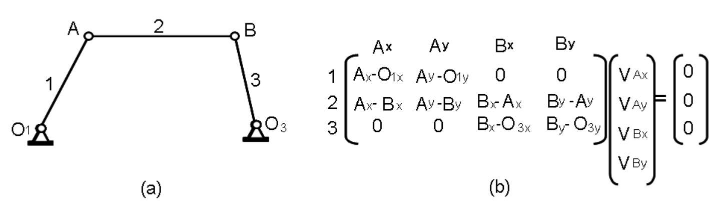


Figure 4 – An example of a mechanism (a) and its corresponding rigidity matrix (b)

The columns of the rigidity matrix span the space of the velocities where the rows span the space of forces. The kernel of the dimension of the columns is actually the dof of the mechanism. If the rows are independent, the kernel of the matrix is then equal to 2*V – N. In the case where the rows are dependent, i.e., the rows of the forces, it follows that there in those rows that are dependent there exist inner self-stresses. Let us denote the kernel of the rows by SS, i.e., there are SS independent self-stress that span the kernel space of the rows. Thus, the dimension of the kernel of the columns also increased by SS, which means the dof are increased by the number SS.
Therefore, we result with the following equation:

$$F = 2*V - N + SS. \quad (4)$$

From the above, since we did not relate to the dimension of the mechanism it follows that Grubler's correction is applied also to spatial mechanisms. It can be verified that in the known example of parallel Cartesian robotic manipulator in [1] where Grubler's equation yields erroneous result, the Grubler's correction number is four, i.e., there are four self-stresses.

Conclusion and Further Research

The paper shows that it is possible to add a correction to the known Grubler's criterion and obtain the correct dof for the given mechanism. As mentioned in the introduction, this corrected dof result is instantaneous, that is, for the given configuration, although there are cases, as shown in Fig. 1 where it also gives the global dof. The idea is to proceed and obtain a correct mobility result for both cases: instantaneous and global. We can expect this objective to be achieved since there is much ongoing reaseach to find the global dof, using a novel algorithm, pebble game. We hope that combining the two approaches will yield the needed result. Both methods work on the mechanisms without the need to write the corresponding matrices or velocity equations.

Acknowledgements

The author would like to thank Dima Mazor, Daniel Rubin and his son Guy Shai for their help.

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